1° *Equation* (1) *possesses infinitely many solutions that are continuous in the set*

$$\bigcup_{\nu=0}^{n-1} \{(a_{\nu+1}, a_\nu) \cup (b_\nu, b_{\nu+1})\} .$$

2° *If, moreover, numbers* c_ν *and* d_ν $(0 \leqslant \nu \leqslant n)$ *fulfill the relations*

$$d_\nu = F(a_\nu, c_\nu) , \quad c_\nu = F(b_\nu, d_\nu) , \tag{15}$$

then equation (1) *may possess solutions that are continuous at the points* a_ν *and* b_ν. *The number of solutions that are continuous in the set* $(a_{\nu+1}, a_{\nu-1}) \cup (b_{\nu-1}, b_{\nu+1})$ *and such that* $\varphi(a_\nu) = c_\nu$ *and* $\varphi(b_\nu) = d_\nu$ *is given, according to assumptions on the function* $f(x)$ *and the derivative* $F_y(x, y) \overset{\mathrm{df}}{=} \partial F / \partial y$, *in table* 3. *As previously, the empty places denote the cases in which we are not able to determine the number of continuous solutions.*

Table 3	$f^2(x) > x$ in $(a_{\nu+1}, a_\nu)$ $f^2(x) > x$ in $(a_\nu, a_{\nu-1})$	$f^2(x) > x$ in $(a_{\nu+1}, a_\nu)$ $f^2(x) < x$ in $(a_\nu, a_{\nu-1})$	$f^2(x) < x$ in $(a_{\nu+1}, a_\nu)$ $f^2(x) > x$ in $(a_\nu, a_{\nu-1})$	$f^2(x) < x$ in $(a_{\nu+1}, a_\nu)$ $f^2(x) < x$ in $(a_\nu, a_{\nu-1})$
$\lvert F_y(a_\nu, c_\nu) F_y(b_\nu, d_\nu)\rvert > 1$	inf. many	exact. one	inf. many	inf. many
$\lvert F_y(x, y) F_y(f(x), F(x, y))\rvert \geqslant 1$ in a neighb. of (a_ν, c_ν)		at most one		
$\lvert F_y(a_\nu, c_\nu) F_y(b_\nu, d_\nu)\rvert < 1$	inf. many	inf. many	exact. one	inf. many
$\lvert F_y(x, y) F_y(f(x), F(x, y))\rvert \leqslant 1$ in a neighb. of (a_ν, c_ν)			at most one	

Remark. In the case $\nu = 0$ one should take $d = d_0 = c_0$ and $(a_1, a_{-1}) \cup (b_{-1}, b_1) = (a_1, b_1)$. Then relations (15) reduce themselves to relations (14). In the case $\nu = n$ the set $(a_{n+1}, a_{n-1}) \cup (b_{n-1}, b_{n+1})$ should be replaced by the set $\langle a_n, a_{n-1}) \cup (b_{n-1}, b_n\rangle$. In both these cases in table 3 only two central columns are to be considered.

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On continuous solutions of some functional equations of the n-th order

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In the present paper we shall consider the following functional equations of the n-th order (for a definition of an order see M. Ghermănescu [2])

$$\varphi(x) = H\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) , \tag{1}$$

$$\varphi[f_n(x)] = G\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_{n-1}(x)]\big) . \tag{2}$$

In these equations $f_i(x)$ $(i = 1, \ldots, n)$, $G(x, y_0, \ldots, y_{n-1})$, $H(x, y_1, \ldots, y_n)$ denote known, real-valued functions of real variables, and $\varphi(x)$ denotes the required function.

Equations (1) and (2) are the particular cases of the equation

$$F\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) = 0 \tag{3}$$

(under suitable assumptions equations (1), (2) and (3) are equivalent).

J. Kordylewski and M. Kuczma proved in [5] that equation (3) possesses an infinite number of solutions that are continuous in the open interval (a, b). In that manner the authors received for the case of equation (3) the result, analogous to a part of their results for the equation

$$F\big(x, \varphi(x), \varphi[f(x)]\big) = 0 , \tag{4}$$

which they had published in [4].

