Die Gleichungen (65) stellen n Identitäten im Raum X_n^ϱ dar; die Gleichungen (66) geben in diesem Raum σ starke potentielle Erhaltungssätze, die einer jeden Lieschen Gruppe G_ϱ entsprechen. Die Gleichungen (67) weisen auf die Existenz von Potentialen.

Die Gleichungen (62) und (63) sind erfüllt für Funktionen $x_i(\xi_j)$ und $\varphi_A(\xi_j)$, die Euler-Lagrangeschen Gleichungen $\delta\Lambda/\delta\varphi_A = 0$ genügen. Da wir $N+n$ unbekannte Funktionen und N unabhängige Gleichungen haben, können wir $x_i = \xi_i$ annehmen. Dann bekommen wir aus (62) und (63)

(73) $$\partial T_{i\beta}^{(w)}/\partial x_i = 0\,,$$

(74) $$T_{i\beta}^{(w)} = -\frac{\partial}{\partial x_k}\Omega_{ik\beta}\,,$$

wo

(75) $$T_{i\beta}^{(w)} = t_{ij}^{(w)}\chi_{j\beta} + \omega_{kij}\partial_k\chi_{j\beta}\,.$$

Die Gleichungen (73) stellen im Raum X_n^ϱ σ schwache potentielle Erhaltungssätze dar, die einer jeden Lieschen Gruppe entsprechen. Die Gleichungen (74) weisen auf die Existenz von Potentialen.

Zum Schluß wollen wir bemerken, daß die kanonischen Erhaltungssätze im Raum X_n^ϱ und die schwachen potentiellen, der Gruppe G_ϱ entsprechenden Erhaltungssätze voneinander unabhängig sind.

Herrn Prof. Ślebodziński möchte ich für wertvolle kritische Hinweise höflichst danken.

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On a certain boundary problem for Laplace equation

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This paper contains the solution of a problem which has been raised by prof. G. Fichera in his course of lectures on the theory of singular integral equations which he have had in Instituto Nazionale di Alta Matematica in Rome in the years 1958-59.

1. Denotations. We denote by Σ a curve lying in the complex z-plane ($z = x+iy$) and we assume that it admits a parametric representation $z = z(s)$, where $z(s)$ is a function of the class C^n on the interval $0 \leqslant s \leqslant L$, satisfying the conditions

$$|z'(s)| = 1 \quad (0 \leqslant s \leqslant L)\,,$$

$$\left.\frac{d^k z(s)}{ds^k}\right|_{s=0} = \left.\frac{d^k z(s)}{ds^k}\right|_{s=L} \quad (k = 0, 1, \ldots, n)$$

and such that $z(s_1) = z(s_2)$ with $s_1 < s_2$, if and only if $s_1 = 0$ and $s_2 = L$.

We denote by Ω a domain whose boundary is the curve Σ; we denote by n_z the normal vector of Σ at the point z, directed towards the interior of Ω, and by ν_z the unit vector of the same direction at the point z of Σ. Σ_r denotes the curve which is parallel to Σ at the distance $|r|$ from Σ and is situated inside Ω when $r > 0$ and outside Ω when $r < 0$. The type ϱ will denote a positive number. We see that $\Sigma_\varrho \subset \Omega$.

The equation of the curve Σ_r can be written briefly

$$z_r = z(s) + r\nu[z(s)] \quad (0 \leqslant s \leqslant L)$$

or by setting $z_r = x_r + iy_r$ we can get the equation of Σ_r in the form

(1) $$x_r = x(s) - ry'(s)\,, \quad y_r = y(s) + rx'(s) \quad (0 \leqslant s \leqslant L)$$

without loosing any generality.

The element of the arc of Σ_r will be denoted by ds_r. If the curve Σ is of the class C^2 we have

(2) $$ds_r = \{[x'(s) - ry''(s)]^2 + [y'(s) + rx''(s)]^2\}^{1/2}\,ds\,.$$

The corresponding domain whose boundary is the curve Σ_r is denoted by Ω_r.

Let us denote by m an arbitrary non-negative integer. We say that a function $f(z)$ is of the class $C^m(\Sigma)$ if the function $\varphi(s) = f[z(s)]$ is of the class C^m on the interval $0 \leqslant s \leqslant L$ and

$$\frac{d^k\varphi(s)}{ds^k}\Big|_{s=0} = \frac{d^k\varphi(s)}{ds^k}\Big|_{s=L} \quad (k = 0, 1, \ldots, m).$$

We say that a function $p(z)$ is of the class C^m, if the function $p^*(x, y) = p(x+iy)$ is of the class C^m.

