ANNALES
POLONICI MATHEMATICI
XI (1961)

Continuous solutions of the functional equation $\varphi[f(x)] = F(x, \varphi(x))$ with the function $f(x)$ decreasing

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The present paper contains results regarding the existence of continuous solutions of the functional equation

$$\varphi[f(x)] = F(x, \varphi(x)), \tag{1}$$

where $\varphi(x)$ denotes the unknown function and $f(x)$ and $F(x, y)$ are given functions, the function $f(x)$ being decreasing. In some very simple particular cases of equation (1) corresponding results have been obtained by W. Chayoth [1].

Analogous results for equation (1) with the function $f(x)$ increasing have been proved by M. Kuczma and the author of this article in [2], [4] and [5]. The present paper has arisen on the one hand in order to generalize the results of W. Chayoth, on the other hand in order to transpose the properties of equation (1) with the function $f(x)$ increasing for the case of the function $f(x)$ decreasing. This last thing at the first moment did not seem to be immediately obtainable. It has turned out, however, that equation (1) with the function $f(x)$ decreasing is in a certain sense equivalent to an analogous equation, but with a function $f(x)$ increasing. Consequently, theorems on continuous solutions of equation (1) with the function $f(x)$ decreasing can be derived from the corresponding theorems on equation (1) with the function $f(x)$ increasing.

§ 1. Since in the sequel we shall make use of properties of the equation

$$\psi[g(x)] = G(x, \psi(x)), \tag{2}$$

where $\psi(x)$ denotes the unknown function and $g(x)$ and $G(x, y)$ are given functions and the function $g(x)$ is increasing, below we shall give theorems concerning the existence of continuous solutions of equation (2).

We make the following assumptions:

(I) The function $g(x)$ is defined, continuous and strictly increasing in an interval $\langle a, \beta\rangle$, moreover $g(a) = a$, $g(\beta) = \beta$, $g(x) \neq x$ for $x \in (a, \beta)$.

(II) The function $G(x, y)$ is defined and continuous in a region Λ of the variables (x, y), normal with respect to the x-axis and possesses the continuous derivative $\partial G/\partial y$ different from zero in the region Λ.

(III) $\Lambda_x \neq 0$, $\Theta_x = \Lambda_{g(x)}$ for $x \in (\alpha, \beta)$, where Λ_x denotes the projection on the y-axis of the intersection of the region Λ with the line $x = \text{const}$ and Θ_x denotes the set of values of the function $G(x, y)$ for $y \in \Lambda_x$, i.e.

$$\Lambda_x = \{y\colon (x, y) \in \Lambda\}, \quad \Theta_x = \Big\{z\colon \sum_y [y \in \Lambda_x, z = G(x, y)]\Big\}.$$

THEOREM I. *If hypotheses* (I)-(III) *are fulfilled, then*

1° *Equation* (2) *possesses infinitely many solutions that are continuous in the interval* (α, β).

2° *If moreover there exist numbers* γ *and* δ *fulfilling the relations*

$$\gamma = G(\alpha, \gamma), \quad \delta = G(\beta, \delta), \tag{3}$$

then equation (2) *may have a solution continuous in the interval* $(\alpha, \beta\rangle$ *or* $\langle\alpha, \beta)$. *The number of solutions that are continuous in the interval* $(\alpha, \beta\rangle$ *and such that* $\psi(\beta) = \delta$ *is given, according to assumptions on the function* $g(x)$ *and the derivative* $G_y(x, y) \overset{\text{df}}{=} \partial G/\partial y$ *in table* 1; *the number of solutions that are continuous in the interval* $\langle\alpha, \beta)$ *and such that* $\psi(\alpha) = \gamma$ *is given in table* 2. *The empty places denote the cases in which we are not able to determine the number of continuous solutions.*