M. Kuczma has expressed the conjecture that for equation (3) are true also theorems about solutions continuous in the one-sided closed interval $(a, b\rangle$ (or $\langle a, b)$)—analogous to the theorems regarding the solutions of equation (4) (see [4] and [6]).

Theorems 1-3 of the present paper (being the contents of § 2) corroborate partially M. Kuczma's conjecture for equations (1) and (2). In the proofs of these theorems we make use (in essential manner) of results contained in the quoted paper [5]. In § 1 we formulate the assumptions and quote the results of the papers [5] and [7], in the formulation as we shall need for the considerations in § 2.

§ 1. Let $f(x)$ be an invertible function. We shall denote by $f^k(x)$ $(k = 0, \pm 1, \pm 2, \ldots)$ the k-th iteration of the function $f(x)$, i.e. we put

$$f^0(x) = x\,, \quad f^{k+1}(x) = f[f^k(x)]\,, \quad f^{k-1}(x) = f^{-1}[f^k(x)]\,,$$

for $k = 0, \pm 1, \pm 2, \ldots$

One can prove the following:

Lemma 1. *Let us suppose that the function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$, and such that $f(a) = a$, $f(b) = b$ and $f(x) > x$ for $x \in (a, b)$. Then, for each $x \in (a, b)$, the sequences $\{f^m(x)\}$ and $\{f^{-m}(x)\}$ are monotone, and*

$$\lim_{m\to\infty} f^m(x) = b\,, \quad \lim_{m\to\infty} f^{-m}(x) = a\,.$$

This lemma was proved in [7].

Now we shall introduce the following assumptions about the functions f_i, G, H appearing in (1) and (2).

(I) The functions $f_i(x)$ $(i = 1, \ldots, n)$ are defined, continuous and strictly increasing in the interval $\langle a, b\rangle$; $f_i(a) = a$, $f_i(b) = b$ $(i = 1, \ldots, n)$ and

$$x < f_1(x) \leqslant f_i(x) \leqslant f_{n-1}(x) < f_n(x) \quad \text{for} \quad x \in (a, b)\,, \quad i = 2, \ldots, n-2\,.$$

(II) The function $H(x, y_1, \ldots, y_n)$ is defined and continuous in a cube $\Omega \overset{\text{df}}{=} \langle a, b\rangle \times \{\langle \alpha, \beta\rangle\}^n$ and fulfills the inequalities

$$\alpha < H < \beta \quad \text{in} \quad \Omega\,.$$

(III) The function $G(x, y_0, \ldots, y_{n-1})$ is defined and continuous in the cube Ω and fulfills the inequalities

$$\alpha < G < \beta \quad \text{in} \quad \Omega\,.$$

Below we shall formulate two lemmas. Those lemmas are in fact somewhat modified variants of the theorem proved in [5]. The proofs of those lemmas, which may be given in quite similar manner as in [5] (compare the remark III in [5]), we omit.

Lemma 2. *Under assumptions* (I) *and* (II), *for any $x_0 \in (a, b)$, there exists an infinite number of solutions of equation* (1) *that are continuous in the interval $(a, f_n(x_0)\rangle$. These solutions are given by the formulae*

$$(5) \qquad \varphi(x) = \begin{cases} \psi(x) & \text{for} \quad x \in \langle x_0, f_n(x_0)\rangle\,, \\ H\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) & \text{for} \quad x \in \langle x_{-i}, x_{-i+1})\,, \end{cases}$$

where $x_{-i} \overset{\text{df}}{=} f_1^{-i}(x_0)$ $(i = 1, 2, \ldots)$, and $\psi(x)$ is an arbitrary function continuous in the interval $\langle x_0, f_n(x_0)\rangle$ and such that

$$\alpha < \psi(x) < \beta \quad \text{for} \quad x \in \langle x_0, f_n(x_0)\rangle$$

and

$$\psi(x_0) = H\big(x_0, \psi[f_1(x_0)], \ldots, \psi[f_n(x_0)]\big)\,.$$

Let us put

$$(6) \qquad k(x) \overset{\text{df}}{=} f_n[f_{n-1}^{-1}(x)]\,.$$

The function $k(x)$ fulfills the assumptions of lemma 1, as one can easily verify (compare [5]).

