Thus in virtue of lemma 10 and (67) with $v = \Delta p$, lemma 11 immediately follows.

Our main theorem is a direct consequence of the above proved lemmas. Indeed, from lemma 2 it follows immediately that this theorem is a consequence of (11), (12), (13), and (14), and these, in turn, follow from lemmas 7, 10, 9, and 11 respectively.

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On the asymptotic coincidence of sets filled up by integrals of two systems of ordinary differential equations

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Introduction. In many papers concerning the asymptotic behaviour of solutions of ordinary differential equations the following problem has been considered.

One has a system of differential equations

$$(0,1) \qquad dy/dt = F(y, t) + \varepsilon(y, t)$$

(y is a vector $y_1, \ldots, y_n$, t is real variable, $F(y,t)$ and $\varepsilon(y,t)$ are vector-functions) which arose from the perturbation of the system

$$(0,2) \qquad dx/dt = F(x, t).$$

The behaviour of solutions of (0,2) is supposed to be known by some means (often system (0,2) is a linear one) and the perturbation $\varepsilon(y,t)$ becomes small as $t \to +\infty$. The problem consists in establishing, under the appropriate assumptions concerning the perturbation, asymptotic relations between the solutions of (0,1) and those of (0,2). More exactly, one wishes to establish that for every solution $x(t)$ of (0,2) there is a solution $y(t)$ of (0,1) which is, what we may call "asymptotically near" to $x(t)$ (as $t \to +\infty$). Of course the term "asymptotically near" has different meanings according to the aims we have in particular considerations.

For instance, we may say that $x(t)$ is asymptotically near to $y(t)$ if their characteristic numbers are equal, i.e. if

$$(0,3) \qquad \limsup_{t\to+\infty} (\ln|y(t)|/t) = \limsup_{t\to+\infty} (\ln|x(t)|/t)$$

(see [2] and [4]), or if the following condition is satisfied

$$(0,4) \qquad y(t) = x(t) + \eta(t) \quad \text{where} \quad |\eta(t)| = o(|x(t)|)$$

(see [9]) or, in the case (0,2) is a linear system, $|\eta(t)| = o(t^\mu e^{\omega t})$ where μ and ω are constants determined by $x(t)$ (see [3]).

The term "asymptotically near" may also express that more exact asymptotic conditions concerning some components of vectors $x(t)$ and $y(t)$ are satisfied (see [7]).

At last, T. Ważewski has introduced a notion of asymptotic coincidence of solutions of two systems. His notion gives another example of the meaning of the term "asymptotically near".

Ważewski's way of comparing asymptotic behaviour of solutions of the two systems differs essentially from the others mentioned above. First of all it has a qualitative character, it is an invariant of topological mapping. On the other hand to any solution $x(t)$ of the one system there may exist at most one solution $y(t)$ of the other which coincide asymptotically with $x(t)$ in the sense of Ważewski. While in the other cases mentioned above there may exist more than one solution of (0,2) satisfying with some solution of (0,1) the condition (0,3) or (0,4). In fact, using (0,4) as a definition of asymptotic nearness, we do not compare single solutions of (0,1) and (0,2) but rather some sets X and Y of solutions of (0,1) and (0,2) respectively. These sets are such that each solution from X is asymptotically near to every solution from Y.

The purpose of the present paper is to present a qualitative way of asymptotically comparing of sets filled up by solutions of the two systems. With this view we will introduce the notion of asymptotic coincidence of sets filled up by integrals. Our notion is a direct generalization of that of Ważewski concerning single integrals.

We formulate the asymptotic coincidence property in terms of filters theory (see acknowledgments at the end of the paper). Thus section 1 deals with filters and their properties. Section 2 and 3 concern the introductory notions such as sets and filters filled up by integrals and the asymptotic boundary of a filter filled up by integrals. In section 4 we define the asymptotic coincidence of filters and sets filled up by integrals. The next sections present the main results of our theory. The last ones concern their applications.

1. By E^n we denote the n-dimensional Euclidean space and by E the Cartesian product $E^n \times R$, where R is a real line. By x, y, z, we denote the points of E^n, by t the real parameter.

DEFINITION 1 ([1], p. 32). We shall call *a filter* on E^n (or E) every family $\mathfrak{F}$ of subsets of E^n (or E) satisfying the following conditions:

(1,1) if $A \in \mathfrak{F}$ and $A \subset B$ then $B \in \mathfrak{F}$,

(1,2) if $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$ then $A \cap B \in \mathfrak{F}$,

(1.3) the empty set 0 does not belong to $\mathfrak{F}$.

EXAMPLE 1. The family of all subsets of E^n containing certain neighbourhood of a fixed point x of E^n is a filter on E^n.

EXAMPLE 2. Let $x(t)$ be a vector-function defined on $\langle 0, +\infty)$. The family $\mathfrak{F}$ of all subsets of E such that if $A \in \mathfrak{F}$ then there exists $t_A > 0$ such that $(x(t), t) \in A$ for $t > t_A$ is a filter on E.

DEFINITION 2 ([1], p. 33). We say that the filter $\mathfrak{F}$ is *stronger* than the filter $\mathfrak{F}'$ (or $\mathfrak{F}'$ is *weaker* than $\mathfrak{F}$) if $\mathfrak{F}' \subset \mathfrak{F}$.

If $\mathfrak{F}'$ is stronger than $\mathfrak{F}$ and $\mathfrak{F}$ is stronger than $\mathfrak{F}'$ than the filters $\mathfrak{F}'$ and $\mathfrak{F}$ are identical.

EXAMPLE 3. Let $\mathfrak{F}'$ be the family of all subsets of E^n containing a fixed point x of E^n. Then $\mathfrak{F}'$ is a filter on E^n and it is stronger than the filter $\mathfrak{F}$ given in Example 1.

DEFINITION 3 ([1], p. 35). We say that the family $\mathfrak{B}$ of subsets of E^n (or E) is *a base of a filter* $\mathfrak{F}$ on E^n (or E) if $\mathfrak{F}$ is composed of all subsets E^n (or E) containing at least one set belonging to $\mathfrak{B}$. The filter $\mathfrak{F}$ is called a filter *generated by the base* $\mathfrak{B}$.

A filter is at the same time a base of itself. Each base $\mathfrak{B}$ generates exactly one filter and one filter may be generated by many different bases. Thus filter $\mathfrak{F}$ is uniquely determined by any of its base.

EXAMPLE 4. The family of neighbourhoods of a fixed point x presents a base of a filter and it generates the filter given by Example 1.

EXAMPLE 5. Let $\mathfrak{B}$ be a denumerable family of sets A_n such that

$$A_n = \{(x, t) \colon x = x(t) \text{ and } t > n\} \qquad (n = 1, 2, \ldots).$$

Then $\mathfrak{B}$ is a base of the filter presented in Example 2.

PROPOSITION 1 ([1], p. 35). *The family $\mathfrak{B}$ of subsets of E^n (or E) is a base of a filter if and only if the following conditions hold*

(1,4) *the product of any two sets of $\mathfrak{B}$ contains another set of $\mathfrak{B}$,*

(1,5) *the empty set 0 does not belong to $\mathfrak{B}$.*

PROPOSITION 2. *A base $\mathfrak{B}$ generates the filter $\mathfrak{F}$ if and only if $\mathfrak{B} \subset \mathfrak{F}$ and for each set $A \in \mathfrak{F}$ there exists a set $B \in \mathfrak{B}$ such that $B \subset A$.*

PROPOSITION 3. *Let $\mathfrak{B}$ be a base of filter and let A contain a set belonging to $\mathfrak{B}$. Then the family $\mathfrak{B}_A$ composed by these sets of $\mathfrak{B}$ which are contained in A is also a base of filter and the filters generated by $\mathfrak{B}$ and $\mathfrak{B}_A$, respectively, are identical.*

PROPOSITION 4. *The filter $\mathfrak{F}$ generated by a base $\mathfrak{B}$ is stronger than the filter $\mathfrak{F}'$ generated by a base $\mathfrak{B}'$ if and only if for every set $B \in \mathfrak{B}$ there exists a set $B' \in \mathfrak{B}'$ such that $B' \subset B$.*

DEFINITION 4 ([1], p. 49). We say that x is *an adherent point* of a filter $\mathfrak{F}$ if each neighbourhood of x meets every set of $\mathfrak{F}$. The set of all adherent points of $\mathfrak{F}$ is called *the adherent set* of $\mathfrak{F}$.

The adherent set of a filter is always closed.

2. Let us consider a system of differential equations

(U) $dx/dt = U(x,t)$, where $x \in E^n$ and $U(x,t) \in E^n$

for $(x,t) \in E$.

We suppose the following conditions concerning the system (U).

HYPOTHESIS $H_1(U)$. 1° The vector-function $U(x,t)$ is continuous on E.

2° There exists a unique solution of (U) passing through a point $M = (x_M, t_M) \in E$ and it may be continued on the whole half-line $\langle t_M, +\infty)$.

Denote by

(2,1) $x = u(t, M)$, where $M = (x_M, t_M) \in E$,

the solution of (U) passing through M, i.e. $u(t_M, M) = x_M$.

By the integral of (U) we will mean the image of (2,1) in E considered in the longest interval in which the solution (2,1) exists. The integral of (U) passing through M we denote by $I_U(M)$.

The part of $I_U(M)$ corresponding $t \geqslant t_M$ we denote by $I_U^+(M)$ and we call it the right hand half-integral of (U) issuing from M. Similarly, by the left hand half-integral of (U) issuing from M we mean that part of $I_U(M)$ which corresponds $t \leqslant t_M$ and we denote it by $I_U^-(M)$.

DEFINITION 5. We say that a subset of E is *filled up by integrals* of (U) (or by left hand half-integrals or by right hand half-integrals of (U)) if for every point $M \in A$ we have

$$I_U(M) \subset A \quad (\text{or } I_U^-(M) \subset A \text{ or } I_U^+(M) \subset A).$$

Now let A be an arbitrary subset of E. We denote by $Z_U(A)$ the zone of emission of A with respect to system (U) and it is the smallest set containing A and filled up by integrals of (U). Similarly, we denote by $Z_U^-(A)$ and by $Z_U^+(A)$ the zone of emission of A to the left and to the right, respectively.

