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On the functional equation $\varphi^n(x) = g(x)$

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Introduction. In the present paper I give the construction of the general continuous solution of the functional equation

$$\varphi^n(x) = g(x) \tag{1}$$

under the assumption that the given function $g(x)$ is monotonic. $\varphi^n(x)$ denotes here the n-th iterate of the function $\varphi(x)$:

$$\varphi^0(x) = x\,,$$

$$\varphi^{k+1}(x) = \varphi[\varphi^k(x)]\,, \quad \varphi^{-k-1}(x) = \varphi^{-1}[\varphi^{-k}(x)]\,, \quad k = 0, 1, 2, \ldots$$

In paper [6] I have proved that in order to find the general solution, or the general continuous solution of the equation

$$\varphi^n(x) = g[\varphi^m(x)]$$

it is enough to know the general solution or the general continuous solution, respectively, of equation (1). The general solution of equation (1) has been given by S. Łojasiewicz [7] (see also [3]). The question arises, how to find the general continuous solution of equation (1). In the present paper I give a partial solution of this problem (the solution under the assumption that the function $g(x)$ is monotonic).

In the particular case, when $g(x) \equiv x$ the general continuous solution of equation (1) is well known (see e.g. [2], [8]). For odd n the unique continuous solution of the equation

$$\varphi^n(x) = x \tag{2}$$

is the function $\varphi(x) \equiv x$, for even n the unique continuous solutions of equation (2) are the continuous solutions of the equation

$$\varphi^2(x) = x$$

(so called involutory functions [1]). Thus in the sequel we shall assume that $g(x) \not\equiv x$. The results are contained in three theorems, corresponding to the three possible cases A, B, C, of the monotonity of the functions $\varphi(x)$ and $g(x)$ (see the following section).

Preliminaries. We begin with some definitions.

DEFINITION I. We shall say that a function $\varphi(x)$ satisfies equation (1) in a set E if for each $x \in E$ the function $g(x)$ and the iterate $\varphi^n(x)$ are defined and both these functions are equal.

It follows from the above definition that if a function $\varphi(x)$ satisfies equation (1) in E, then

$$\varphi(E) \subset E . \tag{3}$$

Similarly, if equation (1) has a solution in E, then

$$g(E) \subset E . \tag{4}$$

In the sequel we shall assume that the set E is an interval. This interval may be open, closed, or one-side closed; one or both of its ends may be infinite.

We shall admit also functions assuming infinite values. As the function $g(x)$ is monotonic in E, it can be continued onto the closure $\bar{E}$ of the interval E. Thus in the sequel we shall assume that the function $g(x)$ is defined, continuous and strictly increasing in a closed interval $\bar{E}$ and that it fulfills condition (4).

In our further considerations the following sets will play an important part:

$$I = \{x\colon x \in E\,,\ g(x) = x\}\,, \qquad I^* = \{x\colon x \in \bar{E}\,,\ g(x) = x\}\,,$$
$$I_2 = \{x\colon x \in E\,,\ g^2(x) = x\}\,, \qquad I_2^* = \{x\colon x \in \bar{E}\,,\ g^2(x) = x\}\,.$$

From the monotonity of the function $g(x)$ it results immediately

LEMMA I. *Every continuous function $\varphi(x)$, satisfying equation* (1) *in E, is strictly monotonic.*

Since all iterates of an increasing function are increasing functions and iterates of a decreasing function are increasing or decreasing, according to the iterative exponent being even or odd, there are three cases possible:

A. $\varphi(x)$ increasing, $g(x)$ increasing, n arbitrary.

B. $\varphi(x)$ decreasing, $g(x)$ decreasing, n odd.

C′. $\varphi(x)$ decreasing, $g(x)$ increasing, n even.

The case C′ can, however, be reduced to a simpler one:

C. $\varphi(x)$ decreasing, $g(x)$ increasing, $n = 2$.

In fact, if a decreasing function $\varphi(x)$ satisfies equation (1) with a function $g(x)$ increasing (and then n must be even), then the function $\psi(x) \overset{\mathrm{df}}{=} \varphi^2(x)$ is increasing and satisfies the equation

$$\psi^{n/2}(x) = g(x)$$

(case A). Thus having solved equation (1) in cases A and B it is enough to give only the solution of the equation

$$\varphi^2(x) = g(x) \tag{5}$$

(with the function $g(x)$ increasing) in the class of the continuous decreasing functions.

Now we are going to investigate separately the cases A, B and C.

A. $\varphi(x)$ increasing, $g(x)$ increasing, n arbitrary.

At first we shall prove the following

LEMMA II. *If $\varphi(x)$ is a continuous and increasing solution of equation* (1) *in E, then $I = \{x\colon x \in E, \varphi(x) = x\}$.*

Proof. The inclusion $\{x\colon x \in E, \varphi(x) = x\} \subset I$ is evident. In order to prove the converse inclusion let us suppose that for a certain $x_0 \in E$ we have $\varphi(x_0) \neq x_0$, e.g.

$$\varphi(x_0) > x_0 . \tag{6}$$

Since the function $\varphi(x)$ is strictly increasing in E, it follows from (6) that

$$\varphi^{\nu+1}(x_0) > \varphi^{\nu}(x_0) \quad \text{for} \quad \nu = 0, 1, 2, \ldots$$

Consequently $\varphi^{\nu}(x_0) \neq x_0$ for $\nu = 1, 2, \ldots$ and in particular $g(x_0) = \varphi^n(x_0) \neq x_0$. Hence the statement of the lemma follows immediately.