Lemma 3. *Under assumptions* (I) *and* (III), *for any $x_0 \in (a, b)$, there exists an infinite number of solutions of equation* (2) *that are continuous in the interval $\langle x_0, b)$. These solutions are given by the formulae*

$$(7) \qquad \varphi(x) = \begin{cases} \psi(x) & \text{for} \quad x \in \langle x_0, f_n(x_0))\,, \\ G\big(f_n^{-1}(x), \varphi[f_n^{-1}(x)], \varphi\big[f_1(f_n^{-1}(x))\big], \ldots, \varphi\big[f_{n-1}(f_n^{-1}(x))\big]\big) & \\ & \text{for} \quad x \in \langle x_i, x_{i+1})\,, \end{cases}$$

where $x_i \overset{\text{df}}{=} k^{i-1}[f_n(x_0)]$ $(i = 1, 2, \ldots)$, and $\psi(x)$ is an arbitrary function continuous in the interval $\langle x_0, f_n(x_0)\rangle$ and such that

$$\alpha < \psi(x) < \beta \quad \text{for} \quad x \in \langle x_0, f_n(x_0)\rangle$$

and

$$\psi[f_n(x_0)] = G\big(x_0, \psi(x_0), \psi[f_1(x_0)], \ldots, \psi[f_{n-1}(x_0)]\big)\,.$$

In the sequel we shall accept one of two following assumptions:

(IV) There exists a number d fulfilling the equation

$$d = H(b, d, \ldots, d)$$

and the inequalities

$$(8) \qquad \alpha < d < \beta\,.$$

(V) There exists a number d fulfilling the equation

$$d = G(b, d, \ldots, d)$$

and inequalities (8).

We shall prove the following:

Lemma 4. *Under hypotheses* (I), (II) *and* (IV), *if there exist numbers $\varepsilon > 0$, $\eta > 0$, $a_i \geqslant 0$ $(i = 1, \ldots, n)$ such that the inequalities*

$$(9) \qquad 0 < \sum_{i=1}^{n} a_i = q < 1\,,$$

$$(10) \qquad |H(x, y_1, \ldots, y_n) - H(x, \tilde{y}_1, \ldots, \tilde{y}_n)| \leqslant \sum_{i=1}^{n} a_i |y_i - \tilde{y}_i|$$

$$\text{for} \quad x \in \langle b-\eta, b\rangle\,, \quad y_i, \tilde{y}_i \in \langle d-\varepsilon, d+\varepsilon\rangle\,, \quad i = 1, \ldots, n$$

hold, and also

$$(11) \qquad |H(x, d, \ldots, d) - H(b, d, \ldots, d)| \leqslant (1-q)\varepsilon$$

holds for $x \in \langle b-\eta, b\rangle$ (¹), *then there exists exactly one solution* $\psi(x)$ *of equation* (1) *which is continuous in the interval* $\langle b-\eta, b\rangle$ *and assumes the value* d *for* $x = b$, *i.e.* $\psi(b) = d$.

Proof. We shall prove that

1° There exists exactly one solution $\psi(x)$ of equation (1), continuous in the interval $\langle b-\eta, b\rangle$, fulfilling the inequality

$$(12)\qquad |\psi(x)-d| \leqslant \varepsilon \quad \text{for} \quad x \in \langle b-\eta, b\rangle$$

and such that $\psi(b) = d$.

2° Each solution $\varphi(x)$ of equation (1) which is continuous in the interval $\langle b-\eta, b\rangle$ and assumes the value d for $x = b$, fulfills inequality (12).

The assertion of lemma 4 follows immediately from 1° and 2°.

