Les ANNALES POLONICI MATHEMATICI constituent une continuation des ANNALES DE LA SOCIÉTÉ POLONAISE DE MATHÉMATIQUE (vol. I-XXV) fondeées en 1921 par Stanisław Zaremba.

Les ANNALES POLONICI MATHEMATICI publient, en langues des congrès internationaux, des travaux consacrés à l'Analyse Mathématique, la Géométrie et la Théorie des Nombres. Chaque volume paraît en 3 fascicules.

Les manuscrits dactylographiés sont à expédier à l'adresse:
Rédaction des ANNALES POLONICI MATHEMATICI
KRAKÓW (Pologne), ul. Solskiego 30.

Toute la correspondance concernant l'échange et l'administration est à expédier à l'adresse:
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ARS POLONA
WARSZAWA 5 (Pologne), Krakowskie Przedmieście 7.

PRINTED IN POLAND

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

ANNALES
POLONICI MATHEMATICI
XI (1962)

On some linear functional equations. II

by J. Kordylewski and M. Kuczma (Kraków)

Equation of the second order

In the present paper we shall discuss the linear functional equation of the second order:

$$\varphi[f^2(x)]+A(x)\varphi[f(x)]+B(x)\varphi(x)=F(x), \tag{1}$$

where $\varphi(x)$ is the required function and $f(x), F(x), A(x), B(x)$ are given functions. We shall denote by $f^k(x)$ the k-th iteration of the function $f(x)$, i.e. we put

$$f^0(x)=x,$$

$$f^{k+1}(x)=f[f^k(x)], \quad f^{-k-1}(x)=f^{-1}[f^{-k}(x)], \quad k=0,1,2,\ldots$$

Equation (1) with constant coefficients A and B has been treated by us previously [2]. It has turned out that it could be easily reduced to a system of equations of the first order. We shall prove that it is also possible in the case of equation (1) with variable coefficients $A(x), B(x)$.

Let $\lambda(x)$ be a solution of the functional equation

$$\lambda[f(x)]\lambda(x)+A(x)\lambda(x)+B(x)=0 \tag{2}$$

and let us put

$$\mu(x)\overset{\text{df}}{=}-A(x)-\lambda[f(x)]. \tag{3}$$

We shall prove the following

Lemma I. *Equation* (1) *is equivalent to the system of equations of the first order with the unknown functions* $\varphi(x), \psi(x)$:

$$\varphi[f(x)]-\lambda(x)\varphi(x)=\psi(x), \quad \psi[f(x)]-\mu(x)\psi(x)=F(x). \tag{4}$$

Proof. Supposing that functions $\varphi(x)$ and $\psi(x)$ satisfy system (4), inserting $\psi(x)$ from the first of the equations (4) to the second and making use of (2) and (3) we get the conclusion that the function $\varphi(x)$ satisfies equation (1). Similarly, if a function $\varphi(x)$ satisfies equation (1), then putting

$$\psi(x)\overset{\text{df}}{=}\varphi[f(x)]-\lambda(x)\varphi(x)$$

we can easily verify that the two functions $\varphi(x)$, $\psi(x)$ satisfy system (4), which was to be proved.

In the sequel we shall consider equation (1) under the supposition that the function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$, where a and b are two consecutive roots of the equation

$$f(x) = x\,. \tag{5}$$

The above lemma enables us to replace the investigation of equation (1) by the investigation of equations of the first order of the form

$$\varphi[f(x)] - \lambda(x)\varphi(x) = G(x)\,. \tag{6}$$

LEMMA II. *Let $f(x)$ be a real-valued function of the real variable x, continuous and strictly increasing in an interval $\langle a, b\rangle$, where a and b are two consecutive roots of equation (5) and let $f(x) > x$ in (a, b). Let further $\lambda(x)$ and $G(x)$ be complex-valued functions of the real variable x, continuous in the interval (a, b), $\lambda(x) \neq 0$ in (a, b). Then equation (6) possesses infinitely many (complex) solutions that are continuous in the open interval (a, b). If, moreover, the functions $\lambda(x)$ and $G(x)$ are continuous in the interval $(a, b\rangle$ ($\langle a, b)$), then:*

(α) *In the case $|\lambda(b)| > 1$ ($|\lambda(a)| < 1$) equation (6) possesses exactly one solution that is continuous in the interval $(a, b\rangle$ ($\langle a, b)$).*

(β) *In the case $|\lambda(b)| < 1$ ($|\lambda(a)| > 1$) every solution of equation (6) that is continuous in the interval (a, b), is also continuous in the interval $(a, b\rangle$ ($\langle a, b)$).* [1]

Proof. The existence of an infinite number of solutions of equation (6) that are continuous in the interval (a, b) has been proved by us in [2]. Similarly, the point (β) of the second part of the lemma has been proved by us in [2]. Thus it remains only to prove the point (α) of the second part of the lemma.