2. We will now formulate the problem, called for brevity problem P, which is the object of the present paper.

Let $D_m(\Sigma)$ be the Banach space of functions $u(z)$ of the class $C^m(\Sigma)$ with the norm

$$\|u\| = \sum_{k=0}^{m} \max_{0\leqslant s\leqslant L} \left| \frac{d^k}{ds^k} u[z(s)] \right|.$$

Let F and G be elements of the conjugate space $D_m^*(\Sigma)$ to the $D_m(\Sigma)$, i.e. let F and G be distributions of order $\leqslant m$, satisfying the condition

(3) $$G(\log|z-\zeta|) - F\left(\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\right) = 0, \text{ whenever } \zeta \in \Sigma,\ z \notin \overline{\Omega} = \Omega \cup \Sigma.$$

We investigate the function of the shape

(4) $$u(z) = \frac{1}{2\pi} G(\log|z-\zeta|) - \frac{1}{2\pi} F\left(\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\right), \quad \zeta \in \Sigma.$$

Of course, u is harmonic inside Ω. Problem P then arises

PROBLEM P. *We have to investigate under certain assumptions on the regularity of Σ, whether the following relations hold:*

(5) $$\lim_{\varrho\to 0} \int_{\Sigma_\varrho} p(z) u(z)\, ds_\varrho = F(p),$$

(6) $$\lim_{\varrho\to 0} \int_{\Sigma_\varrho} p(z) \frac{\partial u}{\partial n}\, ds_\varrho = G(p),$$

$p(z)$ being an arbitrary function of some defined regularity class.

This problem may be considered as a generalisation of a problem solved by L. Amerio [1], on the distributions. Namely, assuming that the summable functions $f(\zeta)$ and $g(\zeta)$ satisfy the condition

(7) $$\int_\Sigma \log|z-\zeta|\cdot g(\zeta)\, ds_\zeta - \int_\Sigma \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\cdot f(\zeta)\, ds_\zeta = 0$$

where $z \notin \overline{\Omega}$, and introducing a function $\tilde{u}(z)$

(8) $$\tilde{u}(z) = \frac{1}{2\pi}\left\{ \int_\Sigma \log|z-\zeta|\cdot g(\zeta)\, ds_\zeta - \int_\Sigma \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\cdot f(\zeta)\, ds_\zeta \right\}$$

harmonic inside Ω, L. Amerio proved (¹) that for almost all $z^* \in \Sigma$ the following conditions hold

$$\lim_{z\to z^*(\text{on } n_{z^*})} \tilde{u}(z) = f(z^*), \qquad \lim_{z\to z^*(\text{on } n_{z^*})} \frac{\partial \tilde{u}(z)}{\partial n_{z^*}} = g(z^*).$$

In the case $m = 0$ we apply the theorem of Riesz which states that the distributions F and G are of the shape

$$F(v) = \int_\Sigma v(\zeta)\, d\alpha, \qquad G(v) = \int_\Sigma v(\zeta)\, d\beta,$$

$\alpha(\zeta)$ and $\beta(\zeta)$ being the two functions of bounded variation such that

$$\int_\Sigma \log|z-\zeta|\, d\beta - \int_\Sigma \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, d\alpha = 0 \quad \text{for} \quad z \notin \overline{\Omega}.$$

Then problem P consists of proving that the function $u(z)$ defined by

$$u(z) = \frac{1}{2\pi}\left\{ \int_\Sigma \log|z-\zeta|\, d\beta - \int_\Sigma \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, d\alpha \right\}$$

satisfies the conditions

$$\lim_{\varrho\to 0} \int_{\Sigma_\varrho} p(z) u(z)\, ds_\varrho = \int_\Sigma p(\zeta)\, d\alpha,$$

$$\lim_{\varrho\to 0} \int_{\Sigma_\varrho} p(z) \frac{\partial u}{\partial n}\, ds_\varrho = \int_\Sigma p(\zeta)\, d\beta$$

$p(z)$ being an arbitrary function of some defined regularity class.

It follows that the problem P in case $m = 0$ constitutes an integral formulation of the mentioned Amerio's problem.