We point out the following relations which are easily seen

(2,2) $$Z_U(A) = \bigcup_{M \in A} I_U(M),$$

(2,3) $$Z_U(\bigcup_\lambda A^\lambda) = \bigcup_\lambda Z_U(A^\lambda),$$

(2,4) $$Z_U(\bigcap_\lambda A^\lambda) \subset \bigcap_\lambda Z_U(A^\lambda).$$

The relations (2,2)-(2,4) are still true if symbols Z_U, I_U are replaced by Z_U^+, I_U^+ or Z_U^-, I_U^-, respectively.

Also the following proposition is easy to verify.

PROPOSITION 5. *A subset A is filled up by integrals (or by left hand half-integrals or by right hand half-integrals) of* (U) *if and only if*

(2,5) $$Z_U(A) = A \quad \left(Z_U^-(A) = A \text{ or } Z_U^+(A) = A\right).$$

DEFINITION 6. We say that *a filter $\mathfrak{F}$ on E is filled up by integrals of* (U) (or by left hand half-integrals or by right hand half-integrals of (U)) if it is generated by a base $\mathfrak{B}$ composed of sets filled up by integrals of (U) (or by left hand half-integrals or by right hand half-integrals of (U)). Such filter will be denoted by $\mathfrak{F}_U$ (or $\mathfrak{F}_U^+$ or $\mathfrak{F}_U^-$).

EXAMPLE 6. Let $\mathfrak{F}$ be an arbitrary filter on E^n. The family of zones of emision with respect to (U) of all sets belonging to $\mathfrak{F}$ satisfies, on the basis of (1,2), (1,3), and (2,4), the conditions (1,4) and (1,5). Hence it is a base of a filter and, owing to Definition 6, of a filter filled up by integrals of (U).

3. Let us consider a system (U) and let us suppose Hypothesis $H_1(U)$. Let A be a subset of E. For arbitrary $\tau > 0$ we put

$$A_\tau = \{(x,t)\colon (x,t) \in A, \text{ and } t > \tau\}.$$

One easily verifies that $A_\tau = A \cap E_\tau$ and

(3,1) if $A \subset A^*$ and $\tau > \tau^*$ then $A_\tau \subset A^*_{\tau^*}$.

Consider now a filter $\mathfrak{F}_U$ filled up by integrals of (U). Let $\mathfrak{B}$ be an arbitrary base of $\mathfrak{F}_U$, and let S be an unbounded set of positive numbers. Put

$$\mathfrak{C}(\mathfrak{B}; S) = \{A_\tau\colon A \in B \text{ and } \tau \in S\}.$$

We will prove that the family $\mathfrak{C}(\mathfrak{B}, S)$ is a filter base. Indeed, by (3,1) we get that $\mathfrak{C}(\mathfrak{B}, S)$ satisfies (1,4) and by Hypothesis $H_1(U)$ each $A_\tau \in \mathfrak{C}(\mathfrak{B}, S)$ is not empty, hence (1,5) also holds, and thus $\mathfrak{C}(\mathfrak{B}, S)$ is a base. It may be also easily seen that any two such bases are equivalent, it means that they generate the same filter.

DEFINITION 7. The filter generated by the base $\mathfrak{C}(\mathfrak{B}, S)$, where $\mathfrak{B}$ is a base of filter $\mathfrak{F}_U$ and S is an unbounded set of positive numbers, we call the *right hand asymptotic boundary* of the filter filled up by integrals of (U) or shortly the asymptotic boundary of $\mathfrak{F}_U$. We denote it by

$$\mathrm{Fr}^+(F_U).$$

Remark 1. The asymptotic boundary of a filter filled up by integrals is closely related to the Ważewski's notion of the asymptotic end of an integral. Consider an integral $I_U(M)$. Let $M_p = (x_p, t_p) \in I_U(M)$, $t_p < t_{p+1}$ $(p = 1, 2, \ldots)$ and $\lim_{p\to\infty} t_p = +\infty$. Further let V_p be a neighbourhood of M_p and suppose

$$Z_U^+(V_p) \subset Z_U^+(V_{p+1}) \quad (p = 1, 2, \ldots), \quad \bigcap_{p=1}^{\infty} Z_U(V_p) = I_U(M).$$

Following Ważewski we denote by $\{\{Z_U^+(V_p)\}\}$ the family of all increasing sequences of sets $\{D_p\}$ which are equivalent to the sequence $\{Z_U^+(V_p)\}$. The last means that every set D_p contains at least one set $Z_U^+(V_r)$ and conversely every $Z_U^+(V_p)$ contains some D_r.

The family $\{\{Z_U^+(V_p)\}\}$ Ważewski called the asymptotic end of $I_U(M)$ ([11], p. 199) and he denote it by $\underset{(U)}{\overset{+}{\mathrm{Extr}}}\big(I_U(M)\big)$. On the other hand let us consider the filter $\mathfrak{F}_U(M)$ generated by the base composed of the zones of emision of all neighbourhoods of M with respect to system (U). It may be easily seen that the sequence $\{Z_U^+(V_p)\}$ (or any other equivalent to this one) is a base of the asymptotic boundary of $\mathfrak{F}_U(M)$. Hence the asymptotic end of an integral as well as the asymptotic boundary of $\mathfrak{F}_U(M)$ are uniquely determined by the same sequence $\{Z_U^+(V_p)\}$, though their logical structures are different.

Now we are going to give some simple facts concerning the asymptotic boundary of a filter filled up by integrals.

Proposition 6. *The asymptotic boundary of a filter filled up by integrals of* (U) *possesses a base* $\mathfrak{B}$ *satisfying the following two conditions*

(3,2) *for arbitrary* $T > 0$ *there exists* $B \in \mathfrak{B}$ *such that* $B \subset E_T$,

(3,3) *for every* $B \in \mathfrak{B}$ *there is a* $T > 0$ *such that* $I_U(M) \cap E_T \subset B$ *for each* $M \in B$.

Proposition 6 is a direct consequence of Definition 7.

Proposition 7. *Suppose a filter* $\mathfrak{G}$ *is generated by a base satisfying* (3,2) *and* (3,3).

Then there is a filter F_U *such that*

$$\mathfrak{G} = \mathrm{Fr}^+(\mathfrak{F}_U).$$

Proof. Denote by $Z_U(\mathfrak{B})$ the family of sets $Z_U(B)$ where $B \in \mathfrak{B}$ and $\mathfrak{B}$ is a base of $\mathfrak{G}$ satisfying (3,2) and (3,3). Of course $Z_U(\mathfrak{B})$ is a base of a filter filled up by integrals of (U). Denote the filter generated by $Z_U(\mathfrak{B})$ by $\mathfrak{F}_U$. By (3,2) and (3,3) there exists an unbounded set of positive numbers S such that $\mathfrak{B} = \mathfrak{C}\big(Z_U(\mathfrak{B}), S\big)$. This finishes the proof of Proposition 7.

Proposition 8. *Suppose filters* $\mathfrak{F}_U$ *and* $\mathfrak{G}_U$ *are filled up by integrals of* (U). *If* $\mathfrak{F}_U$ *is stronger than* $\mathfrak{G}_U$ *then* $\mathrm{Fr}^+(\mathfrak{F}_U)$ *is stronger than* $\mathrm{Fr}^+(\mathfrak{G}_U)$ *and vice versa.*

This proposition follows from Definition 2 and (3,1).

4. Consider now two systems

(U) $$dx/dt = U(x, t),$$

(V) $$dx/dt = V(x, t)$$

and suppose the Hypothesis $\mathrm{H}_1(U)$ and $\mathrm{H}_1(V)$, respectively.

Definition 8. We say that filter $\mathfrak{F}_U$ is *asymptotically incident* into the filter $\mathfrak{G}_V$ if the asymptotic boundary of F_U is stronger than the asymptotic boundary of $\mathfrak{G}_V$ i.e. if

$$\mathrm{Fr}^+(\mathfrak{G}_V) \subset \mathrm{Fr}^+(\mathfrak{F}_U).$$

We say that $\mathfrak{F}_U$ *asymptotically coincides* with $\mathfrak{G}_V$ if $\mathfrak{F}_U$ is asymptotically incident into $\mathfrak{G}_V$ and vice-versa, hence if

$$\mathrm{Fr}^+(\mathfrak{F}_U) = \mathrm{Fr}^+(\mathfrak{G}_V).$$

In the applications we give in Sections 10 and 11 we are interested in asymptotic coincidence of sets filled up by integrals. Now we are going to make this notion precise.

First we need some preliminary notions.

Definition 9. We say that a filter F *is open* if it admits a base composed of open sets. Similarly, a filter $\mathfrak{F}_U$ is open if it admits a base composed of open sets, and filled up by integrals sets.

Consider now the sets P and Q filled up by integrals of (U) and (V), respectively.

Hypothesis $\mathrm{H}_2(P, U)$. Suppose P is a set filled up by integrals of (U) and suppose it is a compact set of integrals; it means that for sufficiently large T the section of P by the hyperplane $t = T$ is a compact set.

Definition 10. We call $\mathfrak{F}_U(P)$ *the filter of neighbourhoods* of a set P filled up by integrals of (U) if it admits a base $\mathfrak{B}(P)$ composed of sets filled up by the integrals of (U) and open and such ones that

(4,1) $$\text{if } B \in \mathfrak{B}(P) \text{ then } P \subset B,$$

(4,2) $$\bigcap_{B \in \mathfrak{B}(P)} B = P.$$

For instance, the filter $\mathfrak{F}_U(M)$ appearing in Remark 1 is a filter of neighbourhoods of a set P composed by a single integral $I_U(M)$.

Remark 2. By (4,1) and (4,2) we easily obtain that P is the adherent set of $\mathfrak{F}_U(P)$. On the other side if we suppose P satisfies $\mathrm{H}_2(P, U)$ then $\mathfrak{F}_U(P)$ is unique and it is the weakest filter for which P is the adherent set. This does not hold if we allow P to be a non-compact set of integrals, but only closed.