In [2] we have proved that in the case $|\lambda(b)| > 1$ ($|\lambda(a)| < 1$) equation (6) possesses at most one solution which is continuous in the interval $(a, b\rangle$ ($\langle a, b)$). Using repeatedly the formula

$$\varphi(x) = \frac{\varphi[f(x)] - G(x)}{\lambda(x)}$$

resp.

$$\varphi(x) = \lambda[f^{-1}(x)]\varphi[f^{-1}(x)] + G[f^{-1}(x)]$$

(1) Every solution of equation (6) defined in the interval (a, b) may be uniquely extended onto the interval $(a, b\rangle$. The exact meaning of the assertion (β) is: If a solution is continuous in (a, b), then the extended solution is continuous in $(a, b\rangle$ (and similarly for the interval $\langle a, b)$).

(both can be easily derived from the relation (6)) we obtain that this unique solution, provided it exists, must have the form

$$\varphi(x) = -\sum_{\nu=0}^{\infty} \frac{G[f^{\nu}(x)]}{\prod_{i=0}^{\nu} \lambda[f^{i}(x)]} = -\frac{G(x)}{\lambda(x)} - \frac{G[f(x)]}{\lambda(x)\lambda[f(x)]} - \frac{G[f^2(x)]}{\lambda(x)\lambda[f(x)]\lambda[f^2(x)]} - \dots \tag{7}$$

or

$$\varphi(x) = G[f^{-1}(x)] + \sum_{\nu=1}^{\infty} G[f^{-\nu-1}(x)] \prod_{i=1}^{\nu} \lambda[f^{-i}(x)] = G[f^{-1}(x)] + G[f^{-2}(x)]\lambda[f^{-1}(x)] + G[f^{-3}(x)]\lambda[f^{-1}(x)]\lambda[f^{-2}(x)] + \dots \tag{8}$$

respectively. We shall prove that if $|\lambda(b)| > 1$ then function (7) is defined and continuous in the interval $(a, b\rangle$ and satisfies equation (6). Similarly one can prove that if $|\lambda(a)| < 1$ then function (8) is defined and continuous in the interval $\langle a, b)$ and satisfies equation (6).

So let be $|\lambda(b)| > 1$. In [3] it has been proved that the sequences $\{f^k(x)\}$ and $\{f^{-k}(x)\}$ are for every $x \in (a, b)$ monotonic and

$$\lim_{k\to\infty} f^k(x) = b\,, \quad \lim_{k\to\infty} f^{-k}(x) = a\,.$$

Consequently for an arbitrary number $\varepsilon > 0$ we can find an index K such that for $k > K$ and $x \in \langle a+\varepsilon, b\rangle$, $|\lambda[f^k(x)]| > M > 1$. Moreover $|G(x)| \leqslant N$ and $|\lambda(x)| \geqslant L > 0$ in $\langle a+\varepsilon, b\rangle$. Thus the series

$$\sum_{\nu=0}^{K} \frac{N}{L^{\nu}} + \sum_{\nu=K+1}^{\infty} \frac{N}{L^K M^{\nu-K}}$$

is a numerical majorant of series (7). Consequently the latter uniformly converges in $\langle a+\varepsilon, b\rangle$ for every $\varepsilon > 0$. Function (7) is then continuous in $(a, b\rangle$ and, as one can easily verify, satisfies equation (6). This completes the proof.

Let d_1, d_2 and c_1, c_2 be roots of the equation

$$d^2 + A(b)d + B(b) = 0\,, \tag{9}$$

and

$$c^2 + A(a)c + B(a) = 0 \tag{10}$$

respectively. We shall prove the following

THEOREM. *Let us assume that the function $f(x)$ fulfills the hypotheses of lemma II and let $A(x)$, $B(x)$ and $F(x)$ be complex-valued functions of the real variable x, continuous in the interval $\langle a, b\rangle$, $B(x) \neq 0$ in $\langle a, b\rangle$.*

Under these hypotheses:

I. *Equation* (1) *possesses infinitely many (complex) solutions that are continuous in the open interval* (a, b).

II. *Let us suppose moreover that* $|d_1| \neq |d_2|$. *Then*

1° *If* $|d_1| > 1$ *and* $|d_2| > 1$, *then equation* (1) *possesses exactly one solution that is continuous in the interval* $(a, b\rangle$.

