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Remark on a certain theorem of H. J. Bremermann

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In paper [1] H. J. Bremermann proved the following theorem:

Let D be a bounded pseudo-convex domain in the space of n complex variables $z = \{z_1, z_2, \ldots, z_n\}$ of the form $D = \{z \,|\, V(z) < 0\}$, $V(z)$ continuous in a neighborhood of D. Then the generalized Dirichlet problem is possible for the upper envelope $\Phi(z)$ of $L\{D, b(z)\}$ ($L\{D, b(z)\}$ is the class of functions that are plurisubharmonic in a neighborhood of D and smaller or equal to the boundary values $b(z)$ wherever these are prescribed) and arbitrary continuous boundary values $b(z)$ if and only if the boundary values $b(z)$ are prescribed on and only on the Šilov boundary $S(D)$ of D.

The Šilov boundary $S(D)$ of a domain D is the smallest closed subset of the boundary $D^{\cdot}$ such that for every function $f(z)$ holomorphic in D and continuous in $\bar{D}$ we have $|f| \leqslant M$ in D if $|f| \leqslant M$ on $S(D)$.

A real valued function $V(z)$ is plurisubharmonic in a domain D if and only if the following conditions are satisfied: (i) $-\infty \leqslant V(z) < \infty$, (ii) $V(z)$ is upper semi-continuous, (iii) the restriction of $V(z)$ to any analytic plane $E = \{z \,|\, z = z_0 + \lambda a\}$ is subharmonic in the intersection ED.

The present note is the generalization of the theorem mentioned above when D is an arbitrary bounded domain and $S^*(D)$ is the smallest closed subset of the boundary of D such that for every function $V(z)$ plurisubharmonic in D and continuous in $\bar{D}$ is $V(z) \leqslant M$ in $\bar{D}$ if $V(z) \leqslant M$ on $S^*(D)$. The existence and uniqueness of $S^*(D)$ is given in [2].

For the sake of brevity we denote by $\mathfrak{M}$ the class of all plurisubharmonic functions $\psi(z)$, $z \in D$, continuous in $\bar{D} = D + D^{\cdot}$ and $\leqslant b(z)$ on $D^{\cdot}$, where $b(z)$ is an arbitrary continuous function defined on $D^{\cdot}$.

From the definition of $S^*(D)$ the proposition follows immediately:

(P) For every point $z_0 \in S^*(D)$ and for every neighborhood $O(z_0)$ of z_0 there exists a function plurisubharmonic in D, continuous in $\bar{D}$ such that $V(z) < M$ in $\bar{D} - O(z_0)$ and $V(z) \geqslant M$ in some points of $D^{\cdot} O(z_0)$.

The proof of the following theorem is similar to that given by H. J. Bremermann.

THEOREM. *Let D be a bounded domain in the space of n complex variables, regular with respect to the Dirichlet problem for harmonic functions. Let $S^*(D)$ be the Šilov boundary of D with respect to the class of all plurisubharmonic functions in D and continuous in $\bar{D}$. Then there exists in D the bounded upper envelope $\Phi(z)$ of the class $\mathfrak{M}$ and* $\overline{\lim}_{z\to z_0\in S^*(D)} \Phi(z) = b(z_0)$.

Proof. 1° The existence of the bounded envelope $\Phi(z) = \overline{\lim}_{z'\to z} \{\sup_{\psi\in\mathfrak{M}} \psi(z')\}$ follows from the fact that every function $\psi(z)\in\mathfrak{M}$ which is $\leqslant M = \max_{z\in S^*(D)} b(z)$ on $S^*(D)$ is $\leqslant M$ in $\bar{D}$. Therefore $\Phi(z)\leqslant M$ in D. On the other hand, the function $\psi(z) = m = \inf_{z\in D^\cdot} b(z)$ is plurisubharmonic and belongs to $\mathfrak{M}$. Hence $\Phi(z)\geqslant m$.