At first we note the following assertion

$$(13)\qquad |H(x, y_1, \ldots, y_n)-d| \leqslant \varepsilon \quad \text{for} \quad x \in \langle b-\eta, b\rangle,\quad y_i \in \langle d-\varepsilon, d+\varepsilon\rangle \quad (i = 1, \ldots, n).$$

Essentially, on account of assumption (V) and inequalities (10) and (11) we have

$$\begin{aligned}|H(x, y_1, \ldots, y_n)-d| &= |H(x, y_1, \ldots, y_n)-H(b, d, \ldots, d)| \\ &\leqslant |H(x, y_1, \ldots, y_n)-H(x, d, \ldots, d)|+|H(x, d, \ldots, d)-H(b, d, \ldots, d)| \\ &\leqslant \sum_{i=1}^{n} a_i|y_i-d|+(1-q)\varepsilon \leqslant \varepsilon\sum_{i=1}^{n} a_i+(1-q)\varepsilon = \varepsilon .\end{aligned}$$

Here we shall give only a sketch of the proof of assertion 1°. This proof is analogous in details to the proof of a theorem of M. Bajraktarević (see [1]).

Let us put $E \overset{\text{df}}{=} \langle b-\eta, b\rangle$, $F \overset{\text{df}}{=} \langle d-\varepsilon, d+\varepsilon\rangle$. Inequality (10) holds in the Cartesian product $E\times F^n$. Let us compose a space of functions $\psi(x)$ which are continuous in the set E and map the set E into the set F. It is a complete metric space (after an introduction of a suitable metric). The transformation

$$\Psi(\psi) \overset{\text{df}}{=} H\big(x, \psi[f_1(x)], \ldots, \psi[f_n(x)]\big)$$

is continuous (considering assumptions (I) and (II)) in this space (on account of (13)). As it follows from inequalities (9) and (10), $\Psi(\psi)$ is also the transformation of a contraction. Hence, on account of Banach's "fixed point principle", we infer that there exists exactly one function $\psi(x)$ defined and continuous in the interval $\langle b-\eta, b\rangle$, fulfilling equation (1) and inequality (12).

(¹) On account of the continuity of the function H at the point $(b, d, \ldots, d)$ there exists a number $\eta_0 > 0$ such that inequality (11) holds for $x \in \langle b-\eta_0, b\rangle$. Lemma 4 will remain valid if we omit assumption (11) and replace the interval $\langle b-\eta, b\rangle$ in the thesis by the interval $\langle b-\eta_1, b\rangle$, where $\eta_1 \overset{\text{df}}{=} \min(\eta, \eta_0)$. We have assumed the inequality (11) in order to simplify the lemma's announcement and proof.

It remains to prove that $\psi(b) = d$. Let us suppose that $\psi(b) = d_1 \neq d$. Since the function $\psi(x)$ fulfills equation (1) for $x = b$, we have (considering assumptions (I)) the equality

$$d_1 = H(b, d_1, \ldots, d_1).$$

On the other hand, it follows from (12) that $d_1 \in \langle d-\varepsilon, d+\varepsilon\rangle$, and the application of inequality (10) yields

$$|d_1-d| = |H(b, d_1, \ldots, d_1)-H(b, d, \ldots, d)| \leqslant q|d_1-d| .$$

Hence $q \geqslant 1$, and we run into a contradiction to inequality (9). This completes the proof of assertion 1°.

Now we pass on the proof of assertion 2°.

Let a solution $\varphi(x)$ of equation (1) be continuous in the interval $\langle b-\eta, b\rangle$ and let $\varphi(x)$ fulfill the condition: $\varphi(b) = d$. The function $\varphi(x)$ is continuous at the point $x = b$, then there exists a positive number $\bar\eta$, such that the inequality

$$(14)\qquad |\varphi(x)-d| \leqslant \varepsilon$$

holds for $x \in \langle b-\bar\eta, b\rangle$. If $\bar\eta \geqslant \eta$, the assertion 2° is proved—consequently, let us suppose that there is $\bar\eta < \eta$. Let us put $x_0 \overset{\text{df}}{=} b-\bar\eta$ and $x_{-i} \overset{\text{df}}{=} f_1^{-i}(x_0)$ $(i = 1, 2, \ldots)$. There exists a natural number l, such that $x_{-l} \leqslant b-\eta$. Consequently (compare formulae (5)) the function $\varphi(x)$ will be expressed for $x \in \langle b-\eta, x_0)$ by the formulae

$$(15)\qquad \varphi(x) = H\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big), \quad x \in \langle x_{-i}, x_{-i+1}) \quad (i = 1, \ldots, l).$$