Now consider two sets P and Q and suppose they satisfy $\mathrm{H}_2(P, U)$ and $\mathrm{H}_2(Q, V)$, respectively.

Definition 11. We say that P *coincides asymptotically* with Q if the filter $\mathfrak{F}_U(P)$ coincides asymptotically with $\mathfrak{F}_V(Q)$; in other words, if the filter of neighbourhoods of P coincides asymptotically with the filter of neighbourhoods of Q.

Remark 3. If P and Q reduce to the single integrals $I_U(M)$ and $I_V(N)$, respectively, then the asymptotic coincidence of P and Q becomes the asymptotic coincidence of $I_U(M)$ and $I_V(N)$ in the strong sense of Ważewski (s. [11]). Indeed, owing to Ważewski the integrals $I_U(M)$ and $I_V(N)$ are said to be asymptotically coincident if

$$\underset{(U)}{\overset{+}{\mathrm{Extr}}}\big(I_U(M)\big) = \underset{(V)}{\overset{+}{\mathrm{Extr}}}\big(I_V(N)\big).$$

On the basis of Remark 1 the last equality is equivalent to the following one

$$\mathrm{Fr}^+\big(\mathfrak{F}_U(M)\big) = \mathrm{Fr}^+\big(\mathfrak{F}_V(N)\big),$$

where by $\mathfrak{F}_U(M)$ and $\mathfrak{F}_V(N)$ we denote the filter of neighbourhoods of $I_U(M)$ and $I_V(N)$, respectively.

Thus our notion of asymptotic coincidence of sets filled up by integrals of two systems is a direct generalization of Ważewski's concept.

Remark 4. Owing to Remark 2 the property of the asymptotic coincidence of P and Q is not only the property of sets but also of the systems in neighbourhoods of P and Q.

5. In this and the next sections we are going to give some results concerning the asymptotic coincidence of filters as well as of sets filled up by integrals. We begin with the following theorem:

Theorem 1. *Besides systems* (U) *and* (V) *consider the third system*

$$\text{(W)} \qquad dx/dt = W(x, t)$$

and suppose the Hypothesis $\mathrm{H}_1(U)$, $\mathrm{H}_1(V)$ *and* $\mathrm{H}_1(W)$, *respectively. Let* $\mathfrak{F}_U$, $\mathfrak{G}_V$ *and* $\mathfrak{H}_W$ *be filters filled up by integrals of* (U), (V) *and* (W) *respectively.*

Suppose $\mathfrak{F}_U$ *is asymptotically incident into* $\mathfrak{G}_V$ *and* $\mathfrak{G}_V$ *is asymptotically incident into* $\mathfrak{H}_W$.

Then $\mathfrak{F}_U$ *is asymptotically incident into* $\mathfrak{H}_W$.

Proof. By the assumptions and Definition 8 we get

$$\mathrm{Fr}^+(\mathfrak{F}_U) \supset \mathrm{Fr}^+(\mathfrak{G}_V) \quad \text{and} \quad \mathrm{Fr}^+(\mathfrak{G}_V) \supset \mathrm{Fr}^+(\mathfrak{H}_W).$$

Hence $\mathrm{Fr}^+(\mathfrak{F}_U) \supset \mathrm{Fr}^+(\mathfrak{H}_W)$ which proves Theorem 1.

It follows from Theorem 1 that the following corollary holds.

Corollary 1. *Under the same assumptions as in Theorem 1 if* $\mathfrak{F}_U$ *coincides asymptotically with* $\mathfrak{G}_V$ *and* $\mathfrak{G}_V$ *coincides asymptotically with* $\mathfrak{H}_W$ *then* $\mathfrak{F}_U$ *coincides asymptotically with* $\mathfrak{H}_W$.

Hence the relation of asymptotic coincidence has the transitive property. Because evidently it is reflexive and symmetric, it is an equivalence relation.

Theorem 2. *The filter* $\mathfrak{F}_U$ *filled up by integrals of* (U) *may coincide with at most one filter* $\mathfrak{G}_V$ *filled up by integrals of* (V).

Proof. Theorem 2 follows directly Proposition 8, Theorem 1 and Corollary 1.

Theorem 3. *Suppose* $\mathfrak{F}_U$ *is open and coincides asymptotically with* $\mathfrak{G}_V$. *Further, suppose* $\mathfrak{F}_U$ *admits a base composed by connected sets.*

Then $\mathfrak{G}_V$ *is open and it admits a base composed by connected sets.*

Proof. Let A be a set filled up by integrals of (U) and suppose it is open and connected. Then for any $\tau > 0$ A_τ is also connected. (Evidently A_τ is open). Indeed, A_τ is filled up by right-hand half integrals of (U) and therefore if we could decompose A_τ into a sum $C \cup D$ such that $(\bar{C} \cap D) \cup (C \cap \bar{D}) = 0$ then C and D would be filled up by right-hand half integrals of (U), also. Then we would have

$$\big(Z_U(\bar{C}) \cap Z_U(D)\big) \cup \big(Z_U(C) \cap Z_U(\bar{D})\big) = 0 \quad \text{and} \quad A = Z_U(C) \cup Z_U(D).$$

The last two relations contradict the assumption A is a connected set. Thus we proved A_τ is connected for every $\tau > 0$.

It follows from the above that the base $\mathfrak{C}(\mathfrak{B}, S)$ where $\mathfrak{B}$ is a base of $\mathfrak{F}_U$ composed of open and connected sets and S is an unbounded set of positive numbers, is also composed of open and connected sets. By Definition 8 $\mathfrak{C}(\mathfrak{B}, S)$ is a base of $\mathrm{Fr}^+(\mathfrak{F}_U)$ and by the assumption $\mathrm{Fr}^+(\mathfrak{F}_U) = \mathrm{Fr}^+(\mathfrak{G}_V)$ it is a base of $\mathrm{Fr}^+(\mathfrak{G}_V)$, also. Therefore $Z_V\big(\mathfrak{C}(\mathfrak{B}, S)\big)$ is a base of $\mathfrak{G}_V$. This, together with an observation that the zone of emission of open and connected sets are always open and connected proves Theorem 3 completely.

The above theorems remain valid if we replace the asymptotic coincidence of filters by that of sets. According to the last notion we mention here only the following noteworthy consequence of Theorem 3.

Corollary 2. *Let the sets* P *and* Q *satisfy Hypothesis* $\mathrm{H}_2(P, U)$ *and* $\mathrm{H}_2(Q; V)$, *respectively. Suppose that* P *coincides asymptotically with* Q *and suppose* P *is connected. Then* Q *is also connected.*

6. Now we are going to discuss the following problem.

Suppose $\mathfrak{F}_U$ is a filter of neighbourhoods of a certain set P satisfying $\mathrm{H}_2(P, U)$ and suppose $\mathfrak{F}_U$ coincides asymptotically with $\mathfrak{G}_V$. Is $\mathfrak{G}_V$ a filter of neighbourhoods of some set Q filled up by integrals of (V)? The positive answer for this problem gives the following theorem.

THEOREM 4. *Let P be a set satisfying $H_2(P, U)$. Suppose the filter of neighbourhoods of P coincides asymptotically with some filter $\mathfrak{G}_V$ filled up by integrals of* (V).

Then there is a set Q satisfying $H_2(Q, V)$ such that one $\mathfrak{G}_V$ is a filter of neighbourhoods of Q, hence

$$\mathfrak{G}_V = \mathfrak{G}_V(Q)\,.$$

Proof. Notice that every filter of neighbourhoods admits a denumerable base $\mathfrak{B} = \{B^{(p)}\}$ satisfying the following condition

(6,1) $\qquad B^{(p)}$ is open and $B^{(p)} \supset B^{(p+1)} \qquad (p = 1, 2, \ldots)\,.$

This condition is also sufficient for the filter generated by $\mathfrak{B}$ to be a filter of neighbourhoods. Now let $\mathfrak{B}$ be a base of $\mathfrak{F}_U(P)$ satisfying (6,1). Without loss of a generality we may suppose that $B^{(p)} \in B$ are filled up by integrals of (U). Then the sequence $C_p = B_p^{(p)} = B^{(p)} \cap E_p$ is a base of $\mathrm{Fr}^+(\mathfrak{F}_U(P))$ and owing to the assumption $\mathrm{Fr}^+(\mathfrak{F}_U(P)) = \mathrm{Fr}^+(\mathfrak{G}_V)$ the sequence C_p is also a base of $\mathrm{Fr}^+(\mathfrak{G}_V)$. By (6,1) we get

(6,2) $$\bar{C}_{p+1} \subset C_p\,.$$

Using Propositions 4 and 7 one can show that the sequence $Z_V^+(C_p)$ is also a base of $\mathrm{Fr}^+(\mathfrak{G}_V)$ and $Z_V(C_p)$ is a base of $\mathfrak{G}_V$. By (6,2) we get that

$$Z_V(\bar{C}_{p+1}) = \bar{Z}_V(C_{p+1}) \subset Z_V(C_p) \qquad (p = 1, 2, \ldots)\,.$$

The last relation proves that $\mathfrak{G}_V$ is a filter of neighbourhoods of a set Q, where

$$Q = \bigcap_{p=1}^{\infty} Z_V(C_p)\,.$$

Evidently Q is filled up by integrals of (V). Q is compact because for sufficiently large p and T the section of $B^{(p)}$ by the hyperplane $t = T$ is bounded. Thus we find Theorem 4 proved.

7. In the present section we prove the invariant property of asymptotic coincidence of filters (or sets) with respect to continuous transformations. First we make precise the kind of transformations with which we will deal.

Consider two systems

(U) $$dx/dt = U(x, t)$$

and

(U$_*$) $$dy/ds = U_*(y, s)$$

and suppose they satisfy Hypothesis $H_1(U)$ and $H_1(U_*)$ respectively.

Further, let us consider the transformation

(T) $$t = h(s)\,, \qquad x = \Phi(y, s)\,.$$

DEFINITION 12 (see [6], p. 39). We say that T carries system (U) into system (U$_*$) if

1° T is an homeomorfism of E onto E $\big(T(E) = E\big)$.

