ANNALES
POLONICI MATHEMATICI
XI (1961)

On the invariant points of a transformation

by A. Pelczar (Kraków)

The reason for undertaking the present paper were some problems concerning certain integral inequalities, which had been considered by Z. Opial. An attempt to obtain more general functional inequalities and equations led me under the suggestions of Prof. T. Ważewski to the application of my reasoning in the partly ordered spaces.

Making use of the qualities of the partly ordered set P, without any topology, I show that the transformation $y = V(x)$, where $V(x)$ is the function defined in P such that $V(P) \subset P$, has the invariant points in P.

I express my hearty thanks to Professor T. Ważewski for his valuable advice and his supervision during the preparation of the paper.

§ 1. Definition 1. Let the transformation

$$y = V(x)$$

be defined in the set B. If $A \neq 0$ and $A \subset B$, then by $V(A)$ we denote an image of the set A.

Definition 2. We call a set P *partly ordered* (G. Birkhoff [1]) if for some pairs of elements $x, y \in P$ a relation $x \leqslant y$ is defined, in such a way, that

(a) for each $x \in P$, $x \leqslant x$,

(b) if $x \leqslant y$ and $y \leqslant x$, then $x = y$,

(c) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

Definition 3. Let Q be a partly ordered set and let $Q \subset P$. We call z the *upper bound* of Q in P, if

(d) $z \in P$,

(e) $x \in Q \Rightarrow x \leqslant z$.

Remark 1. In an analogous way we define the *lower bound* of Q in P.

Definition 4. We call $\hat{z}$ the *supremum* of set Q (we denote short $\sup Q$) if $\hat{z}$ is the upper bound of Q in P and

(f) x is the upper bound of Q in $P \Rightarrow \hat{z} \leqslant x$.

Remark 2. In an analogous way we define the *infimum* of Q ($\inf Q$).

Remark 3. It is easy to see that each partly ordered set can have at most one supremum and at most one infimum.

Definition 5. Let the function $V(x)$ be defined in a partly ordered set P and let $V(P) \subset R$, where R is some partly ordered set. We call $V(x)$ increasing if:

(g) $x \leqslant y \Rightarrow V(x) \leqslant V(y)$.

§ 2. Theorem A. *Assumptions*:

I. *the set P is not empty and is a partly ordered set,*
II. *for each not empty subset $Q \subset P$, there exists* $\sup Q$ *in P,*
III. *$V(z)$ is a function defined in P,*
IV. $V(P) \subset P$,
V. *$V(z)$ is increasing in P,*
VI. *there exists in P a point z_0, such that $z_0 \leqslant V(z_0)$.*

Then we have the thesis:

The transformation

$$y = V(z) \tag{1}$$

has invariant points in P and moreover, among them exists a maximal point. That means that the equation

$$z = V(z) \tag{2}$$

has solutions in P and moreover, among them exists a maximal solution.

Proof. We denote:

$$Q\colon \{z \in P,\ z \leqslant V(z)\}\,.$$

We show that

$$V(Q) \subset Q\,. \tag{3}$$

In order to prove (3) we shall show that

$$z \in P \tag{4}$$

and the inequality

$$z \leqslant V(z) \tag{5}$$

implicates the relation

$$V(z) \in P \tag{6}$$

and the inequality

$$V(z) \leqslant V\big(V(z)\big)\,. \tag{7}$$

Relation (6) follows from (4) and from assumption IV. Inequality (7) follows from (4), (5), (6) and from assumption V. This yields directly (3).

Let R be the set of all upper bounds of Q in P. In particular $\sup Q \in R$. We shall show that

$$V(R) \subset R\,. \tag{8}$$

In order to prove this, we shall show that if z satisfies (4), (5) and if

$$y \in P\,, \tag{9}$$

$$z \leqslant y \tag{10}$$

then

$$V(y) \in P \tag{11}$$

and

$$z \leqslant V(y)\,. \tag{12}$$

Relation (11) follows from (9) and assumption IV. From (10) and assumption V, we obtain

$$V(z) \leqslant V(y)\,. \tag{13}$$

From (13), (5) and (c) we have (12) and this yields us to the inclusion of (8).

We denote $\hat{z} = \sup Q$. Since $\hat{z} \in R$, we have from (8)

$$\hat{y} = V(\hat{z}) \in R\,. \tag{14}$$

By the definition, $\hat{z}$ is $\sup Q$, so it must be (by (f))

$$\hat{z} \leqslant \hat{y} = V(\hat{z})\,. \tag{15}$$

In the consequence

$$\hat{z} \in Q\,. \tag{16}$$

Evidently by (3) also follows

$$\hat{y} = V(\hat{z}) \in Q\,. \tag{17}$$

Because $\hat{z}$ is in particular upper bound of Q in P, so must be

$$\hat{y} = V(\hat{z}) \leqslant \hat{z}\,. \tag{18}$$

In view of (b) and inequalities (15) and (18) it follows that $\hat{z}$ satisfies equation (2). Moreover, from the definition $\hat{z}$ as $\sup Q$, we have in the cosequence that $\hat{z}$ is just the maximal solution.

§ 3. It is possible to prove the following theorem, analogous to theorem A.

Theorem A′. *Assumptions*:

I. *the set P is not empty and is a partly ordered set,*
II′. *for each not empty subset $Q \subset P$, there exists* $\inf Q$ *in P,*
III. *$V(z)$ is a function defined in P,*
IV. $V(P) \subset P$,
V. *$V(z)$ is increasing in P,*
VI′. *there exists in P a point z_1, such that $V(z_1) \leqslant z_1$.*

Then we have the thesis:

Transformation (1) *has invariant points in* P *and moreover, among them exists a minimal invariant point, which means that equation* (2) *has solutions in* P *and moreover, among them exists a minimal solution.*

§ 4. It is possible to formulate theorem A in another equivalent form, as follows:

THEOREM B. *Assumptions*:

Ia. *the set* P *is not empty and is partly ordered,*

IIa. *for each not empty subset* $Q \subset P$, *there exists* $\sup Q$ *in* P,

IIIa. $V(z)$ *is a function defined in* P,

IVa. $V(P) \subset P$,

Va. $V(z)$ *is increasing in* P,

VIa. *there exists in* P *a point* z_0, *such that* $z_0 \leqslant V(z_0)$,

VIIa. *there exists in* P *a point* z_1, *such that* $z_0 \leqslant z_1$ *and* $V(z_1) \leqslant z_1$.

Then we have the thesis:

Equation (2) *has in* P *solutions* z, *such that*

$$z_0 \leqslant z \leqslant z_1$$

and moreover, among them exists a maximal solution.

§ 5. Remark 4. If the set P and the function $V(z)$ fulfill assumptions I-VI of theorem A and if for $y \in P$ is

$$y \leqslant V(y)$$

then it must be

$$y \leqslant \hat{z}$$

where $\hat{z}$ ist the maximal solution of equation (2).

Remark 5. If the set P and the function $V(z)$ fulfill assumptions I-VI and assumptions II′ and VI′ and if moreover it holds for the following relation

$$[z \leqslant V(z) \text{ and } V(u) \leqslant u] \Rightarrow z \leqslant u$$

then equation (2) has exactly one solution in P.

Remark 6. It is possible to prove in an analogous way similiar theorems concerning equation (2) where the function $V(z)$ is decreasing.

Reference

[1] G. Birkhoff, *Lattice Theory*, New York 1948.

Reçu par la Rédaction le 10. 9. 1960