For $x \in \langle x_{-1}, x_0)$, there is $f_j(x) \in \langle x_0, f_n(x_0)) \subset \langle x_0, b)$ $(j = 1, \ldots, n)$, hence inequality (14) holds for $\varphi[f_j(x)]$ $(j = 1, \ldots, n)$. In virtue of (13) and (15), inequality (14) holds for $x \in \langle x_{-1}, b)$. Now we repeat this reasoning consecutively for the intervals $\langle x_{-2}, x_{-1})$; ..., $\langle x_{-l}, x_{-l+1})$. We draw a conclusion that inequality (14) holds also for $x \in \langle x_{-l}, b)$, therewith we have $\langle b-\eta, b) \subset \langle x_{-l}, b)$.

Thus we have proved that the function $\varphi(x)$ fulfills inequality (12). This completes the proof of assertion 2°.

§ 2. Now we shall prove

THEOREM 1. *If the assumptions of lemma* 4 *are fulfilled, then equation* (1) *possesses exactly one solution* $\varphi(x)$ *that is continuous in the interval* $(a, b\rangle$ *and fulfills the condition* $\varphi(b) = d$.

Proof. On account of lemma 4, there is exactly one continuous solution $\psi(x)$ of equation (1) in the interval $\langle b-\eta, b\rangle$, and therewith $\psi(b) = d$.

Let $x_0 \in \langle b-\eta, b)$, then also $f_n(x_0) \in \langle b-\eta, b)$ (lemma 1) and $\langle x_0, f_n(x_0))$ $\subset \langle b-\eta, b)$. In the interval $(a, f_n(x_0)\rangle$ the continuous solution of equation (1) is uniquely determined by the function $\psi(x)$. This follows from lemma 2 (formulae (5)). We put $\varphi(x) \stackrel{\text{df}}{=} \psi(x)$ for $x \in (f_n(x_0), b\rangle$, and thus we obtain the solution of equation (1), continuous in the interval $(a, b\rangle$. It is obvious that we have found all such solutions, which proves the theorem.

It is interesting that for equation (2) we obtain a quite different thesis, while assumptions are analogous to the assumptions of theorem 1. Namely, we shall prove

THEOREM 2. *Under hypotheses* (I), (III) *and* (V), *if there exist numbers* $\varepsilon > 0$, $\eta > 0$, $a_i \geqslant 0$ $(i = 0, 1, \ldots, n-1)$ *such that the inequalities*

$$\sum_{i=0}^{n-1} a_i = \vartheta_1 < 1,$$

$$(16) \qquad |G(x, y_0, \ldots, y_{n-1}) - G(x, \tilde{y}_0, \ldots, \tilde{y}_{n-1})| \leqslant \sum_{i=0}^{n-1} a_i |y_i - \tilde{y}_i|$$

for $x \in \langle b-\eta, b\rangle$, $y_i, \tilde{y}_i \in \langle d-\varepsilon, d+\varepsilon\rangle$ $(i = 0, 1, \ldots, n-1)$ *hold, and also*

$$(17) \qquad |G(x, d, \ldots, d) - G(b, d, \ldots, d)| \leqslant (\vartheta - \vartheta_1)\varepsilon$$

for $x \in \langle b-\eta, b\rangle$ [2], *where*

$$(18) \qquad \vartheta_1 < \vartheta < 1,$$

then every solution $\varphi(x)$ *of equation* (2) *which is continuous in the interval* (a, b) *and for which there exists a point* $x_0 \in \langle b-\eta, b)$ *such that*

$$(19) \qquad |\varphi(x) - d| \leqslant \varepsilon \quad \textit{for} \quad x \in \langle x_0, f_n(x_0))$$

fulfills the condition

$$(20) \qquad \lim_{x \to b-} \varphi(x) = d.$$

Proof. Let us put

$$x_\nu = k^{\nu-1}[f_n(x_0)], \quad \hat{x}_\nu = f_n^\nu(x_0) \quad (\nu = 1, 2, \ldots),$$

where the function $k(x)$ is defined by formula (6). We have $x_1 = \hat{x}_1 = f_n(x_0)$.