Table 1	$g(x) > x$ in (α, β)	$g(x) < x$ in (α, β)
$\lvert G_y(\beta, \delta)\rvert > 1$	exactly one	infinitely many
$\lvert G_y(x, y)\rvert \geqslant 1$ in a neighbourhood of (β, δ)	at most one	
$\lvert G_y(\beta, \delta)\rvert < 1$	infinitely many	exactly one
$\lvert G_y(x, y)\rvert \leqslant 1$ in a neighbourhood of (β, δ)		at most one

Table 2	$g(x) > x$ in (α, β)	$g(x) < x$ in (α, β)
$\lvert G_y(\alpha, \gamma)\rvert > 1$	infinitely many	exactly one
$\lvert G_y(x, y)\rvert \geqslant 1$ in a neighbourhood of (α, γ)		at most one
$\lvert G_y(\alpha, \gamma)\rvert < 1$	exactly one	infinitely many
$\lvert G_y(x, y)\rvert \leqslant 1$ in a neighbourhood of (α, γ)	at most one	

Proof. Since $g(x) \neq x$ for $x \in (\alpha, \beta)$, either $g(x) > x$ in (α, β) or $g(x) < x$ in (α, β).

In the case when $g(x) > x$ in (α, β) the present theorem gathers the results contained in theorems I and IV from [2], in the theorem from [4] and in theorem IV from [5]. The case $g(x) < x$ has not been dealt with in the above papers, but it can be easily reduced to the case $g(x) > x$. Namely, let $g(x) < x$ in (α, β). Equation (2) is equivalent to the equation

$$\psi[h(x)] = H\big(h(x), \psi(x)\big), \tag{4}$$

where $h(x)$ is the function inverse to the function $g(x)$ and $y = H(x, z)$ is the function inverse to the function $z = G(x, y)$ with respect to the second variable. From the fulfillment of hypotheses (I)-(III) and relations (3) for equation (2) follows the fulfillment of those for equation (4). Since moreover apparently $h(x) > x$ in (α, β), we may apply to equation (4) the already verified first part of this theorem. Taking into account the fact that

$$G_y(x, y) = \frac{1}{H_z(x, z)} \quad \text{with the substitution} \quad z = G(x, y)$$

we obtain the validity of the present theorem also in the case $g(x) < x$.

§ 2. Now we pass to equation (1). We introduce the following definition and hypotheses (i)-(iii).

DEFINITION. We call an interval $\langle a, b\rangle$ a *modulus-interval* for the function $f(x)$ if $f(\langle a, b\rangle) = \langle a, b\rangle$.

HYPOTHESES. (i) The function $f(x)$ is defined, continuous and strictly decreasing in a modulus-interval $\langle a, b\rangle$ containing a finite number of points fulfilling the equation

$$f^2(x) = x \tag{5}$$

($f^2(x) = f[f(x)]$ denotes here the second iteration of the function f).

(ii) The function $F(x, y)$ is defined and continuous in a region Ω of the variables (x, y), normal with respect to the x-axis and possesses the continuous derivative $\partial F/\partial y$ different from zero in the region Ω.

(iii) $\Omega_x \neq 0$, $\Gamma_x = \Omega_{f(x)}$ for $x \in \langle a, b\rangle$, where Ω_x denotes the projection on the y-axis of the intersection of the region Ω with the line $x = \text{const}$ and Γ_x denotes the set of values of the function $F(x, y)$ for $y \in \Omega_x$, i.e.

$$\Omega_x = \{y\colon (x, y) \in \Omega\}, \quad \Gamma_x = \Big\{z\colon \sum_y [y \in \Omega_x, z = F(x, y)]\Big\}.$$

LEMMA I. *Suppose that a function* $k(x)$ *is defined, continuous and strictly increasing in an interval* $\langle a, b\rangle$. *The necessary and sufficient condition that the interval* $\langle a, b\rangle$ *be a modulus-interval for the function* $k(x)$ *is that* $k(a) = a$ *and* $k(b) = b$.

The proof of this lemma is to be found in [3].

