Some theorems on double integrals over rectangles

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1. Notation. Suppose \( a \leq x \leq b \) and \( c \leq y \leq d \). We shall denote by \([a; x; e; y]\) and \([a; x; e; y]\), or briefly by \([x; y]\) and \([x; y]\) \(a\) and \(e\) being fixed, the rectangles

\[
((s; t); a \leq x \leq b) \quad \text{and} \quad ((s; t); a \leq x \leq b; c \leq t \leq y),
\]

respectively. By \( \langle b; d \rangle \) we shall denote the polygonal line, formed by two sides of the greatest rectangle \([a; b; c; d] = [b; d]\), joining the points \((a, b)\) and \((a, d)\) and passing through \((a, b)\).

In theorem 1 we consider functions \( f(s; t), g(s; t), \varphi(s; t) \) bounded and Riemann-integrable in the rectangle \([a; b; c; d] = [b; d]\). The function \( \varphi(s; t) \) is non-negative and non-increasing in \([b; d]\) in each variable, separately, and such that for any pair \( t', t'' \), where \( c \leq t' < t'' \leq d \), the difference \( \varphi(s; t') - \varphi(s; t'') \) is non-increasing with respect to \( s \) in the interval \((a, b)\). Moreover, we assume that \( \varphi(s; t) \) is not identical to zero in \([b; d]\). The integrals are taken in the sense of Riemann. If \( a > 0 \) and \( c > 0 \) then for example, \( \varphi(s; t) = (s + t)^{-a} \) and \( \varphi(s; t) = (s)^{-a} (a > 0) \) satisfy the above conditions in any rectangle \([b; d]\).

In theorems 2 and 3 the function \( f(s; t) \) is Lebesgue-integrable over \([0, 1; 0, 1]\), and the function \( \varphi(s; t) \) satisfies the same conditions as before, in this square, and the integrals are taken in the Lebesgue sense.

2. An analogue of a Bieracki theorem. At first we shall give the fundamental

**Lemma 1.** Let \( \omega_{ij} (1 \leq i \leq m, 1 \leq j \leq n) \) be arbitrary numbers and let \( v_{ij} \) and \( w_{ij} \) be such that

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij} > 0 \quad \text{(or < 0)} \quad (1 \leq i \leq m, 1 \leq j \leq n),
\]

\[
0 \leq \omega_{i+1,0} \leq \omega_{i0}, \quad 0 \leq \omega_{m,j+1} \leq \omega_{mj} \quad (1 \leq i \leq m-1, 1 \leq j \leq n-1),
\]

\[
\omega_{i+1,j} - \omega_{i+1,j+1} \leq \omega_{ij} - \omega_{i+1,j} \quad (1 \leq i \leq m-1, 1 \leq j \leq n-1).
\]
Write

\[
M_i = \min_{(a_i)} \left\{ \frac{1}{r} \sum_{i=1}^{r} u_{ij} \right\}, \quad M_i = \max_{(a_i)} \left\{ \frac{1}{r} \sum_{i=1}^{r} u_{ij} \right\}
\]

\[
(1 \leq i \leq m, \quad 1 \leq j \leq n).
\]

If, moreover, \( \sum_{i=1}^{m} u_{ij} \neq 0 \), then

\[
M_i \leq \frac{1}{r} \sum_{i=1}^{r} u_{ij} \leq M_i
\]

(1)

Proof. We replace the numerator of the quotient (1) by the right-hand side of the identity (3), p. 16

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} w_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} U_{ij}(\omega_{ij} - \omega_{i+1,j} - \omega_{i,j+1} + \omega_{i+1,j+1}) +
\]

\[
+ \sum_{i=1}^{n-1} U_{mm}(\omega_{mm} - \omega_{mi+1}) + \sum_{j=1}^{m-1} U_{mm}(\omega_{mm} - \omega_{mj+1}) + U_{mm} \omega_{mm},
\]

where

\[
U_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij};
\]

similarly we write the denominator of this quotient.