The set $E - I$ is a sum of at most an enumerable number of disjoint intervals. These intervals are open, possibly with the only exception of the intervals possessing a common end with the interval E. Solutions of equation (1) will be constructed in each of these intervals separately.

Let (a, b) be one of these intervals and let us suppose that a and b are elements of the set I^*. Moreover let us assume that $g(x) > x$ in (a, b) (if $g(x) < x$, the considerations follow similarly). We shall prove

LEMMA III. *If $\varphi(x)$ is a continuous and increasing solution of equation* (1) *in E, then*

$$\varphi\big((a, b)\big) \subset (a, b) . \tag{7}$$

Proof. If neither a nor b belongs to I, then $(a, b) = E$ and (7) follows from (3). If $a \in I$ and $b \in I$, then on account of lemma II $\varphi(a) = a$ and $\varphi(b) = b$. As has been proved in [5], in such a case $\varphi\big((a, b)\big) = (a, b)$, which in particular implies (7). If $a \notin I$ and $b \in I$, then $\varphi(b) = b$ and a is an end of the interval E. Then we have by (3) $\varphi(a) \geqslant a$, whence relation (7) follows easily. In the case when $a \in I$ and $b \notin I$, we argue analogously.

Now we shall give a construction of an arbitrary continuous and strictly increasing solution of equation (1) in (a, b) when a and b belong to I^*. Let x_0 be an arbitrary point of the interval (a, b) and let us choose points $x_1, \ldots, x_{n-1}$ in such a manner that

$$x_0 < x_1 < \ldots < x_{n-1} < g(x_0) . \tag{8}$$

We put further

$$x_{\nu+n} = g(x_\nu), \quad \nu = 0, \pm 1, \pm 2, \ldots \tag{9}$$

As has been proved in [5], the sequence x_ν ($\nu \geqslant 0$) is increasing and converges to b, the sequence $x_{-\nu}$ ($\nu \geqslant 0$) is decreasing and converges to a.

Let $\varphi_1(x), \ldots, \varphi_{n-1}(x)$ be arbitrary functions which are defined, continuous and strictly increasing in the intervals $\langle x_0, x_1\rangle, \ldots, \langle x_{n-2}, x_{n-1}\rangle$ respectively and fulfill the conditions

$$\varphi_i(x_{i-1}) = x_i, \quad \varphi_i(x_i) = x_{i+1}, \quad i = 1, \ldots, n-1. \tag{10}$$

We put

$$\varphi_{\nu+n}(x) = g\left\{\overset{-1}{\varphi}_{\nu+1}\left[\overset{-1}{\varphi}_{\nu+2}\left(\ldots\left(\overset{-1}{\varphi}_{\nu+n-1}(x)\right)\ldots\right)\right]\right\}, \quad x \in \langle x_{\nu+n-1}, x_{\nu+n}\rangle, \quad \nu = 0, \pm 1, \pm 2, \ldots \tag{11}$$

We shall prove that each function $\varphi_\nu(x)$ is defined, continuous and strictly increasing in the interval $\langle x_{\nu-1}, x_\nu\rangle$ and

$$\varphi_\nu(x_{\nu-1}) = x_\nu, \quad \varphi_\nu(x_\nu) = x_{\nu+1}. \tag{12}$$

The proof follows by induction. For $\nu = 1, \ldots, n-1$ it is so by hypothesis. Let us suppose that the functions $\varphi_\nu(x)$ are defined, continuous and strictly increasing in the intervals $\langle x_{\nu-1}, x_\nu\rangle$ respectively, and that relations (12) hold for $\nu = 1, \ldots, p \geqslant n$. Consequently, all the functions $\varphi_i^{-1}(x)$, $i = p-n+2, \ldots, p$, are defined, continuous and strictly increasing in the intervals $\langle x_{p-n+2}, x_{p-n+3}\rangle, \ldots, \langle x_p, x_{p+1}\rangle$ respectively and assume values from the intervals $\langle x_{p-n+1}, x_{p-n+2}\rangle, \ldots, \langle x_{p-1}, x_p\rangle$ respectively. Thus the function

$$\overset{-1}{\varphi}_{p-n+2}\left[\overset{-1}{\varphi}_{p-n+3}\left(\ldots\left(\varphi_p^{-1}(x)\right)\ldots\right)\right]$$

is defined, continuous and strictly increasing in the interval $\langle x_p, x_{p+1}\rangle$ and its values remain in the interval $\langle x_{p-n+1}, x_{p-n+2}\rangle \subset (a, b)$. Consequently the function $\varphi_{p+1}(x)$ is (by means of relation (11)) defined, continuous and strictly increasing in the interval $\langle x_p, x_{p+1}\rangle$. Using succesively relations (12) for $\nu = p, \ldots, p-n+2$, and then making use of relation (9), we obtain that (12) holds also for $\nu = p+1$.