LEMMA II. *If the function $f(x)$ fulfills hypothesis* (i), *then*

1° *In the interval $\langle a, b\rangle$ exists exactly one root c of the equation*

$$f(x) = x, \tag{6}$$

$c \in (a, b)$ *and* $f(x) > x$ *in* $\langle a, c)$ *and* $f(x) < x$ *in* $(c, b\rangle$.

2° a, b *and* c *are roots of equation* (5).

3° *In the intervals $\langle a, c\rangle$ and $\langle c, b\rangle$ are equally many roots a_ν and b_ν of equation* (5):

$$a = a_n < a_{n-1} < \ldots < a_1 < a_0 = c = b_0 < b_1 < \ldots < b_{n-1} < b_n = b,$$

i.e. $f^2(a_\nu) = a_\nu$, $f^2(b_\nu) = b_\nu$ *and* $f^2(x) \neq x$ *for* $x \neq a_\nu$ *and* $x \neq b_\nu$, $\nu = 0, 1, \ldots, n$. *Moreover* $f(a_\nu) = b_\nu$, $f(b_\nu) = a_\nu$ *for* $\nu = 0, 1, \ldots, n$.

4° *We have for* $\nu = 0, 1, \ldots, n-1$

$$f\big((a_{\nu+1}, a_\nu)\big) = (b_\nu, b_{\nu+1}), \quad f\big((b_\nu, b_{\nu+1})\big) = (a_{\nu+1}, a_\nu), \tag{7}$$

$$f^2\big((a_{\nu+1}, a_\nu)\big) = (a_{\nu+1}, a_\nu), \quad f^2\big((b_\nu, b_{\nu+1})\big) = (b_\nu, b_{\nu+1}). \tag{8}$$

Proof. 1° Since the interval $\langle a, b\rangle$ is a modulus-interval for the decreasing function $f(x)$, necessarily $f(a) > a$ and $f(b) < b$. The function $h(x) \overset{\text{df}}{=} f(x) - x$ is continuous and decreasing in the interval $\langle a, b\rangle$ and $h(a) > 0$, $h(b) < 0$. Consequently there exists exactly one $c \in (a, b)$ such that $h(c) = 0$. Thus $f(c) = c$ and since the function $f(x)$ is decreasing, $f(x) > c$ for $x < c$ and $f(x) < c$ for $x > c$.

2° From hypothesis (i) it follows that the function $f^2(x)$ is increasing in the interval $\langle a, b\rangle$ and the latter is a modulus-interval for it. Thus we have by lemma I $f^2(a) = a$ and $f^2(b) = b$. Since $f(c) = c$, also $f^2(c) = c$.

3° From part 1° of this lemma it follows that $f(\langle a, c\rangle) = \langle c, b\rangle$ and $f(\langle c, b\rangle) = \langle a, c\rangle$. Since from the fulfillment of equation (5) by $\overline{x}$ follows its fulfillment by $\overline{\overline{x}} \overset{\text{df}}{=} f(\overline{x})$ and the relation $f(\overline{\overline{x}}) = \overline{x}$, the intervals $\langle a, c\rangle$ and $\langle c, b\rangle$ must contain equal numbers of roots of equation (5). We have moreover on account of the fact that the function $f(x)$ is decreasing

$$f(a_\nu) = b_\nu \quad \text{and} \quad f(b_\nu) = a_\nu \quad \text{for} \quad \nu = 0, 1, \ldots, n. \tag{9}$$

4° Let $x \in (a_{\nu+1}, a_\nu)$. The function $f(x)$ is decreasing, consequently $f(a_\nu) < f(x) < f(a_{\nu+1})$, and by (9) $f(x) \in (b_\nu, b_{\nu+1})$. Thus $f\big((a_{\nu+1}, a_\nu)\big) \subset (b_\nu, b_{\nu+1})$. Similarly one can prove the inclusions $(b_\nu, b_{\nu+1}) \subset f\big((a_{\nu+1}, a_\nu)\big)$, $f\big((b_\nu, b_{\nu+1})\big) \subset (a_{\nu+1}, a_\nu)$, $(a_{\nu+1}, a_\nu) \subset f\big((b_\nu, b_{\nu+1})\big)$. Hence follow relations (7). Relations (8) are an immediate consequence of relations (7). This completes the proof of the lemma.