The solution of the problem P is given in the following

THEOREM. *Let m be an arbitrary non-negative integer. Let Σ be a closed single curve of the class C^{m+2}, let F and G be elements of the space $D_m^*(\Sigma)$ of distributions of order $\leqslant m$, and let*

(9) $$G(\log|z-\zeta|) - F\left(\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\right) = 0, \quad \textit{whenever} \quad z \notin \overline{\Omega}.$$

Then the function $u(z)$ defined by (4) satisfies the Laplace equation in the interior of Ω and the conditions (5) and (6) for an arbitrary function $p(z)$ which is of the class C^{m+2} if $m > 0$ and of the class C^3 if $m = 0$.

(¹) Cf. [1]. A more simple proof of this result was given by G. Fichera [2].

The proof of this theorem will be given in several steps.

LEMMA 1. *Suppose Σ is a closed single curve of the class C^2, and denote by s_ζ the curvilinear coordinate of the point $\zeta \in \Sigma$. Let $\varphi(z,\zeta)$ be a function defined for $z \in \Sigma_r$, $\zeta \in \Sigma$ and continuous together with its derivatives*

$$\frac{\partial^k \varphi(z,\zeta)}{\partial s_\zeta^k} \qquad (k = 1, \ldots, m) \tag{10}$$

in the cartesian product $\Sigma_r \times \Sigma$.

By these assumptions we have for every continuous function $p(z)$ and every functional $H \in D_m^*(\Sigma)$ the equality

$$\int_{\Sigma_r} p(z) H[\varphi(z,\zeta)]\, ds_r = H\Big[\int_{\Sigma_r} p(z)\varphi(z,\zeta)\, ds_r\Big].$$

The proof of this lemma is based on the definition of the integral. Here we make use of the linearity and continuity of the functional H and the uniform continuity of $\varphi(z,\zeta)$ and its derivatives of the shape (10) in the cartesian product $\Sigma_r \times \Sigma$.

LEMMA 2. *Let F and G be two linear continuous functionals on $D_m(\Sigma)$ satisfying the conditions* (9).

The following conditions expressed by the formulae (11)-(14) *are sufficient so that a function $u(z)$ defined by* (4) *satisfies* (5) *and* (6)

$$\lim_{\varrho\to 0} \frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho} p(z)\log|z-\zeta|\, ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\log|z-\zeta|\, ds_{-\varrho}\Big] = 0, \tag{11}$$

$$\lim_{\varrho\to 0} \frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_{-\varrho}\Big] = -2\pi\frac{\partial^k p(\zeta)}{\partial s_\zeta^k}, \tag{12}$$

$$\lim_{\varrho\to 0} \frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|\, ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|\, ds_{-\varrho}\Big] = 2\pi\frac{\partial^k p(\zeta)}{\partial s_\zeta^k}, \tag{13}$$

$$\lim_{\varrho\to 0} \frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_\zeta}\frac{\partial\log|z-\zeta|}{\partial n_z}\, ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_\zeta}\frac{\partial\log|z-\zeta|}{\partial n_z}\, ds_{-\varrho}\Big] = 0, \tag{14}$$

where $k = 0, 1, \ldots, m$ and the convergence is uniform with respect to $\zeta \in \Sigma$.

Proof. It follows immediately from (9) that the function $u(z)$ defined by (4) satisfies for $\varrho > 0$ the relations

$$\int_{\Sigma_{-\varrho}} p(z)u(z)\, ds_{-\varrho} = 0, \tag{15}$$

$$\int_{\Sigma_{-\varrho}} p(z)\frac{\partial u}{\partial n_z}\, ds_{-\varrho} = 0. \tag{16}$$

By lemma 1 we obtain from (15)

$$\begin{aligned}\int_{\Sigma_\varrho} p(z)u(z)\, ds_\varrho &= \frac{1}{2\pi} G\Big[\int_{\Sigma_\varrho} p(z)\log|z-\zeta|\, ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\log|z-\zeta|\, ds_{-\varrho}\Big] - \\ &\quad -\frac{1}{2\pi} F\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_{-\varrho}\Big].\end{aligned} \tag{17}$$

Starting from the definition of the derivative it is easy to show in view of the continuity and the linearity of the functionals F and G that for $z \in \Sigma_r$ and $|r| > 0$ we have

$$\frac{\partial}{\partial n_z} G(\log|z-\zeta|) = G\Big(\frac{\partial}{\partial n_z}\log|z-\zeta|\Big),$$

$$\frac{\partial}{\partial n_z} F\Big(\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\Big) = F\Big(\frac{\partial}{\partial n_z}\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\Big).$$