2° Between integrals of (U) and (U$_*$) there is a one-to-one correspondence such that to any solution $x = u(t)$ of (U) defined on $(\alpha, +\infty)$ there corresponds a solution $y = u_*(s)$ of (U$_*$) defined on $(\beta, +\infty)$ such that

(7,1) $$u\big(h(s)\big) = \Phi\big(u_*(s), s\big) \quad \text{for} \quad \beta < s < +\infty\,,$$

(7,2) $$h\big((\beta, +\infty)\big) = (\alpha, +\infty)\,.$$

THEOREM 5. *Consider two systems*

(U) $$dx/dt = U(x, t)\,,$$

(V) $$dx/dt = V(x, t)$$

and suppose Hypothesis $H_1(U)$ and $H_1(V)$ respectively.

If the transformation (T) *carries system* (U) *into*

(U$_*$) $$dy/ds = U_*(y, s)$$

and system (V) *into*

(V$_*$) $$dy/ds = V_*(y, s)$$

and if the filter $\mathfrak{F}_U$ filled up by integrals of (U) *coincides asymptotically with the filter $\mathfrak{G}_V$ filled up by integrals of* (V) *then*

(i) *the filter $T(\mathfrak{F}_U)$ is filled up by integrals of* (U$_*$) *and the filter $T(\mathfrak{G}_V)$— by integrals of* (V$_*$),

(ii) *$T(\mathfrak{F}_U)$ coincides asymptotically with $T(\mathfrak{G}_V)$.*

Proof. The part (i) follows the definition of (T) and part (ii) is a consequence of the fact: if $A \subseteq B$ than $T(A) \subset T(B)$.

Remark 5. Under sufficiently general assumptions one can transform any system (U) satisfying Hypothesis $H_1(U)$ into the trivial system

(B) $$dx/dt = 0\,, \qquad \text{where } 0 \text{ is a zero-vector}\,.$$

Theorem 5 allows, in such cases, to reduce the problem of comparing two systems to that one in which one of the systems is of the form (B). This often may give us a simplification of the problem.

8. Before we formulate the main result of this section we need some notions and facts.

DEFINITION 13 ([1], p. 9). We say that the family $\mathfrak{D}$ of subsets of $\mathcal{E}$ determines on $\mathcal{E}$ a *topological structure* if $\mathfrak{D}$ satisfies the following conditions

(8,1) the sum of an arbitrary number of sets of $\mathfrak{D}$ belongs to $\mathfrak{D}$,

(8,2) the product of a finite number of sets of $\mathfrak{D}$ belongs to $\mathfrak{D}$.

The set $\mathcal{E}$ with a topological structure determined on it, we call the topological space and the sets belonging to $\mathfrak{D}$, we call the open subsets of $\mathcal{E}$.

DEFINITION 14. We say that $\mathcal{A} \subset \mathcal{E}$ is *a neighbourhood* of x if $x \in \mathcal{A}$ and $\mathcal{A}$ contains a set belonging to $\mathfrak{D}$ (an open set).

PROPOSITION 9 ([1], p. 11). *A set $\mathcal{A}$ is a neighbourhood of every point belonging to $\mathcal{A}$ if and only if it is open.*

It is easy to be verified that the family $\mathfrak{B}(x)$ of all neighbourhoods of x (x is a point of topological space) fulfills the following conditions:

(8,3) In each subset of $\mathcal{E}$ which contains a set belonging to $\mathfrak{B}(x)$ also belongs to $\mathfrak{B}(x)$,

(8,4) The product of finite number of sets of $\mathfrak{B}(x)$ belongs to $\mathfrak{B}(x)$.

(8,5) The point x belongs to every set of $\mathfrak{B}(x)$.

(8,6) If $\mathcal{A} \in \mathfrak{B}(x)$ then there is $\mathcal{B} \in \mathfrak{B}(x)$, $\mathcal{B} \subset \mathcal{A}$, and for every $y \in \mathcal{B}$, $\mathcal{A} \in \mathfrak{B}(y)$.

These properties of $\mathfrak{B}(x)$ characterize completely the topology on $\mathcal{E}$. More exactly, the following proposition holds.

PROPOSITION 10 ([1], p. 12). *If to any point $x \in \mathcal{E}$ there corresponds a family $\mathfrak{B}(x)$ of subsets of $\mathfrak{E}$ satisfying* (8,3)-(8,6) *then there is a topological structure on $\mathcal{E}$ for which $\mathfrak{B}(x)$ is a family of neighbourhoods of x.*

We define now space of open filters on E. Let $\mathcal{E}$ be some set of open filters on E and let $\mathfrak{F} \in \mathcal{E}$. In order to define a topological structure on $\mathcal{E}$ it suffices, on the basis of Proposition 10, to determine a family $\mathfrak{B}(\mathfrak{F})$ of subsets of $\mathcal{E}$, which would satisfy (8,3)-(8,6).

The family $\mathfrak{B}(\mathfrak{F})$, where $\mathfrak{F} \in \mathcal{E}$. The set $\mathcal{A} \subset \mathcal{E}$ will belong to $\mathfrak{B}(\mathfrak{F})$ if

(8,7) there exists an open set $A \subset E$ such that $A \in \mathfrak{F}$ and if $A \in \mathfrak{G} \in \mathcal{E}$ then $\mathfrak{G} \in \mathcal{A}$.

LEMMA 1. *The family $\mathfrak{B}(\mathfrak{F})$ defined above satisfy conditions* (8,3)-(8,6).

Proof. If $\mathcal{A} \subset \mathcal{E}$ satisfies (8,7) and if $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{B}$ satisfies (8,7), also. Hence $\mathfrak{B}(\mathfrak{F})$ fulfills (8,3). Now let $\mathcal{A}_i \in \mathfrak{B}(\mathfrak{F})$ $(i = 1, \ldots, k)$. By the definition of $\mathfrak{B}(\mathfrak{F})$ we can find open subsets $A^i \subset E$ $(i = 1, \ldots, k)$ such that $A^i \in \mathfrak{F}$ and A^i satisfies (8,7) with respect to $\mathcal{A}_i$. Put $A = \bigcap_{i=1}^{k} A^i$. It is easily seen that A satisfies (8,7) with respect to $\mathcal{A} = \bigcap_{i=1}^{k} \mathcal{A}_i$. Therefore (8,4) holds for $\mathfrak{B}(\mathfrak{F})$.

The condition (8,5) is a direct consequence of (8,7).

We are now going to prove (8,6). Let $\mathcal{A} \in \mathfrak{B}(\mathfrak{F})$ and let A be the set assured by the definition of $\mathfrak{B}(\mathfrak{F})$ and corresponding to $\mathcal{A}$. Denote by $\mathcal{B}$ the sets of all filters of $\mathcal{E}$ containing A. Obviously $\mathcal{B} \in \mathfrak{B}(\mathfrak{F})$ and $\mathcal{B} \subset \mathcal{A}$. Further, for any $\mathfrak{G} \in \mathcal{B}$, $\mathcal{B} \in \mathfrak{B}(\mathfrak{G})$ and therefore $\mathcal{A} \in \mathfrak{B}(\mathfrak{G})$ for any $\mathfrak{G} \in \mathcal{B}$. The last proves (8,6) for $\mathfrak{B}(\mathfrak{F})$ and at the same time finishes the proof of Lemma 1.

We may propose now the following definition.

DEFINITION 15 (see also [13]). A set $\mathcal{E}$ of open filters on E with the topological structure given by the family of neighbourhoods $\mathfrak{B}(\mathfrak{F})$ corresponding to any $\mathfrak{F} \in \mathcal{E}$ we will call the *filter-space*.

DEFINITION 16. We say that *the mapping $x' = f(x)$ of $\mathcal{E}$ onto $\mathcal{E}'$ is continuous* for $x_0 \in \mathcal{E}$ if for arbitrary neighbourhood $\mathcal{V}'$ of $f(x_0)$ we may find a neighbourhood $\mathcal{V}$ of x_0 such that for every $x \in \mathcal{V}$, $f(x) \in \mathcal{V}'$ or that $f(\mathcal{V}) \subset \mathcal{V}'$. If $x' = f(x)$ is continuous for every $x \in \mathcal{E}$ then we say briefly that it is continuous.

PROPOSITION 11 ([1], p. 29). *The mapping $x' = f(x)$ of $\mathcal{E}$ onto $\mathcal{E}'$ is continuous if and only if for every open subset $\mathcal{A}'$ of $\mathcal{E}'$ there is an open subset $\mathcal{A}$ of $\mathcal{E}$ such one that $\mathcal{A}' = f(\mathcal{A})$.*

We are now ready to formulate the main result of the present section.

THEOREM 6. *Let us assume that* (U) *and* (V) *satisfy* $H_1(U)$ *and* $H_1(V)$ *respectively.*

Denote by $\mathcal{E}$ some space of filters on E which are open and filled up by integrals of (U).

Suppose for every $\mathfrak{F}_U \in \mathcal{E}$ there exists a filter $\mathfrak{G}_V$ filled up by integrals of (V) *which coincides asymptotically with $\mathfrak{F}_U$. Denote such filter by $e(\mathfrak{F}_U)$, hence*

$$(8,8) \qquad \mathfrak{G}_V = e(\mathfrak{F}_U),$$

and denote by $\mathcal{E}'$ the image of $\mathcal{E}$ by (8,1), *thus $E' = e(E)$.*

Then the relation (8,8) *represents a homeomorphism of $\mathcal{E}$ onto $\mathcal{E}'$.*

Proof. We point out that $\mathcal{E}'$ is composed of open filters (see Theorem 3) and therefore we may consider $\mathcal{E}'$ as a filter-space.