To obtain the inequality (1) it is sufficient to use the following:

if \( B_k > 0 \) (or < 0), \( a_k \geq 0 \) (\( k = 1, 2, ..., r \)), \( a_k \) are not all equal to zero, and if

\[
S_k \leq \frac{A_k}{B_k} \leq S_k \quad (k = 1, 2, ..., r),
\]

\( S_k \) and \( S_k \) being constants, then

\[
S_k \leq \frac{1}{r} \sum_{i=1}^{r} A_k \leq S_k.
\]

Now, we shall present the following (compare [1], pp. 123-126; [2], pp. 70-81)

\[
\text{Theorem 1. Let}
\]

\[
\int \int f(x, t) \, dt \, dx \quad \text{for} \quad (x, y) \in (b', d),
\]

\[
\int \int g(x, t) \, dt \, dx \quad \text{for} \quad (x, y) \in (b', d),
\]

\[
\int \int \frac{f(x, t)}{g(x, t)} \, dt \, dx \quad \text{for} \quad (x, y) \in (b', d),
\]

\[
H(x, y) = \int \int \frac{f(x, t)}{g(x, t)} \, dt \, dx \quad \text{for} \quad (x, y) \in (b', d'),
\]

\[
\int \int f(x, t) \, dt \, dx \quad \text{for} \quad (x, y) \in (b', d),
\]

\[
\int \int g(x, t) \, dt \, dx \quad \text{for} \quad (x, y) \in (b', d),
\]

\[
f(a, c) \quad \text{for} \quad (x, y) = (a, c).
\]

1° If \( \int \int g(x, t) \, dt \, dx \neq 0 \) for every point \( (x, y) \in (b, d), \) then

\[
\int \int g(x, t) \, dt \, dx \neq 0
\]

(3)

where the infimum and the supremum are taken over all \( (x, y) \in (b, d). \)

2° If, moreover, the functions \( f(x, t) \) and \( g(x, t) \) are continuous in rectangles \([a, a+\delta] \times \delta, \delta, [a, b, c, c+\delta], \) respectively, with \( \delta > 0 \) as small as we please, and if

\[
g(a, c) \neq 0, \quad \int \int g(x, t) \, dt \, dx \neq 0, \quad \int \int f(x, t) \, dt \, dx \neq 0
\]

(a < x < b, c < y < d), then there exists a point \( (\xi, \eta) \in (b, d) \) such that

\[
\int \int f(x, t) \, dt \, dx \quad \int \int g(x, t) \, dt \, dx = H(\xi, \eta).
\]

Proof. 1° Let \( (a, y) \) be the point, lying inside \([a, b, c, d] = [b, d], \)

at which the value of the function \( \varphi(s, t) \) is positive. Taking the normal sequence of partitions

\[
\nu \{ a = a^0 < a^1 < \ldots < a^{m-1} < a^m = b \} \quad \{ k = 1, 2, \ldots, m \}
\]

\[
eq \nu \{ c = c^0 < c^1 < \ldots < c^{m-1} < c^m = d \} \quad \{ k = 1, 2, \ldots, m \}
\]
of the rectangle $[b; d]$ such that $\alpha = \frac{a}{b}$, $\gamma = \frac{b}{c}$ ($0 < \mu_k < m_k$, $0 < \theta_k < \phi_k$), we denote by $I_1(\alpha; f)$, $I_2(\alpha; f)$ and $I_3(\alpha; f)$ the sums

$$
\sum_{p=1}^{n_1} \sum_{t=1}^{n_2} \int \int f(s, t)\varphi(s, t) \, ds \, dt, \quad \sum_{p=1}^{n_1} \sum_{t=1}^{n_2} \int \int \{f(s, t) - \varphi(s, t)\} \, ds \, dt, \\
\sum_{p=1}^{n_1} \sum_{t=1}^{n_2} \varphi(s_{p}, t_{p}) \int \int f(s, t) \, ds \, dt,
$$

(respectively, where $\int \int$ is the rectangle $[a_{p}, b_{p}] \times [c_{p}, d_{p}]$). By $I_1(\alpha; g)$, $I_2(\alpha; g)$ and $I_3(\alpha; g)$ we shall understand the sums (7) in which $f$ is replaced by $g$. Let $Q_1, Q_2$ be the infimum and the supremum of the function $H$ in $[b; d]$.