LEMMA III. *If we put*

$$g(x) \overset{\text{df}}{=} f^2(x) \quad \text{and} \quad \langle \alpha, \beta\rangle \overset{\text{df}}{=} \langle a_{\nu+1}, a_\nu\rangle, \tag{10}$$

then from hypothesis (i) *follows hypothesis* (I).

Proof. Since the function $f(x)$ is decreasing, the function $g(x)$ is increasing. Moreover we have by lemma II $g(a_{\nu+1}) = f^2(a_{\nu+1}) = a_{\nu+1}$, $g(a_\nu) = f^2(a_\nu) = a_\nu$ and $g(x) = f^2(x) \neq x$ for $x \in (a_{\nu+1}, a_\nu)$.

LEMMA IV. *If we put*

$$G(x, y) \overset{\text{df}}{=} F\big(f(x), F(x, y)\big) \quad \text{and} \quad \Lambda \overset{\text{df}}{=} \Omega, \tag{11}$$

then (*with notation* (10)) *from hypotheses* (ii) *and* (iii) *follow hypotheses* (II) *and* (III).

Proof. Let us take an arbitrary point $(x, y) \in \Lambda$. Consequently $(x, y) \in \Omega$ and $F(x, y)$ has a meaning. We have further

$$F(x, y) \in \Gamma_x,$$

whence

$$F(x, y) \in \Omega_{f(x)},$$

i.e.

$$\big(f(x), F(x, y)\big) \in \Omega$$

and

$$F\big(f(x), F(x, y)\big) = G(x, y)$$

has a meaning. Since (x, y) has been an arbitrary point of the region Λ, the function $G(x, y)$ is defined in the region Λ. The continuity of the function $G(x, y)$ and the existence of the continuous derivative $\partial G/\partial y$ in the region Λ follow from the analogous properties of the function $F(x, y)$ in the region Ω and from the continuity of the function $f(x)$. Since $\partial F/\partial y \neq 0$ in Ω we have

$$\frac{\partial G(x, y)}{\partial y} = \frac{\partial F\big(f(x), F(x, y)\big)}{\partial y} \cdot \frac{\partial F(x, y)}{\partial y} \neq 0 \quad \text{in} \quad \Lambda.$$

Thus hypothesis (II) is fulfilled. As it can be easily verified, $\Lambda_x = \Omega_x$ and $\Theta_x = \Gamma_{f(x)}$ and consequently also hypothesis (III) is fulfilled.

LEMMA V. *If hypotheses* (i)-(iii) *are fulfilled and the functions $g(x)$ and $G(x, y)$ are defined with the aid of formulae* (10) *and* (11) *respectively, then equations* (1) *and* (2) *are equivalent in a certain sense. The equivalence is to be understood as follows:*

1° *If a function $\varphi(x)$ is defined and satisfies equation* (1) *in the set* $(a_{\nu+1}, a_\nu) \cup (b_\nu, b_{\nu+1})$, *resp.* $\langle a_{\nu+1}, a_\nu) \cup (b_\nu, b_{\nu+1}\rangle$, *resp.* $(a_{\nu+1}, a_\nu\rangle \cup \langle b_\nu, b_{\nu+1})$, *then the function*

$$\psi(x) \overset{\text{df}}{=} \varphi(x) \quad \text{for} \quad x \in (a_{\nu+1}, a_\nu), \tag{12}$$

$$\text{resp.} \quad x \in \langle a_{\nu+1}, a_\nu), \quad \text{resp.} \quad x \in (a_{\nu+1}, a_\nu\rangle,$$

is defined and satisfies equation (2) *in the interval* $(a_{\nu+1}, a_\nu)$, *resp.* $\langle a_{\nu+1}, a_\nu)$, *resp.* $(a_{\nu+1}, a_\nu\rangle$ $(0 \leqslant \nu \leqslant n-1)$.