Let $\varphi(x)$ be a continuous solution (in the interval (a, b)) of equation (2) and let $\varphi(x)$ fulfill inequality (19). For $x \in \langle x_0, b)$ the function $\varphi(x)$ is given, according to lemma 3, by formulae (7).

We shall prove that for $x \in \langle \hat{x}_1, b)$ the inequality

$$(21) \qquad |\varphi(x) - d| \leqslant \vartheta\varepsilon$$

holds. The proof will be by induction.

[2] For assumption (17) see remark [1].

1) For $x \in \langle \hat{x}_1, x_2)$ we have from (7)

$$(22) \qquad \varphi(x) = G\left(f_n^{-1}(x), \varphi[f_n^{-1}(x)], \varphi[f_1(f_n^{-1}(x))], \ldots, \varphi[f_{n-1}(f_n^{-1}(x))]\right).$$

But $f_n^{-1}(x), f_i[f_n^{-1}(x)] \in \langle x_0, f_n(x_0))$ $(i = 1, \ldots, n-1)$, as it follows from hypothesis (I). Consequently

$$\varphi[f_n^{-1}(x)], \varphi[f_i(f_n^{-1}(x))] \in \langle d-\varepsilon, d+\varepsilon\rangle \quad (i = 1, \ldots, n-1),$$

according to (19). Thus, making use of relation (22), hypothesis (V), and inequalities (16) and (17), we can write the following inequalities:

$$(23) \qquad |\varphi(x) - d| = \left|G\left(f_n^{-1}(x), \varphi[f_n^{-1}(x)], \varphi[f_1(f_n^{-1}(x))], \ldots, \varphi[f_{n-1}(f_n^{-1}(x))]\right) - \right.$$
$$\left. - G(b, d, \ldots, d)\right| \leqslant \left|G\left(f_n^{-1}(x), \varphi[f_n^{-1}(x)], \ldots, \varphi[f_{n-1}(f_n^{-1}(x))]\right) - G\left(f_n^{-1}(x), d, \ldots, d\right)\right| +$$
$$+ \left|G\left(f_n^{-1}(x), d, \ldots, d\right) - G(b, d, \ldots, d)\right| \leqslant a_0 |\varphi[f_n^{-1}(x)] - d| +$$
$$+ \sum_{i=1}^{n-1} a_i |\varphi[f_i(f_n^{-1}(x))] - d| + (\vartheta - \vartheta_1)\varepsilon \leqslant \varepsilon \sum_{i=0}^{n-1} a_i + (\vartheta - \vartheta_1)\varepsilon = \vartheta\varepsilon.$$

In this manner we have proved that inequality (21) holds for $x \in \langle \hat{x}_1, x_2)$.

2) Let us suppose that inequality (21) holds for $x \in \langle \hat{x}_1, x_p)$. We shall prove that this inequality holds also for $x \in \langle \hat{x}_1, x_{p+1})$.

For $x \in \langle x_p, x_{p+1})$ the solution $\varphi(x)$ is expressed by the same formula (22), but now $f_n^{-1}(x), f_i[f_n^{-1}(x)] \in \langle x_0, x_p)$ $(i = 1, \ldots, n-1)$. As it follows from the inductive hypothesis, for $x \in \langle x_0, x_p)$ inequality (19) holds, whence we have

$$\varphi[f_n^{-1}(x)], \varphi[f_i(f_n^{-1}(x))] \in \langle d-\varepsilon, d+\varepsilon\rangle \quad (i = 1, \ldots, n-1).$$

Consequently, we can write again inequalities (23), from whence we infer as in 1) that inequality (21) holds for $x \in \langle x_p, x_{p+1})$, and, as a consequence of the inductive hypothesis, for $x \in \langle \hat{x}_1, x_{p+1})$.

From 1), 2) and lemma 1 we deduce that inequality (21) holds for $x \in \langle \hat{x}_1, b)$.