Evidently,

$$
I(\alpha; f) = \int \int f(s, t)\varphi(s, t) \, ds \, dt, \quad I(\alpha; f) = I_1(\alpha; f) + I_2(\alpha; f).
$$

From the well-known test of integrability it follows

$$
\lim_{k \to \infty} I(\alpha; f) = \lim_{k \to \infty} I(\alpha; g) = 0,
$$

whence

$$
\lim_{k \to \infty} I_2(\alpha; f) = \int \int f(s, t) \, ds \, dt, \quad \lim_{k \to \infty} I_3(\alpha; f) = \int \int g(s, t) \, ds \, dt.
$$

Let $G(x, y) = \int \int g(s, t) \, ds \, dt$. Suppose that $G(x, y) > 0$ for every $(x, y) \in [b; d]$. Then, there exists a constant $l > 0$ in the rectangle $[a, b; c, d]$. Applying the transformation (2), we obtain

$$
I_2(\alpha; g) \geq \sum_{p=1}^{n_1} \sum_{t=1}^{n_2} \{G(s_{p}, t_{p})\}[\varphi(s_{p}, t_{p}) - \varphi(s_{p-1}, t_{p-1}) + \varphi(s_{p}, t_{p}) + \varphi(s_{p}, t_{p}) - \varphi(s_{p-1}, t_{p-1}) + \varphi(s_{p}, t_{p}) - \varphi(s_{p-1}, t_{p-1}) + \varphi(s_{p}, t_{p}) - \varphi(s_{p-1}, t_{p-1})]
$$

Hence

$$
I_2(\alpha; g) \neq 0 \quad \text{and} \quad \int \int g(s, t)\varphi(s, t) \, ds \, dt \neq 0.
$$

The last two inequalities hold also in the case $G(x, y) < 0$.

In virtue of lemma 1,

$$
Q_1 \leq \frac{I_2(\alpha; f)}{I_2(\alpha; g)} \leq Q_2,
$$

and by (8) we obtain (3).

2° If the functions $f$ and $g$ are continuous in $[a + \delta; d]$ and $[b; c + \delta]$, respectively, and if the conditions (4) are satisfied, then, it is easy to verify that the function $H$ is continuous in the rectangle $[b; d]$. From (9), (8) and the Darboux theorem the equality (5) follows.

**Corollary 1.** Replacing the function $f$ in the inequality (3) by $pf$, where $p = p(s, t)$ is positive and Riemann-integrable in $[b; d]$, and the function $g$ by $p$, we obtain, under the assumption

$$
\int \int p(s, t) f(s, t) \, ds \, dt = \inf_{k \to \infty} \int \int p(s, t) f(s, t) \, ds \, dt, \\
\int \int p(s, t) \, ds \, dt \leq \int \int p(s, t) \, ds \, dt \\
\int \int p(s, t) \, ds \, dt = \inf_{k \to \infty} \int \int p(s, t) \, ds \, dt,
$$

the two-dimensional analogue of the Tchebychev inequality (compare [1], pp. 128-139)

$$
\int \int p(s, t) \, ds \, dt \cdot \int \int p(s, t) \, ds \, dt \leq \int \int p(s, t) \, ds \, dt \\
\int \int p(s, t) \, ds \, dt,
$$

it is easy to observe that the assumption that $\varphi(s, t)$ is non-negative can be omitted.

**Corollary 2.** If $g(s, t) = 1$ in (3), we have

$$
\int \int \frac{f(s, t) \, ds \, dt}{(x - \alpha)(y - \beta)} \leq \int \int \frac{f(s, t) \, ds \, dt}{(x - \alpha)(y - \beta)} \cdot \int \int \varphi(s, t) \, ds \, dt,
$$

whence

$$
\int \int f(s, t) \, ds \, dt \leq \sup_{(x,y) \in [a,b] \times [c,d]} \left(\int \int f(s, t) \, ds \, dt \cdot \int \int \varphi(s, t) \, ds \, dt\right),
$$

whence

$$
\int \int f(s, t) \, ds \, dt \leq \sup_{(x,y) \in [a,b] \times [c,d]} \left(\int \int f(s, t) \, ds \, dt \cdot \int \int \varphi(s, t) \, ds \, dt\right),
$$

whence

$$
\int \int f(s, t) \, ds \, dt \leq \sup_{(x,y) \in [a,b] \times [c,d]} \left(\int \int f(s, t) \, ds \, dt \cdot \int \int \varphi(s, t) \, ds \, dt\right),
$$

It is easy to verify that (10) remains true for functions $f$ having finite improper Riemann integral $\int \int f(s, t) \, ds \, dt$, if the integrals $\int \int f(s, t) \, ds \, dt$ and $\int \int f(s, t) \, ds \, dt$ are understood as improper (with respect to $\gamma, \delta$) in the Riemann sense.