Setting in relation (11)

$$x = \varphi_{\nu+n-1}\left[\ldots\left(\varphi_{\nu+2}\left(\varphi_{\nu+1}(y)\right)\right)\ldots\right]$$

we obtain (writing again x in the place of y)

$$\varphi_{\nu+n}\left\{\varphi_{\nu+n-1}\left[\ldots\left(\varphi_{\nu+2}\left(\varphi_{\nu+1}(x)\right)\right)\ldots\right]\right\} = g(x). \tag{13}$$

Basing on relation (13) we can (quite similarly as above) prove by induction also that for $\nu \leqslant 0$ each function $\varphi_\nu(x)$ is defined, continuous and strictly increasing in the interval $\langle x_{\nu-1}, x_\nu\rangle$ and that relations (12) hold.

Now let us put

$$\varphi(x) \overset{\text{df}}{=} \varphi_\nu(x) \quad \text{for} \quad x \in \langle x_{\nu-1}, x_\nu). \tag{14}$$

The function $\varphi(x)$ is by relation (14) defined in the whole interval (a, b). It is obvious that $\varphi(x)$ is continuous and strictly increasing in (a, b). It follows from (13) that $\varphi(x)$ satisfies equation (1).

Since the function $\varphi(x)$ is increasing, we have by (12)

$$\lim_{x\to b} \varphi(x) = b, \quad \lim_{x\to a} \varphi(x) = a. \tag{15}$$

Now we shall prove that taking all possible systems of points $x_1, \ldots, x_{n-1}$, fulfilling condition (8), and all possible systems of functions $\varphi_1(x), \ldots, \varphi_{n-1}(x)$ which are defined, continuous and strictly increasing in the intervals $\langle x_0, x_1\rangle, \ldots, \langle x_{n-2}, x_{n-1}\rangle$ respectively, and fulfill relations (10), we obtain by means of relation (14) all continuous and strictly increasing solutions of equation (1) in E, restricted to the interval (a, b). Let $\varphi(x)$ be an arbitrary solution of equation (1) in E that is continuous and strictly increasing. The assumption $g(x) > x$ in (a, b) implies the inequality

$$\varphi(x) > x \quad \text{in} \quad (a, b).$$

Hence it follows on account of lemma III that the sequence

$$x_\nu \overset{\text{df}}{=} \varphi^\nu(x_0), \quad \nu = \pm 1, \pm 2, \ldots,$$

fulfills relations (8) and (9), and that the functions

$$\varphi_i(x) \overset{\text{df}}{=} \varphi(x) \quad \text{for} \quad x \in \langle x_{i-1}, x_i\rangle, \quad i = 1, \ldots, n-1,$$

are defined, continuous and strictly increasing in the intervals $\langle x_0, x_1\rangle, \ldots, \langle x_{n-2}, x_{n-1}\rangle$ respectively, and fulfill condition (10). Moreover, it is easy to prove that if two functions, $\varphi(x)$ and $\psi(x)$, both satisfy equation (1) in E and

$$\varphi(x) \equiv \psi(x) \quad \text{for} \quad x \in \langle x_0, g(x_0)\rangle,$$

then they have to coincide in the whole interval (a, b). Hence it follows that formula (14) defines the general continuous and strictly increasing solution of equation (1) in (a, b).

It remains to be investigated the case, when one of the ends of the interval (a, b) does not belong to I^*. If $a \notin I^*$, and $b \in I^*$, then we may proceed quite analogously as before, assuming $x_0 = a$ and confining ourselves to $\nu \geqslant 0$ only. Similarly, if $a \in I^*$, but $b \notin I^*$, we may also proceed analogously, assuming this time $x_0 = g^{-1}(b)$ and confining ourselves to $\nu \leqslant n$. Lastly, the case $a \notin I^*$ and $b \notin I^*$ cannot occur. In fact, if $a \notin I^*$ and $b \notin I^*$, then necessarily $\langle a, b\rangle = \bar{E}$. But then no point of the interval $\bar{E}$ would belong to I^*, which contradicts condition (4).

From the above considerations follows

THEOREM I. *If the function $g(x) \not\equiv x$ is defined, continuous and strictly increasing in an interval E and fulfills condition* (4), *then equation* (1) *possesses infinitely many solutions $\varphi(x)$ that are defined, continuous and strictly increasing in E. We can obtain all these solutions, constructing in the above described manner (formula* (14)) *continuous and strictly increasing solutions independently in each of the intervals of the set $E-I$ and putting $\varphi(x) = x$ for $x \in I$. Relations* (15) *guarantee that so obtained functions are continuous in the whole interval E.*

B. $\varphi(x)$ decreasing, $g(x)$ decreasing, n odd.

The procedure in this case is somewhat analogous to that of the case A. Therefore we confine ourselves to a sketch of this procedure only, omitting some oppressive details.