Hence in view of (16) and lemma 1 we have

$$\begin{aligned}\int_{\Sigma_\varrho} p(z)\frac{\partial u}{\partial n_z}\, ds_\varrho &= \frac{1}{2\pi} G\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|\, ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|\, ds_{-\varrho}\Big] - \\ &\quad -\frac{1}{2\pi} F\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_z}\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_z}\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_{-\varrho}\Big].\end{aligned} \tag{18}$$

The assertion of lemma 2 follows immediately from (17), (18) and the assumption that $F \in D_m^*(\Sigma)$, $G \in D_m^*(\Sigma)$.

LEMMA 3. *Assume that for each value of the parameter r from the interval $r_1 \leqslant r \leqslant r_2$, where $r_1 \leqslant 0 \leqslant r_2$, $r_1 < r_2$ and for each value of σ from the interval $a \leqslant \sigma \leqslant b$, $0 \leqslant a < b \leqslant L$, the function $\varphi_r(s,\sigma)$ satisfies the following three conditions:*

(i) *$\varphi_r(s, \sigma)$ is a function in s that is defined and continuous at all points of the interval $0 \leqslant s \leqslant L$ except at most for $s = \sigma$.*

(ii) *There exists such a constant c that*

$$|\varphi_r(s, \sigma)| \leqslant c \quad \text{whenever } r_1 \leqslant r \leqslant r_2,\ a \leqslant \sigma \leqslant b,\ s \neq \sigma .$$

(iii) *For each δ, $0 < \delta < L/2$*

$$\lim_{r\to 0} \varphi_r(s, \sigma) = \varphi_0(s, \sigma)$$

uniformly for $a \leqslant \sigma \leqslant b$, $0 \leqslant s \leqslant L$, $|s-\sigma| \geqslant \delta$.

With these assumptions

$$\lim_{r\to 0} \int_0^L \varphi_r(s, \sigma)\, ds = \int_0^L \varphi_0(s, \sigma)\, ds$$

uniformly on the interval $a \leqslant \sigma \leqslant b$.

The proof of lemma 3 will be omitted.

LEMMA 4. *Assume that $q(s)$ is a function of the class C^n, periodic with the period L, and the functions $g(s, \sigma)$, $\gamma(s, \sigma)$ are of class C^n in the set $0 \leqslant s \leqslant L$, $a \leqslant \sigma \leqslant b$, $a < b$, and further, assume that there exists a constant $B > 0$, such that*

$$g(s, \sigma) \geqslant B \quad \text{for} \quad 0 \leqslant s \leqslant L, \quad a \leqslant \sigma \leqslant b.$$

Let us denote

$$\gamma_0(s, \sigma) = \gamma(s, \sigma), \qquad g_0(s, \sigma) = g(s, \sigma),$$

$$\gamma_j(s, \sigma) = \frac{\partial \gamma_{j-1}}{\partial s} + \frac{\partial \gamma_{j-1}}{\partial \sigma}, \qquad g_j(s, \sigma) = \frac{\partial g_{j-1}}{\partial s} + \frac{\partial g_{j-1}}{\partial \sigma} \qquad (j = 1, \ldots, n).$$

Assume that

$$\text{(19)} \qquad \begin{aligned} &\frac{\gamma_j(L, \sigma)}{g_0(L, \sigma)} = \frac{\gamma_j(0, \sigma)}{g_0(0, \sigma)}, \quad \text{for} \quad a \leqslant \sigma \leqslant b, \quad j = 0, 1, \ldots, n, \\ &\frac{g_j(L, \sigma)}{g_0(L, \sigma)} = \frac{g_j(0, \sigma)}{g_0(0, \sigma)}, \quad \text{for} \quad a \leqslant \sigma \leqslant b, \quad j = 1, \ldots, n. \end{aligned}$$