3. **An analogue of a Natanson theorem.** We shall prove that the inequality (10) is true for any Lebesgue-integrable function $f$ if integrals $\int \int f(s, t) \, ds \, dt$ and $\int \int f(s, t) \, ds \, dt$ are considered in the Lebesgue
sense. Of course, it is interesting only the case when the supremum in (10) is finite. For the sake of brevity we assume \( a = c = 0, b = d = 1 \) and denote by \( L^* \) the class of functions \( f(x, t) \), Lebesgue-integrable in the square \([0, 1; 0, 1] = [1, 1]\), such that
\[
\frac{1}{h_i h_i} \int_{(h_i h_i)} f(x, t) \mathrm{d} s \mathrm{d} t \leq M(f) \quad \text{for} \quad 0 < h_i \leq 1 \quad (i = 1, 2),
\]
where \( M(f) \) is a constant depending only on \( f \).

**Lemma 2.** For every function \( f \in L^* \) and for every positive \( \varepsilon \) there is a function \( f_\varepsilon \in L^* \) continuous in any closed domain included in \([1, 1]\) which does not contain the line \( y = x \), eventually with the exception of a finite number of segments parallel to the axes \( x, y \), such that the improper (with respect to \( y = x \)) Riemann integral
\[
\int \int |f(x, t)| \mathrm{d} x \mathrm{d} t \mathrm{d} s \mathrm{d} t
\]
is finite and
\[
(11) \quad \sup_{\varepsilon > 0, h_i, h_i} \frac{1}{h_i h_i} \int_{(h_i h_i)} |f(x, t)| \mathrm{d} s \mathrm{d} t \leq \varepsilon.
\]

**Proof.** Let \( h_i \) and \( h_i \) be two fixed positive numbers less than or equal to 1 and let \( n_i \) and \( n_i \) denote positive integers such that
\[
2^{-n_i} < h_i \leq 2^{1-n_i}, \quad 2^{-n_i} < h_i \leq 2^{1-n_i}.
\]

Divide the square \([0, 1; 0, 1]\) into rectangles by straight lines, parallel to axes of coordinates, passing through points \( 2^{1-n_i} (r = 1, 2, \ldots) \) on both axes.

In view of the two-dimensional analogue of the Weierstrass theorem, in every rectangle \( P_\mu = [2^{-\mu}, 2^{-\mu}; 2^{-\mu}, 2^{-\mu}] \) there exists an algebraic polynomial \( W_\mu(x, t) \) of two variables such that
\[
\int \int |f(x, t) - W_\mu(x, t)| \mathrm{d} x \mathrm{d} t \mathrm{d} s \mathrm{d} t < \frac{\varepsilon}{2^{n_i+n_i}} \quad (\mu, \nu = 1, 2, \ldots).
\]

Let \( f_\mu(x, t) \) be defined as \( W_\mu(x, t) \) inside each rectangle \( P_\mu \), and zero at the remaining points of the square \([0, 1; 0, 1]\). Then
\[
\frac{1}{h_i h_i} \int_{(h_i h_i)} |f(x, t) - f_\mu(x, t)| \mathrm{d} s \mathrm{d} t \leq 2^{-n_i-n_i} \sum_{\mu = 0}^{n_i} \int_{P_\mu} |f(x, t) - W_\mu(x, t)| \mathrm{d} x \mathrm{d} t \leq \frac{\varepsilon}{2^{n_i+n_i}} < \varepsilon.
\]

So we have proved the inequality (11) from which it follows that the Lebesgue integral
\[
\int \int |f_\mu(x, t)| \mathrm{d} s \mathrm{d} t
\]
is finite, i.e. the Riemann improper integral
\[
\int \int |f(x, t)| \mathrm{d} s \mathrm{d} t
\]
being equal to it, is finite also, and \( f \in L^* \).

Now, an analogue of the Natanson lemma ([4], pp. 243-245) for double integrals will be given.