By Theorem 2 we get that the mapping (8,8) is one-to-one. In order to prove that (8,8) is continuous we will use Proposition 11. Therefore let $\mathcal{A}'$ be an open set of $\mathcal{E}'$ and let $\mathcal{A}' = e(\mathcal{A})$. We ought to prove that $\mathcal{A}$ is an open subset of $\mathcal{E}$. Let $\mathfrak{F}_U^0$ be an arbitrary element of $\mathcal{A}$ and let $\mathfrak{G}_V^0 = e(\mathfrak{F}_U^0)$. Evidently $\mathfrak{G}_V^0 \in \mathcal{A}'$ and since $\mathcal{A}'$ is an open set of $\mathcal{E}'$ we find that $\mathcal{A}'$ is a neighbourhood of $\mathfrak{G}_V^0$. Thus there is a set $B \subset E$ which is open and filled up by integrals of (V) and such one that if $B \in \mathfrak{G}_V \in \mathcal{E}'$ then $\mathfrak{G}_V \in \mathcal{A}'$ (see (8,7)). By the assumption $\mathrm{Fr}^+(\mathfrak{G}_V^0) = \mathrm{Fr}^+(\mathfrak{F}_U^0)$ there is a constant T and a set $A \in \mathfrak{F}_U^0$ which is open and filled up by integrals of (U), such that

$$(8,9) \qquad A_T \subset B_T.$$

Denote by $\mathcal{B} = \{\mathfrak{F}_U: \mathfrak{F}_U \in \mathcal{E}$ and $A \in \mathfrak{F}_U\}$. Obviously $\mathcal{B}$ is a neighbourhood of $\mathfrak{F}_U^0$. For an arbitrary filter $\mathfrak{F}_U \in \mathcal{B}$ there exists a set $B^* \in \mathfrak{G}_V$, where $\mathfrak{G}_V = e(\mathfrak{F}_U)$, and a constant $T^* > T$ such that $B^*_{T^*} \subset A_T$. This and (8,9) shows that $B^*_{T^*} \subset B_T$ and in consequence $B^* \subset B$. The last means that $B \in \mathfrak{G}_V$ for any $\mathfrak{F}_U \in \mathcal{B}$. Therefore $e(\mathcal{B}) \subset \mathcal{A}'$ and $\mathcal{B} \subset \mathcal{A}$. Hence (see (8,3)) $\mathcal{A}$ is a neighbourhood of every element of itself, therefore $\mathcal{A}$ is open what was to be proved.

To illustrate Theorem 6 let us consider the filter space $\mathcal{E}$ composed by filters of neighbourhoods of single integrals of (U). Hence if $\mathfrak{F}_U \in \mathcal{E}$ then there is a point M such that $\mathfrak{F}_U = \mathfrak{F}_U(M)$ or, in other words, $\mathfrak{F}_U$ is a filter of neighbourhoods of $I_U(M)$. One can easily see that the convergence in this filter-space means the almost uniform convergence of corresponding integrals. Suppose now that to every integral $I_U(M)$ there is an integral $I_V(N)$ which coincides asymptotically with $I_U(M)$. By Theorem 6 we conclude that if $I_U(M_p)$ tends almost uniformly to $I_U(M_0)$ as $p \to \infty$, $I_U(M_p)$ coincides asymptotically with $I_V(N_p)$ $(p = 1, 2, \ldots)$ and $I_U(M_0)$—with $I_V(N_0)$ then $I_V(N_p)$ tends almost uniformly to $I_V(N_0)$ as $p \to \infty$.

This consequence of Theorem 6 have been proved by T. Ważewski (see [11], p. 200).

9. In this section we prove the following theorem.

THEOREM 7. *Consider two system* (U) *and* (V) *and suppose they satisfy* $\mathrm{H}_1(U)$ *and* $\mathrm{H}_1(V)$, *respectively.*

Let $\mathfrak{F}_U$ *be a filter filled up by integrals of* (U).

The sufficient and necessary condition for $\mathfrak{F}_U$ *is asymptotically coincident with some filter* $\mathfrak{G}_V$ *filled up by integrals of* (V), *is that there exist two bases* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *of* $\mathrm{Fr}^+(\mathfrak{F}_U)$ *satisfying the following conditions*

(i) *If* $B \in \mathfrak{B}_1$ *then* B *is filled up by right-hand half integrals of* (V).

(ii) *If* $B \in \mathfrak{B}_2$ *then* B *may be obtained as a product of a set filled up by left-hand half integrals of* (V) *and the half-space* E_T $(E_T = \{(x, t): T < t, x \in E^n\})$.

Proof. On the basis of Proposition 8 it suffices to show that there is a base $\mathfrak{B}$ of $\mathrm{Fr}^+(\mathfrak{F}_U)$ satisfying (3,2) and (3,3) with respect to system (V).

As a base $\mathfrak{B}$ let us take the family of right-hand zone of emission of sets belonging to $\mathfrak{B}_2$. Owing to (ii) $\mathfrak{B}$ satisfies (3,2) and (3,3). We are now going to prove that $\mathfrak{B}$ is a base of $\mathrm{Fr}^+(\mathfrak{F}_U)$. Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ generate the same filter, thus to any $B^1 \in \mathfrak{B}_1$ there is $B^2 \in \mathfrak{B}_2$ such that $B^2 \subset B^1$. But because of (i), $Z_V^+(B^2) \subset Z_V^+(B^1) = B^1$. Thus we have shown that to any $B^1 \in \mathfrak{B}_1$ there is $B \in \mathfrak{B}$ such that $B \subset B^1$ $\big(B = Z_V^+(B^2)\big)$. On the other hand to any $B^2 \in \mathfrak{B}_2$ there is $B^1 \in \mathfrak{B}_1$ such that $B^1 \subset B^2$. The last and (i) implies that $B^1 = Z_V^+(B^1) \subset Z_V^+(B^2) = B$. Thus to any $B \in \mathfrak{B}$ there is $B^1 \in \mathfrak{B}_1$ such that $B^1 \subset B$. Therefore we have proved that $\mathfrak{B}$ is equivalent to $\mathfrak{B}_1$, hence $\mathfrak{B}$ is a base of $\mathrm{Fr}^+(\mathfrak{F}_U)$ which was to be proved.

In this way we proved the sufficient condition, the necessity is obvious. Thus Theorem 7 is completely proved.

10. In this section we present the first application.

THEOREM 8. *Consider system*

$$dx/dt = U(x, t) \tag{W}$$

and suppose that

$$|W(x, t)| \leqslant g(t)|x| \quad \textit{for } t > 0 \textit{ and every } x, \tag{10,1}$$

where

$$\int^{\infty} g(s)\,ds < +\infty. \tag{10,2}$$

Under these assumptions every integral $I_Z(M)$ *of the trivial system*

$$dx/dt = 0 \tag{Z}$$

coincides asymptotically with some set Q (¹) *filled up by integrals of* (W).

Proof. Each integral $I_Z(M)$ is a straight line

$$x = x_M, \quad \text{where} \quad (x_M, t_M) = M. \tag{10,3}$$

The filter of neighbourhoods of $I_Z(M)$ is composed by subsets of E containing at least one cylinder surrounding (10,3) that is the set of the form $|x - x_M| < \varepsilon$ and t arbitrary, ε is a positive number. Similarly, the family $\mathfrak{B}$

$$\mathfrak{B} = \{B_{\varepsilon\tau}:\ 0 < \varepsilon < 1,\ \tau > 0\} \quad \text{where} \quad B_{\varepsilon\tau} = \{(x, t):\ |x - x_M| < \varepsilon,\ t > \tau\}$$

is a base of $\mathrm{Fr}^+\big(\mathrm{Fr}_Z(\mathfrak{F}_Z(M))\big)$, where by $F_Z(M)$ we denote the filter of neighbourhoods of integral $I_Z(M)$.

In order to prove Theorem 8 we apply Theorem 7. Thus we define below two bases of $\mathrm{Fr}^+\big(\mathfrak{F}_Z(M)\big)$ which satisfy (i) and (ii) of Theorem 7, respectively.

First, notice that there is function $h(t)$ such that

$$|W(x, t)| < h(t) \quad \text{for} \quad |x - x_M| < 1 \quad \text{and} \quad 0 \leqslant t < +\infty, \tag{10,4}$$

and

$$\int^{\infty} h(s)\,ds < +\infty. \tag{10,5}$$

(¹) Under some additional assumptions the set Q reduces to a single integral (see [6], p. 51). However, if we suppose (10,1) and (10,2) only then Q may contain more than one integral (see example below).

Suppose $\int\limits_{\tau_0}^{+\infty} h(s)\,ds < 1/2$. Let us put

$$\mathfrak{B}_1 = \{C_{\varepsilon\tau}\colon\ 0 < \varepsilon < 1/2,\ \tau > \tau_0\}$$

where $C_{\varepsilon\tau} = \left\{(x, t)\colon |x - x_M| < \varepsilon - \int\limits_t^{\infty} h(s)\,ds,\ t > \tau\right\}$, and

$$\mathfrak{B}_2 = \{D_{\varepsilon\tau}\colon\ 0 < \varepsilon < 1/2,\ \tau > \tau_0\}$$

where $D_{\varepsilon\tau} = \left\{(x, t)\colon |x - x_M| < \varepsilon + \int\limits_t^{\infty} h(s)\,ds,\ t > \tau\right\}$.

Since $\int\limits_t^{\infty} h(s)\,ds \to 0$ as $t \to +\infty$ thus each of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is a base of $\mathrm{Fr}^+\big(\mathfrak{F}_Z(M)\big)$. We are going to prove that $\mathfrak{B}_1$ satisfies (i) and $\mathfrak{B}_2$ satisfies (ii) with respect to system (W). Let $x = w(t)$ be arbitrary solution of (W). Suppose $N = \big(w(t_0), t_0\big) \in C_{\varepsilon\tau}$ for some $0 < \varepsilon < 1$ and $\tau > \tau_0$. By the inequality $D_+|w(t) - x_M| \leqslant \big|W\big(w(t), t\big)\big|$ and by (10,4) we get

$$-h(t) < D_+|w(t) - x_M| < +h(t) \quad \text{if} \quad \big(w(t), t\big) \in B_{10}. \tag{10,6}$$

By (10,6) we easily obtain that

$$\begin{aligned} |w(t) - x_M| &\leqslant |w(t_0) - x_M| + \int\limits_{t_0}^{t} h(s)\,ds \\ &= |w(t_0) - x_M| + \int\limits_{t_0}^{\infty} h(s)\,ds - \int\limits_{t}^{\infty} h(s)\,ds. \end{aligned} \tag{10,7}$$

Since $\big(w(t_0), t_0\big) \in C_{\varepsilon\tau}$ thus $|w(t_0) - x_M| < \varepsilon - \int\limits_{t_0}^{\infty} h(s)\,ds$ and therefore by (10,7) we get that

$$|w(t) - x_M| < \varepsilon - \int\limits_t^{\infty} h(s)\,ds \quad \text{for} \quad t > t_0.$$

The last means that $I_W^+(N) \subset C_{\varepsilon\tau}$ if $N \in C_{\varepsilon\tau}$, that is $\mathfrak{B}_1$ verifies condition (i) of Theorem 7.