Under these assumptions the function $\Psi(\sigma)$ defined by the formula

$$\Psi(\sigma) = \int_0^L q(s) \frac{\gamma(s, \sigma)}{g(s, \sigma)}\, ds$$

has the continuous derivatives up to the order n. It will be shown that the following formula for the derivatives $\dfrac{d^k\Psi(\sigma)}{d\sigma^k}$ $(k = 1, \ldots, n)$ holds

$$\text{(20)} \qquad \frac{d^k\Psi(\sigma)}{d\sigma^k} = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \sum_{a_1,\ldots,a_k} A^{ij}_{a_1\ldots a_k} \int_0^L \frac{d^i q}{ds^i} \cdot \frac{\gamma_j}{g_0} \left(\frac{g_1}{g_0}\right)^{a_1} \cdots \left(\frac{g_k}{g_0}\right)^{a_k} ds ,$$

where $A^{ij}_{a_1\ldots a_k}$ are the appropriate constants [2], *$0 \leqslant a_h \leqslant k$, for $h = 1, \ldots, k$, $a_1 + 2a_2 + \ldots + ka_k = k - (i+j)$.*

Proof. When $k = 1$ and coefficients A^{00}_1, A^{01}_0, A^{10}_0 are defined by

$$A^{00}_1 = -1, \qquad A^{01}_0 = 1, \qquad A^{10}_0 = 1,$$

the validity of (20) follows from the relation

$$\text{(21)} \qquad \begin{aligned} \frac{d\Psi(\sigma)}{d\sigma} &= \int_0^L q(s) \left[\frac{\partial}{\partial \sigma}\frac{\gamma_0}{g_0} + \frac{\partial}{\partial s}\frac{\gamma_0}{g_0}\right] ds - \int_0^L q(s) \frac{\partial}{\partial s}\frac{\gamma_0}{g_0}\, ds \\ &= \int_0^L q(s) \left[\frac{\gamma_1}{g_0} - \frac{\gamma_0}{g_0} \cdot \frac{g_1}{g_0}\right] ds + \int_0^L \frac{dq}{ds} \cdot \frac{\gamma_0}{g_0}\, ds \end{aligned}$$

and this, in turn, is true in virtue of (19) and of the periodicity of the function $q(s)$ with the period L.

Let us assume that (20) is valid for $k = 1, \ldots, m$, where $m < n$. Let us differentiate both sides of (20) with $k = m$ and let us apply the partial integration as in (21). Writing for the sake of simplicity Σ in place of $\sum_{i=0}^{m} \sum_{j=0}^{m-i} \sum_{a_1,\ldots,a_m}$ with $a_1 + 2a_2 + \ldots + ma_m = m - (i+j)$, the following equation will be obtained:

$$\begin{aligned} \frac{d^{m+1}\Psi(\sigma)}{d\sigma^{m+1}} &= \sum A^{ij}_{a_1\ldots a_m} \int_0^L \frac{d^{i+1}q}{ds^{i+1}} \cdot \frac{\gamma_j}{g_0} \left(\frac{g_1}{g_0}\right)^{a_1} \cdots \left(\frac{g_m}{g_0}\right)^{a_m} ds + \\ &+ \sum A^{ij}_{a_1\ldots a_m} \int_0^L \frac{d^i q}{ds^i} \left\{ \left(\frac{\gamma_{j+1}}{g_0} - \frac{\gamma_j}{g_0} \cdot \frac{g_1}{g_0}\right) \left(\frac{g_1}{g_0}\right)^{a_1} \cdots \left(\frac{g_m}{g_0}\right)^{a_m} + \right. \\ &+ \frac{\gamma_j}{g_0} \sum_{h=1}^{m} a_h \left(\frac{g_1}{g_0}\right)^{a_1} \cdots \left(\frac{g_{h-1}}{g_0}\right)^{a_{h-1}} \left(\frac{g_h}{g_0}\right)^{a_h - 1} \left(\frac{g_{h+1}}{g_0}\right)^{a_{h+1}} \cdots \left(\frac{g_m}{g_0}\right)^{a_m} \times \\ &\times \left. \left(\frac{g_{h+1}}{g_0} - \frac{g_h}{g_0} \cdot \frac{g_1}{g_0}\right) \right\} ds . \end{aligned}$$

It is easy to show that the right-hand side of this equation can be written in the form

$$\sum_{i=0}^{m+1} \sum_{j=0}^{m+1-i} \sum_{a_1,\ldots,a_{m+1}} A^{ij}_{a_1\ldots a_{m+1}} \int_0^L \frac{d^i q}{ds^i} \cdot \frac{\gamma_j}{g_0} \left(\frac{g_1}{g_0}\right)^{a_1} \cdots \left(\frac{g_{m+1}}{g_0}\right)^{a_{m+1}} ds ,$$