2° From the regularity of D with respect to the Dirichlet's problem follows the existence of a function $h(z)$ harmonic in D, continuous in $\bar{D}$ and equal to $b(z)$ on $D^\cdot$. Since every plurisubharmonic function $\psi(z)\in\mathfrak{M}$ is subharmonic in D from the inequality $\psi(z)\leqslant b(z)$ on $D^\cdot$ follows $\psi(z)\leqslant h(z)$ in $\bar{D}$ and therefore $\Phi(z)\leqslant h(z)$ in D. Hence

$$\overline{\lim}_{z\to z_0\in D^\cdot} \Phi(z) \leqslant b(z_0)\,. \tag{1}$$

3° Let z_0 be an arbitrary point of $S^*(D)$ and let $O_1(z_0)$ be a neighborhood of z_0 such that $b(z_0)+\varepsilon > b(z) > b(z_0)-\varepsilon$, $\varepsilon>0$. From the proposition (P) follows that for every neighborhood $O(z_0)\subset O_1(z_0)$ there exists a function $V(z)$ plurisubharmonic in D, continuous in $\bar{D}$ such that for $z\in D-O(z_0)$ we have $V(z) < \sup_{z\in D^\cdot O(z_0)} V(z) = V(z_1) = 1$, where $z_1\in D^\cdot O(z_0)$.

Let $u(z)$ be the following function

$$u(z) = b(z_0)-\varepsilon+c[V(z)-1]\,,\qquad c=\text{const}>0\,.$$

$u(z_1) = b(z_0)-\varepsilon$ and $u(z)\leqslant b(z_0)-\varepsilon < b(z)$ for $z\in D^\cdot O(z_0)$. Taking $c>0$ large enough we achieve that $u(z)<b(z)$ for $z\in D^\cdot - O(z_0)$. Therefore $u(z)\in\mathfrak{M}$ and $\underline{\lim}_{z\to z_1} \Phi(z) \geqslant \lim_{z\to z_1} u(z) = b(z_0)-\varepsilon > b(z)-2\varepsilon$. Since $\varepsilon>0$ is arbitrarily small we have

$$\underline{\lim}_{z\to z_1} \Phi(z) \geqslant b(z_1)\,. \tag{2}$$

Let $O_n(z_0)\subset O_1(z_0)$ be a sequence of neighborhoods of z_0 with radius $r_n\to 0$. For every neighborhood $O_n(z_0)$ we can construct the plurisubharmonic function $u_n(z)\in\mathfrak{M}$ such that $u_n(z_n) = b(z_0)-\varepsilon > b(z_n)-2\varepsilon$, $z_n\in D^\cdot O_n(z_0)$. Hence there exists a sequence of points $\{z_n\}$, $z_n\in D^\cdot O_1(z_0)$, $z_n\to z_0$ such that $\underline{\lim}_{z\to z_n} \Phi(z)\geqslant b(z_n)$. Therefore in every neighborhood of $z_0\in S^*(D)$ there are points $z\in D$ such that $\Phi(z)\geqslant b(z_0)-\varepsilon$. Hence

$$\overline{\lim}_{z\to z_0\in S^*(D)} \Phi(z) \geqslant b(z_0)\,. \tag{3}$$

It follows from (1) and (3) that $\overline{\lim}_{z\to z_0\in S^*(D)} \Phi(z) = b(z_0)\,.$

Remark. 1° If every point $z_0\in S^*(D)$ is regular, for example, if there exists a plurisubharmonic function $V(z)$, $z\in D$, continuous in $\bar{D}$ such that $V(z_0)=1$ and $V(z)<1$ in $\bar{D}-(z_0)$ then we have $\lim_{z\to z_0\in S^*(D)} \Phi(z) = b(z_0)$.

2° If the function $b(z)$ is prescribed on $D^\cdot - O(z_0)$, $z_0\in S^*(D)$ then $\Phi(z)$ will be unbounded in D. In fact, from (P) follows that there is a plurisubharmonic function $V(z)\geqslant M$ in some points $\in D^\cdot O(z_0)$ and $\psi(z)<M = \max_{z\in S^*(D)-O(z_0)} b(z)$ in $D^\cdot - O(z_0)$. For sufficiently large n the function $V_1(z) = M\big(V(z)/M\big)^n$ would be plurisubharmonic in D, continuous in $\bar{D}$, $\leqslant b(z)$ in $D^\cdot - O(z_0)$ and arbitrarily large in $D^\cdot O(z_0)$. Hence $\Phi(z)$ will not be bounded in D.

3° $\Phi(z)$ cannot be larger in D than $\sup_{z\in S^*(D)} b(z)$. If $\sup_{z\in D^\cdot - S^*(D)} b(z) > \sup_{z\in S^*(D)} b(z)$ then $\Phi(z)$ does not assume the values $b(z)$ on the whole boundary $D^\cdot$.

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