The set $I = I^*$ contains now only a single element $x = c$. But in the present case the sets I_2 and I_2^* will play an analogous part as the sets I and I^* in the case A. In particular we shall prove the following

LEMMA IV. *If $\varphi(x)$ is a continuous, strictly decreasing solution of equation* (1) *in E, then*

$$\varphi(x) = g(x) \quad \text{for} \quad x \in I_2 . \tag{16}$$

Proof. Let $\varphi(x)$ be a continuous and strictly decreasing solution of equation (1) in E. Then the function $\psi(x) \overset{\text{df}}{=} \varphi^2(x)$ is a continuous and strictly increasing solution of the equation

$$\psi^n(x) = g^2(x)$$

in E, in which the function $g^2(x)$ is increasing. Thus it follows from lemma II that

$$\psi(x) = x \quad \text{for} \quad x \in I_2 .$$

Hence we have

$$\varphi^n(x) = \varphi(x) \quad \text{for} \quad x \in I_2 ,$$

whence, according to (1), relation (16) follows immediately.

Thus it is enough now to construct the solution of equation (1) in the set $E-I_2$. The set $E-I_2$ is a sum of at most an enumerable number of disjoint intervals. These intervals are open, possibly with the only exception of the intervals possessing a common end with the interval E. Solutions of equation (1) will be constructed in suitably chosen couples of these intervals separately.

Let (a, b) be one of the intervals of the set $E-I_2$ and let us assume that a and b are elements of the set I_2^*. Let us assume moreover that $g^2(x) > x$ in (a, b). The interval $\big(g(b), g(a)\big)$ is then also one of those of the set $E-I_2$, $g(b)$ and $g(a)$ belong to I_2^* and $g^2(x) < x$ in $\big(g(b), g(a)\big)$.

Let x_0 be an arbitrary point of the interval (a, b) and x_1 an arbitrary point of the interval $\big(g(x_0), g^{-1}(x_0)\big)$. Let us choose points $x_2, \ldots, x_{n-1}$ in such a manner that

$$\begin{aligned} &x_0 < x_2 < x_4 < \ldots < x_{n-1} < g(x_1) , \\ &x_1 > x_3 > x_5 > \ldots > x_{n-2} > g(x_0) , \text{ (1)} \end{aligned} \tag{17}$$

and let the sequence x_ν be defined by formula (9). The sequence $x_{2\nu}$ $(\nu \geqslant 0)$ is increasing and converges to b, the sequence $x_{-2\nu}$ $(\nu \geqslant 0)$ is decreasing and converges to a. Similarly the sequence $x_{2\nu+1}$ $(\nu \geqslant 0)$ is decreasing and converges to $g(b)$, the sequence $x_{-2\nu-1}$ $(\nu \geqslant 0)$ is increasing and converges to $g(a)$.

Let $\varphi_1(x), \ldots, \varphi_{n-1}(x)$ be arbitrary functions which are defined, continuous and strictly decreasing in the intervals $\langle x_0, x_2\rangle$, $\langle x_3, x_1\rangle$, ..., $\langle x_{n-3}, x_{n-1}\rangle$, $\langle x_n, x_{n-2}\rangle$ respectively, and fulfill the conditions

$$\varphi_i(x_{i-1}) = x_i , \quad \varphi_i(x_{i+1}) = x_{i+2} , \quad i = 1, \ldots, n-1 .$$

Further, let the sequence of functions $\varphi_\nu(x)$, $\nu = 0, \pm 1, \pm 2, \ldots$ be defined by formula (11). It can be easily proved (quite analogously as in the case A) that each function $\varphi_\nu(x)$ is defined, continuous and strictly decreasing in the interval $\langle x_{\nu-1}, x_{\nu+1}\rangle$ or $\langle x_{\nu+1}, x_{\nu-1}\rangle$ (according to ν being odd or even) and that

$$\varphi_\nu(x_{\nu-1}) = x_\nu , \quad \varphi_\nu(x_{\nu+1}) = x_{\nu+2} . \tag{18}$$

Now we put

$$\varphi(x) \overset{\text{df}}{=} \varphi_\nu(x) \quad \text{for} \quad x \in A_\nu , \tag{19}$$

where $A_\nu = \langle x_{\nu-1}, x_{\nu+1})$ for odd ν and $A_\nu = (x_{\nu+1}, x_{\nu-1}\rangle$ for even ν. The function $\varphi(x)$ is by relation (19) defined in the whole set $A = (a, b) \cup \cup \big(g(b), g(a)\big)$. It is obvious that $\varphi(x)$ is continuous and strictly decreasing in A. It follows from relation (13), which is equivalent with relation (11), that $\varphi(x)$ satisfies equation (1). Quite similarly, as in the preceding case, one can prove that in this manner all solutions of equation (1) in E restricted to the set A have been obtained.

Since the function $\varphi(x)$ is decreasing, it follows by (18) that

$$\begin{aligned} &\lim_{x\to a} \varphi(x) = g(a) , \qquad \lim_{x\to b} \varphi(x) = g(b) , \\ &\lim_{x\to g(a)} \varphi(x) = a , \qquad \lim_{x\to g(b)} \varphi(x) = b . \end{aligned} \tag{20}$$

The cases when a or b is an end of the interval E must be thoroughly investigated. At first we shall prove

(1) Since the function $g(x)$ is decreasing, it follows from the condition $g^2(x) > x$ that $g^{-1}(x) > g(x)$ for $x \in (a, b)$; thus the interval $\big(g(x_0), g^{-1}(x_0)\big)$ is not empty. Moreover $x_1 < g^{-1}(x_0)$ implies $g(x_1) > x_0$. Consequently conditions (17) can be realized. (Let us notice that the function $g^{-1}(x)$ is defined in (a, b) because a and b belong to I_2^*.)