Similarly if $N \in D_{\varepsilon\tau}$ for some $0 < \varepsilon < 1/2$ and $\tau > \tau_0$, then by (10,6) we get the inequality

$$|w(t) - x_M| \leqslant |w(t_0) - x_M| + \int\limits_t^{t_0} h(s)\,ds \tag{10,8}$$

for $t \leqslant t_0$ if $|w(t) - x_M| < 1$. It follows from (10,8) that

$$|w(t) - x_M| \leqslant |w(t_0) - x_M| + \int\limits_t^{\infty} h(s)\,ds - \int\limits_{t_0}^{\infty} h(s)\,ds. \tag{10,9}$$

Since $N \in D_{\varepsilon\tau}$ thus $|w(t_0) - x_M| < \varepsilon + \int\limits_{t_0}^{\infty} h(s)\,ds$ and therefore by (10,9) we get the inequality

$$|w(t) - x_M| < \int\limits_t^{\infty} h(s)\,ds + \varepsilon. \tag{10,10}$$

The last inequality is valid in an interval (t, t_0) where $t \geqslant \tau$. But it is easy to see that t is equal τ. Hence the left-hand half integral $I_W^-(N)$ remains in D for $\tau < t \leqslant t_0$ and any $N \in D_{\varepsilon\tau}$. The last proves (ii) for B_2.

By Theorem 7 we get that there is a filter $\mathfrak{G}_W$ filled up by integrals of (W) which coincides asymptotically with $\mathfrak{F}_Z(M)$ and by Theorem 4 we deduce that $\mathfrak{G}_W$ is the filter of neighbourhoods of some set Q filled up by integrals of (W). The last finishes the proof of Theorem 8.

Remark 6. Notice that if the solution $x = w(t)$ of (W) belongs to Q then $\lim\limits_{t\to\infty} w(t) = x_M$. On the other hand, it is easy to prove that if $\lim\limits_{t\to\infty} w(t) = x_M$ then the solution $x = w(t)$ belongs to Q. Hence Q is composed of all integrals of (W) having x_M as a limit at infinity. Also by (10,1) and (10,2) one may conclude that every integral of (W) has a limit as $t \to +\infty$. Therefore the set of all integrals of (W) we may divide into a sum of closed sets $Q(M)$, $M \in E^n$, such that $Q(M)$ coincides asymptotically with $I_Z(M)$. Theorem 6 shows some continuous dependence of $Q(M)$ with respect to M. More exactly, if $M_p \to M_0$ as $p \to +\infty$ $(p = 1, 2, \ldots)$ then every open set V filled up by integrals of (W) and containing $Q(M_0)$ contains also $Q(M_p)$ for sufficiently large p.

By Theorem 5 we get that every set $Q(M)$ is compact.

Example 7. The following example shows that the set Q in Theorem 8 may contain more than one integral. In the following x is a real variable. Consider the equation

$$x' = \begin{cases} -\dfrac{1}{1+t^2}\,x & \text{if} \quad x \geqslant \exp(\pi - \operatorname{arc\,tan} t), \\[2mm] \dfrac{1}{1+t^2} \cdot \dfrac{2\big(x - \exp(\pi - \operatorname{arc\,tan} t)\big)}{\exp(\operatorname{arc\,tan} t) - \exp(\pi - \operatorname{arc\,tan} t)}\,x, & \\ \qquad \text{if} \quad \exp(\operatorname{arc\,tan} t) < x < \exp(\pi - \operatorname{arc\,tan} t), & \\[2mm] \dfrac{1}{1+t^2}\,x & \text{if} \quad x \leqslant \exp(\operatorname{arc\,tan} t). \end{cases} \tag{10,11}$$

One can easily verify that

$$|x'| \leqslant \frac{1}{1+t^2}|x|$$

and therefore the assumptions (10,1) and (10,2) hold. However every solution of (10,11) issuing from $(x_0, 0)$, where $1 < x_0 < \exp\pi$, tends to $\exp\pi/2$ as $t \to +\infty$.

At last we point out the following result as a simple consequence of Theorems 8 and 5.

THEOREM 9. *Consider two systems*

$$dx/dt = A(t)x \tag{L}$$

and

$$dx/dt = A(t)x + \varepsilon(x, t)\,. \tag{P}$$

Let $X(t)$ *be the matrix-solution of* (L) *that is it satisfies the conditions*

$$dX(t)/dt = A(t)X(t) \quad \text{and} \quad X(0) = I\,,$$

where I *denotes the unit-matrix.*

If

$$|X^{-1}(t)\,\varepsilon(X(t)y, t)| \leqslant g(t)|y|\,, \tag{10,12}$$

where

$$\int^{\infty} g(s)\,ds < +\infty \tag{10,13}$$

then every integral of (L) *coincides asymptotically with some set filled up by integrals of* (P).

Proof. One can see immediately that the transformation

$$x = X(t)y\,, \quad t = t \tag{T}$$

carries system (L) into system (Z) and system (P) into the following one,

$$dy/dt = W(y, t) = X^{-1}(t)\,\varepsilon(X(t)y, t)\,.$$

Hence by (10,12) we find ourselves in the case considered by Theorem 8 and owing to this theorem and Theorem 5 we get Theorem 9.

11. In this section we deal with the special system of two differential equations

$$dx/dt = R(x) \tag{R}$$

where $x = (x_1, x_2)$ and $R(x) = (R_1(x_1, x_2), R_2(x_1, x_2))$. We suppose the following hypothesis concerning (R).

HYPOTHESIS H_3. 1. $R(x)$ is of class C^1 for $x \in E^2$ and $R(x) \neq 0$ for $|x| \neq 0$, where $|x| = \sqrt{x_1^2 + x_2^2}$.

2. Each solution of (R) is periodic.

On other words (R) besides one singular point admits only cycles surrounding (0,0).

Some condition concerning the existence of systems (R) satisfying H_3 is due to Z. Opial [8]. Opial's condition is a generalization of some other due to Filippov (for references see [8]).

Besides (R) we consider system

$$dx/dt = S(x, t) \tag{S}$$

and we suppose the following assumptions.

HYPOTHESIS H_4. I. $S(x, t)$ is of class C^1 for $x \in E^2$ and $t > 0$.

II. For every $x \in E^2$ there exist the limits

$$\lim_{t\to+\infty} S(x, t) = R(x)\,, \quad \lim_{t\to+\infty} \frac{\partial S(x, t)}{\partial x_i} = \frac{\partial R(x)}{\partial x} \quad (i = 1, 2)$$

and the above covergence is almost uniform on E^2.

III. For every fixed $s > s_0$ (s_0 is an appropriate constant) the autonomous system

$$dx/dt = S(x, s)\ (^2) \tag{Q}$$

satisfies Hypothesis H_3 with the exception that the only one singular point of (Q) must not be (0,0).

EXAMPLE 8. The linear system

$$dx_1/dt = a(t)x_1 + b(t)x_2\,, \quad dx_2/dt = c(t)x_1 - a(t)x_2 \tag{L_1}$$

where $a(t), b(t), c(t)$ are of class C^1 and there exist the limits $\lim_{t\to+\infty} a(t) = a$, $\lim_{t\to+\infty} b(t) = b$, $\lim_{t\to+\infty} c(t) = c$ and $a^2 + bc < 0$, satisfies Hypothesis H_4. Indeed, there is s_0 such that for $s > s_0$ the characteristic roots of

$$\begin{bmatrix} a(t) & b(t) \\ c(t) & -a(t) \end{bmatrix}$$

are purely imaginary and therefore any solution of linear system with constant coefficients

$$dx_1/dt = a(s)x_1 + b(s)x_2\,, \quad dx_2/dt = c(s)x_1 - a(s)x_2$$

is periodical. Under slightly general assumptions system (L_1) was investigated by T. Ważewski [12] and our result given below is closely connected with these of Ważewski's note [12].

Let $\mathcal{G}$ denote a trajectory of (R). Then $P = \mathcal{G} \times R$ ($R = (-\infty, +\infty)$) is a surface filled up by integrals of (R) when the last is considered in $E^2 \times R$.

We prove now the following result.

(2) (Q) is a one-parameter family of autonomous systems depending on s—the parameter. The solutions of (Q) are functions of t.

THEOREM 10. *Consider systems* (R) *and* (S) *and suppose they satisfy* H_3 and H_4, *respectively.*

Further suppose that

$$|\partial S(x,t)/\partial t| \leqslant a(t)|x| \tag{11,1}$$

where

$$\int^{\infty} a(s)\,ds < +\infty . \tag{11,2}$$

Under these assumptions the following assertions hold.

a. *To any set* $P = \mathcal{G}\times R$ [3] *there exists a set* Q *filled up by integrals of* (S) *which coincides asymptotically with* P.

b. Q *is a compact set of integrals.*

c. *If a sequence of trajectories* $\mathcal{G}_k$ *tends to* $\mathcal{G}_0$ *and* $P_k = \mathcal{G}_k\times R$ $(k = 1, 2, \ldots)$ *then the sequence of sets* Q_k *coinciding asymptotically with* P_k *has the following property: every open set filled up by integrals of* (S) *containing* Q_0 *contains also* Q_k *for sufficiently large* k.