2° *If a function* $\psi(x)$ *is defined and satisfies equation* (2) *in the interval* $(a_{\nu+1}, a_\nu)$, *resp.* $\langle a_{\nu+1}, a_\nu)$, *resp.* $(a_{\nu+1}, a_\nu\rangle$, *then the function*

$$(13)\qquad \varphi(x) \overset{\mathrm{df}}{=} \begin{cases} \psi(x) \quad \textit{for} \quad x \in (a_{\nu+1}, a_\nu)\,, \\ \quad \textit{resp.} \quad x \in \langle a_{\nu+1}, a_\nu)\,, \quad \textit{resp.} \quad x \in (a_{\nu+1}, a_\nu\rangle\,, \\ F\big(f^{-1}(x), \psi[f^{-1}(x)]\big) \quad \textit{for} \quad x \in (b_\nu, b_{\nu+1})\,, \\ \quad \textit{resp.} \quad x \in (b_\nu, b_{\nu+1}\rangle\,, \quad \textit{resp.} \quad x \in \langle b_\nu, b_{\nu+1})\,, \end{cases}$$

is defined and satisfies equation (1) *in the set* $(a_{\nu+1}, a_\nu) \cup (b_\nu, b_{\nu+1})$, *resp.* $\langle a_{\nu+1}, a_\nu) \cup (b_\nu, b_{\nu+1}\rangle$, *resp.* $(a_{\nu+1}, a_\nu\rangle \cup \langle b_\nu, b_{\nu+1})$ $(0 \leqslant \nu \leqslant n-1)$. *For the interval* $(a_1, c\rangle$ *we must assume additionally that the value* $d = \psi(c)$ *fulfills the relation*

$$(14)\qquad d = F(c, d)\,.$$

Proof. 1° Let us suppose that a function $\varphi(x)$ is defined and satisfies equation (1) in the set $(a_{\nu+1}, a_\nu) \cup (b_\nu, b_{\nu+1})$. Let $x \in (a_{\nu+1}, a_\nu)$. On account of lemma II $f(x) \in (b_\nu, b_{\nu+1})$ and $f^2(x) \in (a_{\nu+1}, a_\nu)$. Since the function $\varphi(x)$ is defined at the points x, $f(x)$ and $f^2(x)$ and equation (1) is satisfied at the points x and $f(x)$, we have

$$\varphi[f(x)] = F\big(x, \varphi(x)\big)$$

and

$$\varphi[f^2(x)] = F\big(f(x), \varphi[f(x)]\big)\,.$$

Consequently

$$\varphi[f^2(x)] = F\Big(f(x), F\big(x, \varphi(x)\big)\Big)$$

and taking into account (10), (11) and (12) we obtain

$$\psi[g(x)] = G\big(x, \psi(x)\big)\,.$$

Consequently the function $\psi(x)$ is defined and satisfies equation (2) at the point x. Since x has been an arbitrary point from the interval $(a_{\nu+1}, a_\nu)$, the function $\psi(x)$ is defined and satisfies equation (2) in the interval $(a_{\nu+1}, a_\nu)$. If we replace the open interval by the closed one, the proof is analogous.