Lemma V. *If one of the ends of the interval E belongs to I_2^*, then so also does the other.*

Proof. Let $a < \beta$ be the ends of the interval E and let us suppose that $a \in I_2^*$:

$$g[g(a)] = a .$$

$g(a)$ must be the second end of the interval E:

$$g(a) = \beta ,$$

for otherwise for $x \in (g(a), \beta)$ the function $g(x)$ should assume values less than a, which contradicts relation (4). Consequently, $g(\beta) = g^2(a) = a$ and $g^2(\beta) = g(a) = \beta$, which was to be proved.

Now let us suppose that a is an end of the interval E (the discussion of the case when b is an end of the interval E follows similarly and leads to similar conclusions). We must distinguish two subcases:

1. $a \in I_2^*$. Then $g(a)$ has to be the second end of the interval E. The solution of equation (1) in $(a, b) \cup (g(b), g(a))$ is given by formula (19). But if $a \in E$, then according to (16) $\varphi(a) = g(a)$ and on account of (3) $g(a) \in E$. Thus the interval E must be on both sides closed, or on both sides open.

2. $a \notin I_2^*$. Let us denote by β the second end of the interval E. Of course, $\beta \notin I_2^*$. Moreover let us assume for the present that the interval E is closed. We shall prove

Lemma VI. *If $a \notin I_2^*$ and if there exists a solution $\varphi(x)$ of equation* (1) *continuous and strictly decreasing in E, then $g(a) \neq \beta$ and $g(\beta) \neq a$.*

Proof. Let us suppose that $g(a) = \beta$. Then $g(\beta) > a$, for otherwise a should belong to I_2^*. Since $g^2(x) - x$ has a constant sign in $\langle a, b)$ and $g^2(a) = g[g(a)] = g(\beta) > a$,

$$g^2(x) > x \quad \text{in} \quad \langle a, b) ,$$

which implies

$$\varphi^2(x) > x \quad \text{in} \quad \langle a, b) .$$

Consequently the sequence $\varphi^{2\nu}(a)$ is strictly increasing, and thus $\varphi^{n-1}(a) > a$. Hence $\varphi^n(a) < \varphi(a)$. From relation (3) the inequality $\varphi(a) \leqslant \beta$ results. Hence we have

$$\beta = g(a) = \varphi^n(a) < \varphi(a) \leqslant \beta ,$$

which is impossible. Consequently $g(a) \neq \beta$. Similarly one can prove that $g(\beta) \neq a$.

In order to build the continuous solution of equation (1) in the set $\langle a, b) \cup (g(b), \beta\rangle$ we may adopt the above described procedure, choosing $x_0 = a$ and $x_1 \in (g(a), \beta\rangle$. We must show that it is possible to choose points $x_2, \ldots, x_{n-1}$ so that relations (17) were fulfilled. It follows from lemma VI that the interval $(g(x_0), \beta\rangle$ is not empty. The inequality $x_1 \leqslant \beta$ implies $g(x_1) \geqslant g(\beta)$, whence $g(x_1) > a = x_0$. On the other hand we have evidently $x_1 > g(x_0)$.

But we must choose the points $x_2, \ldots, x_{n-1}$ in such a manner that besides relation (17) also the condition

$$x_{n-1} \leqslant g(\beta) \tag{21}$$

may be fulfilled [2]. In relation (21) the equality is possible if and only if $g(\beta) < g(x_1)$, i.e. if $x_1 < \beta$.

Thus we may construct the solution of equation (1) with the aid of formula (19) for $\nu \geqslant 0$ and for odd $\nu < 0$ till we reach the point $x = \beta$. If $\beta > x_{-\nu_0+1}$, and $x_{-\nu_0-1}$ is not defined, then we put in formula (19) $A_{-\nu} = (x_{-\nu+1}, x_{-\nu-1}\rangle$ for even $\nu < \nu_0+1$, $A_{-\nu_0-1} = (x_{-\nu_0+1}, \beta\rangle$, and $A_{-\nu} = 0$ for even $\nu > \nu_0+1$ and for odd $\nu > 0$.

Now, if the interval E is not closed, we can continue the function $g(x)$ onto $\bar{E}$ and then construct the solution of equation (1) in $\bar{E}$. If at this construction we choose $x_1 = \beta$, i.e. $\varphi(a) = \beta$, then we must take $x_{n-1} < g(\beta)$, i.e. $\varphi^{n-1}(a) < \varphi^n(\beta)$, which is equivalent to the relation $\varphi(\beta) > a$. Thus we have in this case

$$\varphi(\bar{E}) = (a, \beta\rangle .$$

Consequently, if interval E is open or closed on the right, we shall obtain the solution of equation (1) in E by the restriction of the solution of equation (1) in $\bar{E}$ to the interval E. But if $E = \langle a, \beta)$, then, according to condition (3), we must choose $x_1 \neq \beta$.