The sequence $\{\hat{x}_\nu\}$ is increasing and $\lim_{\nu \to \infty} \hat{x}_\nu = b$ (lemma 1). The function $G(x, d, \ldots, d)$ is continuous at the point $x = b$. Consequently we can choose from the sequence $\{\hat{x}_\nu\}$ the sequence $\{x_\nu^*\}$, $x_\nu^* \stackrel{\text{df}}{=} \hat{x}_{n_\nu}$, $\hat{x}_1 \leqslant x_1^* < \ldots < x_j^* < x_{j+1}^* < \ldots$ so that the inequalities

$$(24) \qquad |G(x, d, \ldots, d) - G(b, d, \ldots, d)| < (\vartheta - \vartheta_1)\vartheta^\nu \varepsilon$$

hold for $x \in \langle x_\nu^*, b)$ and $\nu = 1, 2, \ldots$

Now we can prove by induction the following inequalities

$$(*)\qquad |\varphi(x)-d|\leqslant \vartheta^{\nu}\varepsilon \quad \text{for} \quad x\in\langle x_\nu^*, b), \quad \nu=1,2,\ldots$$

For $\nu=1$ inequality $(*)$ follows from inequality (21), because $x_1^*\geqslant \hat{x}_1$. Let us assume inequality $(*)$ for $\nu=p>1$. In a similar manner as in the proof of inequality (21), making use of this hypothesis and inequality (24) for $\nu=p$, we obtain according to (23) the inequality

$$|\varphi(x)-d|\leqslant \vartheta^p\varepsilon\vartheta_1+(\vartheta-\vartheta_1)\vartheta^p\varepsilon=\vartheta^{p+1}\varepsilon$$

for $x\in\langle f_n(x_p^*), b)$. But $x_p^*<f_n(x_p^*)<x_{p+1}^*$ hence $\langle x_{p+1}^*, b)\subset\langle f_n(x_p^*), b)$ and inequality $(*)$ holds for $\nu=p+1$. Consequently it holds for each natural ν.

At last, let us take an arbitrary number $\bar{\varepsilon}>0$. We can find (according to (18)) a natural number $\nu(\bar{\varepsilon})$ such that

$$\vartheta^{\nu}\varepsilon<\bar{\varepsilon} \quad \text{for} \quad \nu\geqslant\nu(\bar{\varepsilon}).$$

Of course, the sequence $\{x_\nu^*\}$ is also increasing and $\lim\limits_{\nu\to\infty}x_\nu^*=b$. Whence, and from inequality $(*)$ we infer that

$$|\varphi(x)-d|<\bar{\varepsilon} \quad \text{for} \quad x\in\langle x_{\nu(\bar{\varepsilon})}^*, b).$$

This completes the proof of the theorem.

COROLLARY. *Under the assumptions of theorem 2 there exists an infinite number of solutions of equation* (2), *continuous in the interval* $(a, b\rangle$.

Proof. Let us take an arbitrary solutions $\varphi(x)$ of equation (2) that fulfills the assumptions of theorem 2. Let us define as a supplement this solution as equal d for $x=b$, i.e. let us put

$$\varphi(b)=d.$$

In this manner we have obtained the solution of equation (2) that is continuous in the interval $(a, b\rangle$, as it follows from hypothesis (V) and equality (20).

For equation (1) and the interval $\langle a, b)$ one can prove a theorem analogous to the theorem 2. Namely, let us assume that

(IV′) There exists a number c which fulfills the equality

$$c=H(a, c, \ldots, c)$$

and the inequalities

$$a<c<\beta.$$

We have

THEOREM 3. *Under the hypotheses* (I), (II) *and* (IV′), *if there exist numbers* $\varepsilon>0$, $\delta>0$, $a_i\geqslant 0$ $(i=1,\ldots,n)$ *such that the inequalities*

$$\sum_{i=1}^{n}a_i=\theta_1<1,$$

$$|H(x, y_1, \ldots, y_n)-H(x, \tilde{y}_1, \ldots, \tilde{y}_n)|\leqslant\sum_{i=1}^{n}a_i|y_i-\tilde{y}_i|$$

$$\text{for} \quad x\in\langle a, a+\delta\rangle, \quad y_i, \tilde{y}_i\in\langle c-\varepsilon, c+\varepsilon\rangle \quad (i=1,\ldots,n)$$