Proof. Without loss of generality we may suppose that $\mathcal{G}$ passes through $(\xi_0, 0)$ where $\xi_0 > 0$ and that

$$R_2(\xi_0, 0) > 0 .$$

There is a positive constance Δ, $\Delta < \xi_0$, such that

$$R_2(x_1, 0) > \gamma > 0 \quad \text{for} \quad \xi_0 - \Delta < x_1 < \xi_0 + \Delta . \tag{11,3}$$

We introduce into consideration two auxiliary functions. The first one $K(x_1, x_2)$ is determined by the following conditions.

A. $K(x_1, x_2)$ is determined in the zone of emission of interval $\xi_0 - \Delta < x_1 < \xi_0 + \Delta$, $x_2 = 0$ with respect to the autonomous system (R).

B. $K(x_1, x_2)$ is constant along the trajectories of (R).

C. $K(x_1, 0) = x_1$ for $\xi_0 - \Delta < x_1 < \xi_0 + \Delta$.

The second function $L(x_1, x_2, s)$ for any fixed and suitably large s is determined by A, B and C provided that system (R) is replaced by (Q). Hence $L(x_1, x_2, s)$ is determined for such $s > s_0$ for which

$$S_2(x_1, 0, s) > 0 \quad \text{on} \quad \xi_0 - \Delta < x_1 < \xi_0 + \Delta .$$

Owing to H_4.II such s_0 exists.

Since $R(x)$ is of class C^1 thus $K(x_1, x_2)$ is also of class C^1 with respect to x_1 and x_2 and owing to assumption B we have

$$\frac{\partial K(x_1, x_2)}{\partial x_1} R_1(x_1, x_2) + \frac{\partial K(x_1, x_2)}{\partial x_2} R_2(x_1, x_2) = 0 .$$

[3] $\mathcal{G}$ may be a singular point (0,0). Then P is the straight line $x_1 = x_2 = 0$.

Similarly, $L(x_1, x_2, s)$ is of class C^1 with respect to x_1 and x_2 and for every fixed $s > s_0$ we have

$$\frac{\partial L(x_1, x_2, s)}{\partial x_1} S_1(x_1, x_2, s) + \frac{\partial L(x_1, x_2, s)}{\partial x_2} S_2(x_1, x_2, s) = 0 . \tag{11,4}$$

According to Hypothesis H_4.II and to A we get that if for some $\bar{x} = (\bar{x}_1, \bar{x}_2)$ $K(x_1, x_2)$ is defined then there is $\bar{s}$ such that for $s > \bar{s}$ and $x = \bar{x}$ $L(x_1, x_2, s)$ is defined also and there exist the limits

$$\lim_{s\to+\infty} L(\bar{x}_1, \bar{x}_2, s) = K(\bar{x}_1, \bar{x}_2), \qquad \lim_{s\to+\infty} \frac{\partial L(\bar{x}_1, \bar{x}_2, s)}{\partial x_i} = \frac{\partial K(\bar{x}_1, \bar{x}_2)}{\partial x_i} \tag{11,5}$$

$(i = 1, 2)$ and the above convergence is almost uniform with respect to $\bar{x}$.

Denote by $\varphi(t, \xi_1, \xi_2, s)$ the solution of (Q) satisfying the initial conditions

$$\varphi_1(0, \xi_1, \xi_2, s) = \xi_1 \quad \text{and} \quad \varphi_2(0, \xi_1, \xi_2, s) = \xi_2 .$$

Since $\partial S/\partial s$ exists; thus the derivative

$$\frac{\partial \varphi(t, x_1, 0, s)}{\partial s} = \psi(t, x_1, s)$$

exists also, and (see for example [5], p. 158) for every fixed x_1 and s $\psi(t, x_1, s)$ is a solution of linear system

$$du_i/dt = \sum_{k=1}^{2} \frac{\partial S_i(x_1, x_2, s)}{\partial x_k} u_k + \frac{\partial S_i(x_1, x_2, s)}{\partial s} \qquad (i = 1, 2) \tag{11,6}$$

where $x = \varphi(t, x_1, 0, s)$ and $\psi(0, x_1, s) = 0$. By the general form of solution of (11,6), by (11,1) and by H_0.III we get the inequality

$$|\partial\varphi/\partial s| = |\psi(t, x, s)| \leqslant M a(s) \tag{11,7}$$

where M is constant common for all $\xi_0 - \Delta \leqslant x_1 \leqslant \xi_0 + \Delta$.

Owing to C we have

$$L(\varphi_1(t, x_1, 0, s), \varphi_2(t, x_1, 0, s), s) \equiv x_1 . \tag{11,8}$$

Since there exist $\partial L/\partial x_i$ $(i = 1, 2)$ and $\partial\varphi/\partial s$ thus it follows from (11,8) that $\partial L/\partial s$ exists also and

$$\frac{\partial L}{\partial s} = -\left(\frac{\partial L}{\partial x_1}\cdot\frac{\partial \varphi_1}{\partial s} + \frac{\partial L}{\partial x_2}\cdot\frac{\partial \varphi_2}{\partial s}\right). \tag{11,9}$$

By (11,5) we get that $\partial L/\partial x_i$ are bounded, therefore (11,9) and (11,7) imply the following estimation

$$|\partial L/\partial s| \leqslant M^* a(s) \tag{11,10}$$

where M^* is the appropriate constant common for all (x_1, x_2, s) for which $L(x_1, x_2, s)$ is defined.

With the aid of $K(x_1, x_2)$ and $L(x_1, x_2, s)$ we defined now a base of $\mathfrak{F}_R(P)$ and two bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of $\mathrm{Fr}^+(\mathfrak{F}_R(P))$, satisfying the assumptions (i) and (ii) of Theorem 7, respectively.

Let us notice that the set P is determined by

$$K(x_1, x_2) = \xi_0, \quad -\infty < t < +\infty . \tag{11,11}$$

The family of sets B_ε for $0 < \varepsilon < \Delta$ where

$$B_\varepsilon = \{(x, t): |K(x_1, x_2) - \xi_0| < \varepsilon, \ -\infty < t < +\infty\}$$

is a base of the filter of neighbourhoods of P and the family $\mathfrak{B} = \{B_{\varepsilon\tau}: 0 < \varepsilon < \Delta \text{ and } \tau > 0\}$ where

$$B_{\varepsilon\tau} = \{(x, t): (x, t) \in B_\varepsilon \text{ and } t > \tau\}$$

is a base of $\mathrm{Fr}^+(\mathfrak{F}_R(P))$. Put

$$\beta(t) = M^* \int_t^\infty a(s)\,ds . \tag{11,12}$$

Consider now another families of sets

$$\mathfrak{B}_1 = \{C_{\varepsilon\tau}: 0 < \varepsilon < \Delta, \ \tau > \tau(\varepsilon)\} \quad \text{and} \quad B_2 = \{D_{\varepsilon\tau}: 0 < \varepsilon < \Delta, \ \tau > \tau(\varepsilon)\}$$

where

$$C_{\varepsilon\tau} = \{(x, t): \xi_0 - \varepsilon + \beta(t) < L(x_1, x_2, t) < \xi_0 + \varepsilon - \beta(t), \ t > \tau\}$$

and

$$D_{\varepsilon\tau} = [(x, t): \xi_0 - \varepsilon - \beta(t) < L(x_1, x_2, t) < \xi_0 + \varepsilon + \beta(t), \ t > \tau\} ,$$

$\tau(\varepsilon)$ is so chosen that $\varepsilon + \beta(t) < \Delta$ and $\varepsilon - \beta(t) > 0$ for $t > \tau(\varepsilon)$ and $0 < \varepsilon < \Delta$.

One can easily see that $\mathfrak{B}_1$ as well as $\mathfrak{B}_2$ satisfies (1,4) and (1,5) and therefore they are bases of filter. We prove now that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are bases of $\mathrm{Fr}^+(\mathfrak{F}_R(P))$. Indeed, let $B_{\varepsilon\tau}$ be an arbitrary set of $\mathfrak{B}$. By (11,4) we can find $\bar{\varepsilon} < \varepsilon$ and $\bar{\tau} > \tau$ such that

$$C_{\bar{\varepsilon}\bar{\tau}} \subset B_{\varepsilon\tau} \quad \text{and} \quad D_{\bar{\varepsilon}\bar{\tau}} \subset B_{\varepsilon\tau} . \tag{11,13}$$

By the same arguments to any ε, τ there are $\varepsilon^* < \varepsilon$ and $\tau^* > \tau$ such that

$$B_{\varepsilon^*\tau^*} \subset C_{\varepsilon\tau} \quad \text{and} \quad B_{\varepsilon^*\tau^*} \subset D_{\varepsilon\tau} . \tag{11,14}$$

Proposition 4, (11,13) and (11,14) imply that $\mathfrak{B}_1$ as well as $\mathfrak{B}_2$ is a base of $\mathrm{Fr}^+(F_R(P))$ which was to be proved.

To finish the proof of Theorem 10 we will show that $\mathfrak{B}_1$ satisfies (i) and $\mathfrak{B}_2$ satisfies (ii) of Theorem 7. If we do so then the part a. of Theorem 10 will follow Theorem 7 and 4, the part b. is a consequence of Theorem 3 and part a., at last part c. results from a. and Theorem 6.