Similarly, if we choose $x_{n-1} = g(\beta)$ (and then already necessarily $x_1 \neq \beta$), which implies $\varphi(\beta) = a$, we shall have

$$\varphi(\bar{E}) = \langle a, \beta) .$$

Consequently, if the interval E is open or closed on the left, we shall obtain the solution of equation (1) in E by the restriction of the solution of equation (1) in $\bar{E}$ to the interval E. But if $E = (a, \beta\rangle$, then, according to condition (3), we must choose $x_{n-1} \neq g(\beta)$.

Gathering the above considerations, we obtain the following

Theorem II. *If the function $g(x)$ is defined, continuous and strictly decreasing in an interval E, $g^2(x) \not\equiv x$ in E, and if condition* (4) *and one of the following conditions is fulfilled:*

1. *the ends a and β of the interval E both belong to I_2^* and the interval E is either on both the sides closed, or open;*

[2] This follows from the fact that if $\varphi(x)$ is a monotonic solution of equation (1), then according to (3) $\varphi(\beta) \geqslant a$, and hence (in view of the fact that the function $\varphi^{n-1}(x)$ is increasing) $g(\beta) = \varphi^n(\beta) \geqslant \varphi^{n-1}(a)$.

2. *neither of the ends α and β belongs to I_2^*, $g(\alpha) < \beta$, $g(\beta) > \alpha$; then equation (1) possesses infinitely many solutions $\varphi(x)$ that are defined, continuous and strictly decreasing in E. We can obtain all these solutions, constructing in the above described manner (formula (19)) continuous and strictly decreasing solutions independently in each couple of the corresponding intervals of the set $E-I_2$ and putting $\varphi(x) = g(x)$ for $x \in I_2$. Relations (20) guarantee that so obtained functions are continuous in the whole interval E.*

If the relation $g^2(x) = x$ holds identically in E and E is on both sides closed or open, then equation (1) has exactly one solution defined, continuous and strictly decreasing in E:

$$\varphi(x) = g(x) .$$

C. $\varphi(x)$ decreasing, $g(x)$ increasing, $n = 2$.

As we have shown in the beginning of this paper, looking for the decreasing solutions of equation (1) with the function $g(x)$ increasing, we may confine our considerations to equation (5).

For an arbitrary point $c \in I$ we shall denote by A_c and B_c respectively the sets

$$A_c \stackrel{\text{df}}{=} \{x\colon x \in I^*,\ x \leqslant c\} , \qquad B_c \stackrel{\text{df}}{=} \{x\colon x \in I^*,\ x \geqslant c\} .$$

DEFINITION II. A point $c \in I$ will be called *semiregular* if there exists a function $f(x)$, defined and strictly decreasing on the set A_c and such that

$$f(A_c) = B_c . \tag{22}$$

The set $E-I$ is a sum of at most enumerable number of disjoint intervals. These intervals are open, possibly with the only exception of the intervals possessing a common end with the interval E.

Let $c \in I$ be a semiregular point and let $f(x)$ be a decreasing function, fulfilling condition (22). Further, let a and b be two consecutive elements of the set A_c. It is evident that then $f(b)$ and $f(a)$ are two consecutive elements of the set B_c.

DEFINITION III. If (a, b) is an interval of the set $E-I$ and $a \in I^*$, $b \in I^*$, then the interval $\big(f(b), f(a)\big)$ will be called *conjugate* to (a, b) by the function $f(x)$. If $a \notin I^*$ or $b \notin I^*$ (then a resp. b is an end of the interval E), then by the interval *conjugate* to (a, b) by the function $f(x)$ will be meant the other of the intervals of $E-I$ having a common end with the interval E.

DEFINITION IV. A semiregular point $c \in I$ will be called *regular*, if there exists a function $f(x)$, defined and decreasing in the set A_c such that relation (22) is fulfilled and in intervals conjugate by the function $f(x)$ the expression $g(x)-x$ has a converse sign. Such a function $f(x)$ will be said to map regularly the set A_c onto B_c.

It is easy to show that if $\varphi(x)$ is a continuous, strictly decreasing solution of equation (5) in E, then the point c such that $\varphi(c) = c$ is a regular point of the set I. Thus the existence of a regular point in the set I is a necessary condition of the existence of the continuous and strictly decreasing solution of equation (5) in E.

Solutions of equation (5) will be constructed in the couples of conjugate intervals of $E-I$. So let $c \in I$ be a regular point, $f(x)$ a function that maps regularly A_c onto B_c and let (a, b) be an interval of $E-I$. Moreover we suppose for the present that a and b belong to A_c. Then the interval conjugate to (a, b) by $f(x)$ is $\big(f(b), f(a)\big)$.