hold, and also

$$|H(x, c, \ldots, c)-H(a, c, \ldots, c)|\leqslant(\theta-\theta_1)\varepsilon \quad \text{for} \quad x\in\langle a, a+\delta\rangle \text{ (}^3\text{)},$$

where

$$\theta_1<\theta<1;$$

then every solution $\varphi(x)$ *of equation* (1) *which is continuous in the interval* (a, b) *and for which there exists a point* $x_0\in(a, b)$ *such that* $f_n(x_0)\in(a, a+\delta\rangle$ *and*

$$|\varphi(x)-c|\leqslant\varepsilon \quad \text{for} \quad x\in\langle x_0, f_n(x_0))$$

fulfills the condition

$$\lim_{x\to a+}\varphi(x)=c.$$

This theorem may be proved in a quite similar manner, as theorem 2. We obtain also

COROLLARY. *Under the hypotheses of theorem* 3 *there exists an infinite number of solutions of equation* (1), *continuous in the interval* $\langle a, b)$.

For equation (2) and the interval $\langle a, b)$ one cannot obtain, making use of the method applied in the present paper, a theorem analogous to theorem 1. Namely, in this case, lemma 4 is false. This problem remains open.

Remark. J. Kordylewski [3] considered the equation

$$(25)\qquad F\big(x, \varphi(x), \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^n(x)]\big)=0.$$

In this equation instead of the functions $f_i(x)$ $(i=1,\ldots,n)$, as in equation (3), there are succesive iterations of the same function $f(x)$ (fulfilling the assumptions of lemma 1).

(³) See remark (¹).

J. Kordylewski proved in [3] a theorem about the existence of continous solutions of equation (25), under assumptions (on the function F) weaker than in the quoted paper [5].

If we shall accept assumptions named in [3], then our theorems will stay true for suitable particular cases of equation (25). Proofs do not change in any essential manner.

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On some extensions of Cauchy's condensation theorem

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1. Introduction. A. Alexiewicz ([1], p. 85) has proved by functional analysis the following theorem, and has shown that the case $b_n\equiv 1$ of the theorem at once completes and extends the familiar condensation theorem (test) of Cauchy ([1], p. 80).

THEOREM I. [1] *Let* $a_n\geqslant 0, b_n\geqslant 0$ $(n=1,2,\ldots)$ *be given positive sequences and* $\{\varepsilon_n\}$ *an arbitrary positive sequence tending monotonically to* 0. *Then the series*

$$\text{(A)}\quad \sum_{n=1}^{\infty} a_n\varepsilon_n, \qquad \text{(B)}\quad \sum_{n=1}^{\infty} b_n\varepsilon_n$$

are either both convergent or both divergent if and only if

$$0<\varlimsup_{n\to\infty}\frac{b_1+b_2+\ldots+b_n}{a_1+a_2+\ldots+a_n}<\infty, \tag{1}$$

i.e. the two limits in (1) *are both finite and strictly positive.*

This note gives a simple proof of Theorem I not depending on functional analysis and deduces from the theorem the following extension of Cauchy's condensation test, due in effect to O. Szász ([4], p. 1397, Theorem 1).

THEOREM II. *Let* $\{f(n)\}$ *be a positive sequence quasi-monotonic decreasing in the sense that there is an* $\alpha\geqslant 0$ *such that*

$$f(n+1)\leqslant(1+\alpha/n)f(n) \quad \text{for} \quad n>n_0(\alpha). \tag{2}$$

Let $\{\lambda_n\}$ *be any sequence of positive integers such that*

$$\lambda_n\nearrow\infty, \quad \varlimsup_{n\to\infty}\lambda_n/\lambda_{n-1}<\infty. \tag{3}$$

(1) Added 17th October 1960. After I had sent the MS of this paper to Professor Alexiewicz, I found in the review of his paper [1] by D. Gaier, in Zbl. für Math. 77 (1958), p. 277, a proof of the simpler 'if' or sufficiency part of Theorem I in the case $b_n\equiv 1$ which is the same as the proof in this paper.