Let $(x_1(t), x_2(t))$ be a solution of (S) and let

$$(x_1(t), x_2(t), t) \in C_{\varepsilon\tau} \quad \text{for} \quad t = t_0 . \tag{11,15}$$

We are going to show that (11,15) holds for $t > t_0$, that is, that $C_{\varepsilon\tau}$ is filled up by right-hand half integrals of (S) (see (i)). Owing to the definition of $C_{\varepsilon\tau}$ we must show the inequality

$$-\varepsilon + \beta(t) < \lambda(t) < \varepsilon - \beta(t) \quad \text{for} \quad t > t_0 \tag{11,16}$$

where $\lambda(t) = L(x_1(t), x_2(t), t) - \xi_0$. By (11,15) we find (11,16) valid for $t = t_0$. Suppose for $t = t_1 > t_0$ (11.16) does not hold. There is t_2, $t_0 < t_2 \leqslant t_1$, such that for $t_0 \leqslant t < t_2$ (11.16) holds and for $t = t_2$ we have

$$\lambda(t_2) = -\varepsilon + \beta(t_2) \quad \text{or} \quad \lambda(t_2) = \varepsilon - \beta(t_2) . \tag{11,17}$$

Since

$$\lambda'(t) = \frac{\partial L}{\partial s}(x_1(t), x_2(t), t) ,$$

thus by (11,10) and (11,12) we get the inequality

$$\beta'(t) < \lambda'(t) < -\beta'(t) \tag{11,18}$$

which is true for $t_0 \leqslant t < t_2$. By (11,18) we conclude that $\lambda(t) + \beta(t)$ is decreasing and $\lambda(t) - \beta(t)$ is increasing for $t_0 \leqslant t < t_2$ and therefore by (11,15) we get the inequality $-\varepsilon + \beta(t_2) < \lambda(t_2) < \varepsilon - \beta(t_2)$ which contradicts (11,17). Hence (11,16) holds for all $t > t_0$ and $\mathfrak{B}_1$ satisfies condition (i) of Theorem 7.

Consider now an arbitrary set $D_{\varepsilon\tau}$ and suppose $(x_1(t_0), x_2(t_0), t_0) \in D_{\varepsilon\tau}$. In this case we have the inequality

$$-\beta(t) - \varepsilon < \lambda(t) < \beta(t) + \varepsilon . \tag{11,19}$$

Using the analogous arguments as above one can prove that for $\tau < t \leqslant t_0$ $\lambda(t) - \beta(t)$ is increasing and $\lambda(t) + \beta(t)$ is decreasing function. Therefore (11,19) holds also for $\tau < t \leqslant t_0$. The last shows that $D_{\varepsilon\tau}$ is filled up by left-hand half integrals of (S) provided (S) is considered for $t > \tau$. That is, $\mathfrak{B}_2$ satisfies condition (ii) of Theorem 7.

Thus using Theorem 7 we conclude that there is a filter $\mathfrak{G}_S$ filled up by integrals of (S) which coincides asymptotically with $\mathfrak{F}_R(P)$. It follows from Theorem 5 that $\mathfrak{G}_S$ is a filter of neighbourhoods and therefore the adherent set Q of $\mathfrak{G}_S$ coincides asymptotically with P. Because of Theorem 3 Q is compact and by Theorem 6 the property c. is fullfilled. Hence we find Theorem 10 completely proved.

EXAMPLE 9. The present example shows that under the assumptions of Theorem 10 any single non-trivial integral of (R) may not coincide asymptotically with any integral of (S).

As a system (R) let us take the following one

$$dx_1/dt = -x_2, \qquad dx_2/dt = x_1 \tag{11,20}$$

and as a system (S) this one

$$dx_1/dt = -(1+1/t)x_2, \qquad dx_2/dt = (1+1/t)x_1. \tag{11,21}$$

One can easily verify that (11,20) satisfies H_3 and (11,21) satisfies H_4 and the assumptions of Theorem 10. The sets P and Q are the same and they present a cylinder

$$x_1^2+x_2^2 = m, \qquad -\infty < t < +\infty.$$

We apply now to system (11,20) and (11,21) the transformation

$$x_1 = u_1\cos(t+u_2), \qquad x_2 = u_1\sin(t+u_2), \qquad t = t. \tag{T}$$

(T) carries integrals of (11,20) into the straight lines and integrals of (11,21) into the curves of the form

$$u_1 = a, \qquad u_2 = b+\ln t.$$

Since for any b u_2 does not tend to finite limit as $t\to+\infty$, thus any single integral of (11,21) does not coincide asymptotically with some of (11,20).

Below we give a generalization of Theorem 10.

THEOREM 11. *Suppose system* (R) *satisfies* H_3.

The assertions of Theorem 10 *remain valid if*:

(i) *There is a sequence* $S_n(x,t)$ *such that*

$$S_n(x,t)\to S(x,t) \qquad \text{as} \qquad n\to+\infty$$

almost uniformly with respect to x *and uniformly with respect to* t.

(ii) $S_n(x,t)$ $(n=1,2,\dots)$ *satisfy* H_4 *and the convergences*

$$S_n(x,t)\to R(x), \qquad \frac{\partial S_n(x,t)}{\partial x_i}\to\frac{\partial R(x)}{\partial x_i}$$

$(i=1,2;\ n=1,2,\dots)$ *are uniform with respect to* n.

(iii)
$$\left|\frac{\partial S_n}{\partial t}\right| \leqslant a_n(t)|x|$$

where

$$\int^{\infty} a_n(s)\,ds < +\infty \qquad (n=1,2,\dots).$$

(iv) *The functions*

$$\beta_n(t) = M\int_t^{\infty} a_n(s)\,ds$$

tend to zero as t *approach infinity uniformly with respect to* n.

Proof. Notice that each of systems

$$dx/dt = S_n(x,t) \qquad (n=1,2,\dots) \tag{S_n}$$

fulfills hypothesis of Theorem 10. Therefore, as in the proof of Theorem 10, we can define functions $L_n(x_1,x_2,t)$, and the sets $C^n_{\varepsilon\tau}$ and $D^n_{\varepsilon\tau}$ in the same way as $L(x_1,x_2,t)$, $C_{\varepsilon\tau}$ and $D_{\varepsilon\tau}$. Owing to (ii) $L_n(x_1,x_2,t)$ tends, uniformly with respect to n, to $K(x_1,x_2)$ as $t\to+\infty$. This and (iii) imply that the relations, analogous to (11,13) and (11,14),

$$C^n_{\bar\varepsilon\bar\tau}\subset B_{\varepsilon\tau}, \qquad D^n_{\bar\tau\bar\delta}\subset B_{\varepsilon\tau}, \tag{11,22}$$

$$B_{\varepsilon^*\tau^*}\subset C^n_{\varepsilon\tau}, \qquad B_{\varepsilon^*\tau^*}\subset D^n_{\varepsilon\tau} \tag{11,23}$$

hold for every n and $\bar\varepsilon, \bar\tau$ or ε^*, τ^* do not depend on n but only on ε, τ.

Now let us put

$$B^*_{\varepsilon\tau} = Z^+_S(B_{\varepsilon\tau}).$$

By (11,22) and (11,23) to any ε, τ, there are $\bar\varepsilon, \bar\tau$ ε^*, τ^* such that

$$B_{\varepsilon^*\tau^*}\subset C^n_{\bar\varepsilon\bar\tau}\subset B_{\varepsilon\tau} \qquad (n=1,2,\dots).$$

Thus

$$Z^+_{S_n}(B_{\varepsilon^*\tau^*})\subset Z^+_{S_n}(C^n_{\bar\varepsilon\bar\tau}) = C^n_{\bar\varepsilon\bar\tau}\subset B_{\varepsilon\tau}.$$

Hence

$$Z^+_{S_n}(B_{\varepsilon^*\tau^*})\subset B_{\varepsilon\tau} \qquad \text{for} \qquad n=1,2,\dots$$

The last and (i) imply that

$$B^*_{\varepsilon^*\tau^*} = Z^+_S(B_{\varepsilon^*\tau^*})\subset B_{\varepsilon\tau}. \tag{11,24}$$

Since for every ε, τ $B_{\varepsilon\tau}\subset B^*_{\varepsilon\tau}$ thus (11,25) shows that the family $\mathfrak{B}^* = \{B^*_{\varepsilon\tau}\colon 0<\varepsilon<\Delta,\ \tau>\tau(\varepsilon)\}$ is a base of $\mathrm{Fr}^+(F_R(P))$. Because $\mathfrak{B}^*$ satisfy (3,2) and (3,3) with respect to system (S) thus $\mathfrak{B}^*$ is a base of asymptotic boundary of a filter filled up by integrals of (S). Hence there is a filter $\mathfrak{G}_S$ filled up by integrals of (S) which coincides asymptotically with $\mathfrak{F}_R(P)$. This fact and some previous results of this paper imply Theorem 11.

Let us observe that system (L_1) when $a(t)$, $b(t)$ and $c(t)$ are continuous and of bounded variation in $(0,+\infty)$ satisfies the hypothesis of Theorem 11. Therefore the last result contains, as a special case, that of Ważewski [12].

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Sur l'existence et l'unicité des solutions de certaines équations différentielles du type $u_{xyz} = f(x, y, z, u, u_x, u_y, u_z, u_{xy}, u_{xz}, u_{yz})$

par M. Kwapisz, B. Palczewski et W. Pawelski (Gdańsk)

Introduction. Le but de ce mémoire est d'étudier certains cas du problème d'existence et d'unicité, dans le domaine V

$$V\{0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b, 0 \leqslant z \leqslant c\}$$

des solutions de l'équation

$$u_{xyz} = f(x, y, z, u, u_x, u_y, u_z, u_{xy}, u_{xz}, u_{yz}) \tag{1}$$

avec les conditions initiales

$$u(0, y, z) = \psi_1(y, z), \quad u(x, 0, z) = \psi_2(x, z), \quad u(x, y, 0) = \psi_3(x, y). \tag{2}$$

On supposera la fonction f assujettie à des conditions analogues à celles qui ont été introduites dans les travaux de W. Walter [1] et [2]. Ces conditions sont une certaine généralisation de celles que Nagumo et Osgood avaient admises pour l'étude de l'unicité des solutions des équations différentielles ordinaires. Nos recherches seront basées sur les mémoires [1] et [2]. Aussi adoptons-nous plusieurs définitions et théorèmes auxiliaires qui y figurent. Quant aux méthodes introduites par ces auteurs, quelques-unes ont pu être étendues, avec quelques modifications, au problème considéré.

Notre mémoire se compose de deux parties principales. Dans la première nous occupons du problème d'unicité des solutions de l'équation (1) lorsque les conditions (2) sont vérifiées. Ce problème sera appelé dans la suite problème (A). La seconde partie du mémoire contient les démonstrations d'existence des solutions relatives à des cas particuliers de l'équation (1).

On trouvera dans les travaux [1] et [2] de W. Walter un compte rendu de la bibliographie des problèmes respectifs concernant l'équation différentielle du second ordre de la forme

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{*}$$