Let be e.g. $g(x) > x$ in (a, b). Then $g(x) < x$ in $\big(f(b), f(a)\big)$. As has been proved in [4], the functional equation

$$\psi[g(x)] = g[\psi(x)] \tag{23}$$

has infinitely many solutions $\psi(x)$ that are continuous in the interval (a, b) and assume values from the interval $\big(f(b), f(a)\big)$. Namely, for an arbitrary function $\psi_0(x)$ which is defined, continuous in the interval $\langle x_0, g(x_0)\rangle$ and fulfills the conditions

$$\psi_0(x) \in \big(f(b), f(a)\big) \quad \text{for} \quad x \in \langle x_0, g(x_0)\rangle ,$$
$$\psi_0[g(x_0)] = g[\psi_0(x_0)] ,$$

there exists a function $\psi(x)$, defined and continuous in the interval (a, b), assuming values from the interval $\big(f(b), f(a)\big)$, satisfying equation (23) and such that $\psi(x) = \psi_0(x)$ for $x \in \langle x_0, g(x_0)\rangle$. It is easy to verify that if we choose the function $\psi_0(x)$ strictly decreasing in $\langle x_0, g(x_0)\rangle$, then the function $\psi(x)$ will be strictly decreasing in (a, b) and will fulfill the conditions

$$\lim_{x\to b} \psi(x) = f(b) , \qquad \lim_{x\to a} \psi(x) = f(a) . \tag{24}$$

Consequently equation (23) possesses infinitely many solutions $\psi(x)$ that are defined, continuous and strictly decreasing in the interval (a, b) and fulfill conditions (24).

Let $\psi(x)$ be such an arbitrary solution of equation (23). We put

$$\varphi(x) \stackrel{\text{df}}{=} \begin{cases} \psi(x) & \text{for} \quad x \in (a, b) , \\ \psi^{-1}[g(x)] & \text{for} \quad x \in \big(f(b), f(a)\big) . \end{cases} \tag{25}$$

The function $\varphi(x)$ is defined, continuous and strictly decreasing in the set $(a, b) \cup \big(f(b), f(a)\big)$ and fulfills the conditions

$$\begin{aligned} &\lim_{x\to b} \varphi(x) = f(b) , && \lim_{x\to a} \varphi(x) = f(a) , \\ &\lim_{x\to f(b)} \varphi(x) = b , && \lim_{x\to f(a)} \varphi(x) = a . \end{aligned} \tag{26}$$

We shall show that $\varphi(x)$ satisfies equation (5).

Let us take an arbitrary $x \in (a, b)$. Then $\varphi(x) = \psi(x) \in (f(b), f(a))$. Consequently

$$\varphi^2(x) = \varphi[\psi(x)] = \psi^{-1}\{g[\psi(x)]\},$$

whence by (23)

$$\varphi^2(x) = \psi^{-1}\{\psi[g(x)]\} = g(x).$$

Now let us take an arbitrary $x \in (f(b), f(a))$. Then $\varphi(x) = \psi^{-1}[g(x)] \in (a, b)$. Consequently

$$\varphi^2(x) = \varphi\{\psi^{-1}[g(x)]\} = \psi\{\psi^{-1}[g(x)]\} = g(x).$$

Thus the function $\varphi(x)$ actually satisfies equation (5).

We supposed that both a and b belong to A_c. The case when a and b belong to B_c can be reduced to the former by taking the interval conjugate to (a, b) instead of (a, b). Thus only the case remained to be considered when a or b is an end of interval E and does not belong to I^*. We shall prove

Lemma VII. *If equation* (5) *possesses a continuous and strictly decreasing solution* $\varphi(x)$ *in* E, *then either both the ends of the interval* E *belong to* I^*, *or none do.*

Proof. We may assume that the interval E is closed, for otherwise we can continue the function $\varphi(x)$ onto $\bar{E}$. Let $\alpha < \beta$ be the ends of the interval E and suppose that $\beta \in B_c$. This means that

$$\varphi^2(\beta) = \beta. \tag{27}$$

Since the function $\varphi(x)$ is strictly decreasing in E and fulfills condition (3), the equality $\varphi(x) = \beta$ may be realized in E only for $x = \alpha$. Thus we have by (27)

$$\varphi(\beta) = \alpha \quad \text{and} \quad \varphi(\alpha) = \beta, \tag{28}$$

whence $g(\alpha) = \varphi^2(\alpha) = \alpha$ and $\alpha \in A_c \subset I^*$. Similarly one can prove that if $\alpha \in A_c$, then $\beta \in B_c$.

Corollary. *It results from relations* (3) *and* (28) *that if the ends of the interval* E *belong to* I^*, *then equation* (5) *may have a continuous and strictly decreasing solution in* E *only if the interval* E *is either open, or closed.*

Now we shall investigate the case when $a \notin A_c$. Then a is an end of the interval E; let β be the second end of E. Thus $(f(b), \beta\rangle$ is the interval conjugate to (a, b). From the relation $a \notin A_c$ and condition (4) we obtain the inequality $g(a) > a$. Thus $g(x) > x$ in $\langle a, b)$ and $g(x) < x$ in $(f(b), \beta\rangle$.