2° Now let us suppose that a function $\psi(x)$ is defined and satisfies equation (2) in the interval $(a_{\nu+1}, a_\nu)$. Let $x \in (a_{\nu+1}, a_\nu)$. On account of lemma II $f(x) \in (b_\nu, b_{\nu+1})$. At the point x the function $\psi(x)$ is defined and satisfies equation (2), consequently $\psi(x)$ and $G\big(x, \psi(x)\big)$ have a meaning. By lemma IV also $F\big(x, \psi(x)\big)$ has a meaning. We have according to (13)

$$\varphi(x) = \psi(x)$$

and

$$\varphi[f(x)] = F\big(x, \psi(x)\big)\,.$$

Hence

$$\varphi[f(x)] = F\big(x, \varphi(x)\big)$$

and thus the function $\varphi(x)$ is defined and satisfies equation (1) at the point x. Since x has been an arbitrary point from the interval $(a_{\nu+1}, a_\nu)$, the function $\varphi(x)$ is defined and satisfies equation (1) in the interval $(a_{\nu+1}, a_\nu)$.

Now let $x \in (b_\nu, b_{\nu+1})$. On account of lemma II $f^{-1}(x) \in (a_{\nu+1}, a_\nu)$ and $f(x) \in (a_{\nu+1}, a_\nu)$. At the points $f^{-1}(x)$ and $f(x)$ the function $\psi(x)$ is defined and satisfies equation (2), consequently $\psi[f(x)]$, $\psi[f^{-1}(x)]$ and $G\big(f^{-1}(x), \psi[f^{-1}(x)]\big)$ have a meaning and

$$\psi\big[g\big(f^{-1}(x)\big)\big] = G\big(f^{-1}(x), \psi[f^{-1}(x)]\big)\,.$$

By lemma IV also $F\big(f^{-1}(x), \psi[f^{-1}(x)]\big)$ has a meaning and

$$\psi[f(x)] = F\Big(x, F\big(f^{-1}(x), \psi[f^{-1}(x)]\big)\Big)\,.$$

We have further according to (13)

$$\varphi(x) = F\big(f^{-1}(x), \psi[f^{-1}(x)]\big)$$

and

$$\varphi[f(x)] = \psi[f(x)]\,.$$

Hence

$$\varphi[f(x)] = F\big(x, \varphi(x)\big)$$

and thus the function $\varphi(x)$ is defined and satisfies equation (1) at the point x. Since x has been an arbitrary point from the interval $(b_\nu, b_{\nu+1})$, the function $\varphi(x)$ is defined and satisfies equation (1) in the interval $(b_\nu, b_{\nu+1})$. Thus the function $\varphi(x)$ is defined and satisfies equation (1) in the set $(a_{\nu+1}, a_\nu) \cup (b_\nu, b_{\nu+1})$. If we replace the open intervals by the closed ones, the proof is analogous.

For the interval $(a_1, c\rangle$ relations (13) twice define the value

$$\varphi(c) = \begin{cases} \psi(c)\,, \\ G\big(c, \psi(c)\big)\,. \end{cases}$$

But according to relation (14) it does not matter. This completes the proof.

An immediate consequence of lemma V and theorem I is the following

THEOREM II. *Let us assume that hypotheses* (i)-(iii) *are fulfilled and let the sequences* a_ν, b_ν *be those occurring in lemma* II. *Then*

1° *Equation* (1) *possesses infinitely many solutions that are continuous in the set*

$$\bigcup_{\nu=0}^{n-1}\{(a_{\nu+1}, a_\nu) \cup (b_\nu, b_{\nu+1})\}\,.$$

2° *If, moreover, numbers* c_ν *and* d_ν $(0 \leqslant \nu \leqslant n)$ *fulfill the relations*

$$d_\nu = F(a_\nu, c_\nu)\,, \qquad c_\nu = F(b_\nu, d_\nu)\,, \tag{15}$$

then equation (1) *may possess solutions that are continuous at the points* a_ν *and* b_ν. *The number of solutions that are continuous in the set* $(a_{\nu+1}, a_{\nu-1}) \cup (b_{\nu-1}, b_{\nu+1})$ *and such that* $\varphi(a_\nu) = c_\nu$ *and* $\varphi(b_\nu) = d_\nu$ *is given, according to assumptions on the function* $f(x)$ *and the derivative* $F_y(x, y) \overset{\mathrm{df}}{=} \partial F/\partial y$, *in table* 3. *As previously, the empty places denote the cases in which we are not able to determine the number of continuous solutions.*