**Theorem 2.** If
\[
K = \sup_{(x, t) \in (h_i h_i)} \left. \frac{1}{h_i h_i} \int_{(h_i h_i)} f(x, t) \mathrm{d} s \mathrm{d} t \right| < \infty,
\]
then
\[
(12) \quad \left. \int \int f(x, t) \varphi(x, t) \mathrm{d} s \mathrm{d} t \right| \leq K \int \int \varphi(x, t) \mathrm{d} s \mathrm{d} t.
\]

**Proof.** Let \( f_\varepsilon \) be the same as above; from (11) there follows
\[
\frac{1}{h_i h_i} \int_{(h_i h_i)} |f_\varepsilon(x, t)| \mathrm{d} s \mathrm{d} t \leq K + \varepsilon \quad (0 < h_i \leq 1, i = 1, 2),
\]

hence, by corollary 2, § 2,
\[
\left. \int \int f(x, t) \varphi(x, t) \mathrm{d} s \mathrm{d} t \right| \leq \left. \int \int f_\varepsilon(x, t) \varphi(x, t) \mathrm{d} s \mathrm{d} t \right| + \left. \int \int (f - f_\varepsilon(x, t)) \varphi(x, t) \mathrm{d} s \mathrm{d} t \right|
\]
\[
\leq (K + \varepsilon) \left. \int \int \varphi(x, t) \mathrm{d} s \mathrm{d} t \right| + \varphi(0, 0) \varepsilon,
\]

and \( \varepsilon > 0 \) being arbitrary, we obtain the inequality (12).

**Remark.** One may also consider unbounded functions \( \varphi \), satisfying the conditions given in §1 in the square \([0, 1; 0, 1]\), the points of unboundedness which lie on coordinate axes. In this case the inequality (12) remains true if both Lebesgue integrals in (12) are finite, or, if the second integral is finite and the function \( f \) is non-negative. This result may be obtained by theorem 2 and a suitable approximation of the function \( \varphi \) by bounded functions of two variables.

Applying the inequality (12), the two theorems of Romanovski and Padé type ([4], pp. 245-247) concerning the convergence of singular integrals
\[
I_\varepsilon(f) = \int \int \frac{f(x, t) \varphi(x, t)}{t} \mathrm{d} s \mathrm{d} t
\]
may be easily obtained for the classes \( L^* \) and \( L^2 \) of Lebesgue-integrable functions \( f \) in the square \([0, 1; 0, 1] = [1, 1]\) such that
\[
\lim_{h_i \to 0} \frac{1}{h_i h_i} \int_{(h_i h_i)} f(x, t) \mathrm{d} s \mathrm{d} t = f(0, 0),
\]
\[
\lim_{h_i \to 0} \frac{1}{h_i h_i} \int_{(h_i h_i)} |f(x, t) - f(0, 0)| \mathrm{d} s \mathrm{d} t = 0.
\]
We shall formulate one of these theorems, namely

**Theorem 3.** If the functions \( \varphi_n(s, t) \) \((n = 1, 2, \ldots)\) subject to the same assumption as \( \varphi(s, t) \) (given in §1), in \([1, 1]\), and if

\[
\lim_{n \to \infty} \int \int \varphi_n(s, t) \, ds \, dt = 1
\]

and

\[
\lim_{n \to \infty} \varphi_n(a, 0) < \infty, \quad \lim_{n \to \infty} \varphi_n(0, \gamma) < \infty,
\]

with any positive \( a \) and \( \gamma \) \((a, \gamma \leqslant 1)\), then for every function \( f \in L^p \) we have

\[
\lim_{n \to \infty} I_n(f) = f(0, 0).
\]

Similar theorems hold when \( \varphi \) (or \( \varphi_n \)) is non-negative and non-decreasing in \([b, d]\) with respect to each variable separately, and such that the difference \( \varphi(s, t') - \varphi(s, t) \) is non-decreasing with respect to \( s \) for any pair \( t' < t' \); in this case the function \( \varphi \) (or \( \varphi_n \)) attains its maximum at the point \((b, d)\) (formerly at \((a, c)\)). It is also evident how to formulate theorems of this type, when the maximum-points of \( \varphi \) (or \( \varphi_n \)) are the remaining vertices of the rectangle \([a, b] \times [c, d]\).

**References**


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