We can construct solutions $\psi(x)$ of equation (23) as above, assuming $x_0 = a$ and $\psi_0(a) \in \langle g(\beta), \beta\rangle$. We define the solution $\varphi(x)$ of equation (5) by the formula

$$\varphi(x) \overset{\mathrm{df}}{=} \begin{cases} \psi(x) & \text{for} \quad x \in \langle a, b), \\ \psi^{-1}[g(x)] & \text{for} \quad x \in (f(b), \beta\rangle. \end{cases} \tag{29}$$

If $\varphi(a) = \psi(a) = \beta$, then

$$\varphi(\beta) = \psi^{-1}[g(\beta)] > \psi^{-1}(\beta) = a,$$

for the function $\psi^{-1}(x)$ is decreasing and $g(\beta) < \beta$. And if $\varphi(a) = \psi(a) = g(\beta)$, then

$$\varphi(\beta) = \psi^{-1}[g(\beta)] = a.$$

Hence it follows that if the interval E is not on the both sides closed, we obtain the solution of equation (5) in the set $F \overset{\mathrm{df}}{=} E \cap \{\langle a, b) \cup (f(b), \beta\rangle\}$ by the restriction of the function $\varphi(x)$, defined by formula (29), to the set F, however under the condition that if $F = \langle a, b) \cup (f(b), \beta)$, then we must choose $\varphi(a) = \psi_0(x_0) \neq \beta$, and if $F = (a, b) \cup (f(b), \beta\rangle$, then we must choose $\varphi(a) = \psi_0(x_0) \neq g(\beta)$.

Thus we have defined the solution $\varphi(x)$ of equation (5) in the set $E - I$. Now, for $x \in I$ we put

$$\varphi(x) \overset{\mathrm{df}}{=} \begin{cases} f(x) & \text{for} \quad x \in A_c \cap E, \\ c & \text{for} \quad x = c, \\ f^{-1}(x) & \text{for} \quad x \in B_c \cap E. \end{cases} \tag{30}$$

It is easy to verify that if both the ends of the interval E belong to I^* and E is either open, or closed; or if neither of the ends of the interval E belongs to I^*, then the function $\varphi(x)$ defined by formula (30) is decreasing and satisfies equation (5) in I.

Now we shall prove that every continuous and strictly decreasing solution $\varphi(x)$ of equation (5) in E can be expressed with the aid of formulae (30) for $x \in I$, and (25) or (29) for $x \in E - I$. The former assertion is a consequence of the fact that the point c such that $\varphi(c) = c$ is a regular point of the set I and the function $\varphi(x)$ maps regularly the set A_c onto B_c. Moreover, if we put

$$f(x) \overset{\mathrm{df}}{=} \varphi(x) \quad \text{for} \quad x \in A_c \cap E,$$

then we have by (5) for $x \in B_c \cap E$

$$\varphi(x) = \varphi^{-1}[g(x)] = \varphi^{-1}(x) = f^{-1}(x).$$

To prove the second assertion let us notice that on account of the monotonity of the function $\varphi(x)$ we have for $x \in (a, b)$

$$\varphi(x) \in (\varphi(b), \varphi(a)) = (f(b), f(a))$$

where a and b are two elements of the set A_c. From the relation

$$\varphi^3(x) = \varphi[\varphi^2(x)] = \varphi^2[\varphi(x)]$$

it follows that every solution of equation (5) satisfies also equation (23). We have further by (5)

$$\varphi(x) = \varphi^{-1}[g(x)],$$

whence it follows that if two solutions of equation (5) are identical in the interval (a, b), then they are also identical in the interval $\big(f(b), f(a)\big)$.

Now we need only to show that if the ends α and β of the interval E do not belong to I^*, then

$$\varphi(\alpha) \in \langle g(\beta), \beta\rangle . \tag{31}$$

The inequalities

$$\varphi(\alpha) \leqslant \beta \quad \text{and} \quad \varphi(\beta) \geqslant \alpha \tag{32}$$

follow from condition (3). We have further from the second of inequalities (32)

$$g(\beta) = \varphi^2(\beta) \leqslant \varphi(\alpha). \tag{33}$$

Relations (32) and (33) imply (31).

Thus we have the following

THEOREM III. *If the function $g(x)$ is defined, continuous and strictly increasing in an interval E, if the set I contains at least one regular point and if condition* (4) *and one of the following conditions is fulfilled*:

1. *the ends of the interval E both belong to I^* and E is either open, or closed*;

2. *neither of the ends of the interval E belong to I^**;

then equation (5) *possesses infinitely many* [3] *solutions $\varphi(x)$ that are defined, continuous and strictly decreasing in E. We can obtain all these solutions, constructing continuous and strictly decreasing solutions independently in each couple of the conjugate intervals of the set $E - I$ (formulae* (25) *and* (29)) *and defining the function $\varphi(x)$ for $x \in I$ with the aid of formula* (30). *Relations* (26) *guarantee that so obtained functions are continuous in the whole interval E.*

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Note added in proof. Recently we have learned that equation (1) has been investigated also by P. I. Chajdukow (П. И. Хайдуков, *Об отыскании функций по заданной итерации*, Уч. зап. Бурятск. гос. пед. ин., вып. 15 (1958), p. 3-28. See Реферативный Журнал, Математика, (1961) 9Б, p. 69). Unfortunately the paper by Chajdukow is not available for us, so we have not been able to determine what exactly has been proved in it.