Table 3	$f^2(x) > x$ in $(a_{\nu+1}, a_\nu)$ $f^2(x) > x$ in $(a_\nu, a_{\nu-1})$	$f^2(x) > x$ in $(a_{\nu+1}, a_\nu)$ $f^2(x) < x$ in $(a_\nu, a_{\nu-1})$	$f^2(x) < x$ in $(a_{\nu+1}, a_\nu)$ $f^2(x) > x$ in $(a_\nu, a_{\nu-1})$	$f^2(x) < x$ in $(a_{\nu+1}, a_\nu)$ $f^2(x) < x$ in $(a_\nu, a_{\nu-1})$
$\lvert F_y(a_\nu, c_\nu)F_y(b_\nu, d_\nu)\rvert > 1$	inf. many	exact. one	inf. many	inf. many
$\lvert F_y(x, y)F_y(f(x), F(x, y))\rvert \geqslant 1$ in a neighb. of (a_ν, c_ν)		at most one		
$\lvert F_y(a_\nu, c_\nu)F_y(b_\nu, d_\nu)\rvert < 1$	inf. many	inf. many	exact. one	inf. many
$\lvert F_y(x, y)F_y(f(x), F(x, y))\rvert \leqslant 1$ in a neighb. of (a_ν, c_ν)			at most one	

Remark. In the case $\nu = 0$ one should take $d = d_0 = c_0$ and $(a_1, a_{-1}) \cup (b_{-1}, b_1) = (a_1, b_1)$. Then relations (15) reduce themselves to relations (14). In the case $\nu = n$ the set $(a_{n+1}, a_{n-1}) \cup (b_{n-1}, b_{n+1})$ should be replaced by the set $\langle a_n, a_{n-1}) \cup (b_{n-1}, b_n\rangle$. In both these cases in table 3 only two central columns are to be considered.

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Reçu par la Rédaction le 1. 6. 1960

ANNALES
POLONICI MATHEMATICI
XI (1961)

On continuous solutions of some functional equations of the n-th order

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In the present paper we shall consider the following functional equations of the n-th order (for a definition of an order see M. Ghermănescu [2])

$$\varphi(x) = H\big(x, \varphi[f_1(x)]\,, \ldots, \varphi[f_n(x)]\big)\,, \tag{1}$$

$$\varphi[f_n(x)] = G\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_{n-1}(x)]\big)\,. \tag{2}$$

In these equations $f_i(x)$ $(i = 1, \ldots, n)$, $G(x, y_0, \ldots, y_{n-1})$, $H(x, y_1, \ldots, y_n)$ denote known, real-valued functions of real variables, and $\varphi(x)$ denotes the required function.

Equations (1) and (2) are the particular cases of the equation

$$F\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) = 0 \tag{3}$$

(under suitable assumptions equations (1), (2) and (3) are equivalent).

J. Kordylewski and M. Kuczma proved in [5] that equation (3) possesses an infinite number of solutions that are continuous in the open interval (a, b). In that manner the authors received for the case of equation (3) the result, analogous to a part of their results for the equation

$$F\big(x, \varphi(x), \varphi[f(x)]\big) = 0\,, \tag{4}$$

which they had published in [4].

M. Kuczma has expressed the conjecture that for equation (3) are true also theorems about solutions continuous in the one-sided closed interval $(a, b\rangle$ (or $\langle a, b)$)—analogous to the theorems regarding the solutions of equation (4) (see [4] and [6]).

Theorems 1-3 of the present paper (being the contents of § 2) corroborate partially M. Kuczma's conjecture for equations (1) and (2). In the proofs of these theorems we make use (in essential manner) of results contained in the quoted paper [5]. In § 1 we formulate the assumptions and quote the results of the papers [5] and [7], in the formulation as we shall need for the considerations in § 2.