2° *If* $|d_1| < 1$ *and* $|d_2| < 1$, *then every solution of equation* (1) *that is continuous in the interval* (a, b) *is also continuous in the interval* $(a, b\rangle$.

3° *If* $|d_1| < 1$ *and* $|d_2| > 1$, *then equation* (1) *possesses infinitely many solutions that are continuous in the interval* $(a, b\rangle$.

III. *Now let us suppose that* $|c_1| \neq |c_2|$. *Then*

1° *If* $|c_1| < 1$ *and* $|c_2| < 1$, *then equation* (1) *possesses exactly one solution that is continuous in the interval* $\langle a, b)$.

2° *If* $|c_1| > 1$ *and* $|c_2| > 1$, *then every solution of equation* (1) *that is continuous in the interval* (a, b) *is also continuous in the interval* $\langle a, b)$.

3° *If* $|c_1| > 1$ *and* $|c_2| < 1$, *then equation* (1) *possesses infinitely many solutions that are continuous in the interval* $\langle a, b)$.

Proof. I. It has been proved in [4] that under the hypotheses of the present theorem equation (2) has a solution that is continuous in the interval (a, b) and fulfills the conditions

$$\lambda(x) \neq 0, \quad \lambda(x) \neq -A[f^{-1}(x)] \quad \text{for} \quad x \in (a, b). \tag{11}$$

Consequently, on account of lemma I, equation (1) is equivalent to the system of equations (4), where the functions $\lambda(x)$ and $\mu(x)$ are continuous and different from zero in (a, b), and the first part of the theorem follows from lemma II.

II. When $|d_1| \neq |d_2|$, equation (2) possesses a solution that is continuous in the interval $(a, b\rangle$ and fulfills conditions (11) (cf. [4]). Consequently equation (1) is equivalent to the system of equations (4), where the functions $\lambda(x)$ and $\mu(x)$ are continuous and different from zero in $(a, b\rangle$. Putting $x = b$ in equation (2) we obtain that $\lambda(b)$ is a root of equation (9), say $\lambda(b) = d_1$, and hence by (3) $\mu(b) = d_2$. Thus assertions 1°, 2°, 3° follow from lemma II.

III. The proof is quite analogical to the proof of part II.

Remark I. If the functions $A(x), B(x)$ and $F(x)$ are real-valued, then under the hypotheses of the above theorem equation (1) possesses infinitely many real solutions that are continuous in the interval (a, b). This follows from a general theorem on the existence of an infinite number of (real) solutions continuous in the interval (a, b) for the equation

$$F(x, \varphi(x), \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^n(x)]) = 0.$$

This theorem is to be found in [1]. Similarly, assertions II, 1°, 2° and III, 1°, 2° will remain valid if the word "solution" is replaced by "real solution". (For II, 2° and III, 2° it is quite evident, for II, 1° and III, 1° it follows from the fact that if a complex-valued function $\varphi(x)$ satisfies equation (1), then the real-valued function $\mathrm{Re}\,\varphi(x)$ also satisfies equation (1)). But it remains an open problem, whether also assertions II, 3° and III, 3° remain valid for real solutions.

Remark II. It is not quite sure whether the hypothesis $|d_1| \neq |d_2|$ (resp. $|c_1| \neq |c_2|$) is essential. Although in [4] it has been shown by example that if $|d_1| = |d_2|$, then equation (2) may happen to have no solution which would be continuous in the interval $(a, b\rangle$. Namely if we put

$$A(x) \equiv 0 \tag{12}$$

and

$$B(x) = -e^{\beta + \sigma(x)}, \tag{13}$$

where $\sigma(x)$ is a suitably chosen real continuous function with the property $\sigma(b) = 0$ and β is an arbitrary real number, then equation (2) has no solution which would be continuous in $(a, b\rangle$. Thus the hypothesis

$$|d_1| \neq |d_2| \tag{14}$$

is essential *for the method of proof* of our theorem. Nevertheless this example does not prove the necessity of assumption (14) for the validity of the theorem. In fact, if we write $g(x) \stackrel{\text{df}}{=} f^2(x)$ in equation (1) with the functions $A(x)$ and $B(x)$ defined by (12) and (13), then it becomes an equation of the first order and the number of its continuous solutions depends only on the sign of β. Thus in this case assertion II of our theorem is true, although assumption (14) is not fulfilled. The question arises whether this hypothesis can be generally omitted. We are not able to answer this question.

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Reçu par la Rédaction le 15. 3. 1960