[2] The numerical values and the number of the constants $A^{ij}_{a_1\ldots a_k}$ are here indifferent. We need only the fact that these constants are finite and that for every k the linear combination on the right-hand side of (20) contains a finite number of terms.

where $A^{ij}_{a_1\dots a_{m+1}}$ are the appropriate constants, $a_1+2a_2+\dots+(m+1)a_{m+1} = m+1-(i+j)$, $a_i \geqslant 0$ for $i=1,\dots,m+1$. Hence it follows that (20) is valid with $k=m+1$, which completes the proof of lemma 4.

LEMMA 5. *Assume that for each r, $|r|\leqslant \varrho_0$, $\varrho_0>0$, the function $q_r(s)$ is a periodic function with the period L and of the class C^n and the functions $g_r(s,\sigma)$, $\gamma_r(s,\sigma)$ are periodic functions in s with the period L and of the class C^n on the domain $-\infty<s<\infty$, $a\leqslant\sigma\leqslant b$, $0\leqslant a<b\leqslant L$. Further, we assume that*

$$\lim_{r\to 0}\frac{d^j q_r(s)}{ds^j}=\frac{d^j q_0(s)}{ds^j} \qquad (j=0,1,\dots,n), \tag{22}$$

uniformly with respect to s, $0\leqslant s\leqslant L$, and that there exist functions $M(s,\sigma)$, $N(s,\sigma)$, $P(s,\sigma)$, $Q(s,\sigma)$, $R(s,\sigma)$ of the class C^n on the domain $-\infty<s<\infty$, $a\leqslant\sigma\leqslant b$ and a positive constant A such that

$$\gamma_r(s,\sigma)=M(s,\sigma)(s-\sigma)^2+N(s,\sigma)(s-\sigma)r+P(s,\sigma)r^2, \tag{23}$$

$$g_r(s,\sigma)=[R(s,\sigma)+2rQ(s,\sigma)](s-\sigma)^2+r^2, \tag{24}$$

$$R(s,\sigma)+2rQ(s,\sigma)\geqslant A \quad \text{for} \quad 0\leqslant s\leqslant L,\quad a\leqslant\sigma\leqslant b,\quad |r|\leqslant\varrho_0. \tag{25}$$

Let

$$\Psi_r(\sigma)=\int_0^L q_r(s)\frac{\gamma_r(s,\sigma)}{g_r(s,\sigma)}ds \qquad (0<|r|\leqslant\varrho_0).$$

Then

$$\lim_{\varrho\to 0}\left[\frac{d^k\Psi_\varrho(\sigma)}{d\sigma^k}-\frac{d^k\Psi_{-\varrho}(\sigma)}{d\sigma^k}\right]=0 \qquad (k=0,1,\dots,n), \tag{26}$$

uniformly with respect to σ, $a\leqslant\sigma\leqslant b$.

Proof. Let us denote

$$\begin{gathered}\gamma_{r0}(s,\sigma)=\gamma_r(s,\sigma), \qquad g_{r0}(s,\sigma)=g_r(s,\sigma),\\ \left.\begin{aligned}\gamma_{rj}(s,\sigma)&=\frac{\partial\gamma_{r,j-1}}{\partial s}+\frac{\partial\gamma_{r,j-1}}{\partial\sigma},\\ g_{rj}(s,\sigma)&=\frac{\partial g_{r,j-1}}{\partial s}+\frac{\partial g_{r,j-1}}{\partial\sigma},\end{aligned}\right\}\quad (j=1,\dots,n),\end{gathered} \tag{27}$$

for $-\infty<s<\infty$, $a\leqslant\sigma\leqslant b$.

We note that all the assumptions of lemma 4 concerning the functions $q(s)$, $g(s,\sigma)$, $\gamma(s,\sigma)$ are satisfied also by the functions $q_r(s)$, $g_r(s,\sigma)$, $\gamma_r(s,\sigma)$, $(0<|r|\leqslant\varrho_0)$, correspondingly.

Thus, setting for $|r|>0$

$$\begin{gathered}\varphi_{r0}(s,\sigma)=q_r(s)\frac{\gamma_r(s,\sigma)}{g_r(s,\sigma)},\\ \varphi_{rk}(s,\sigma)=\sum_{i=0}^{k}\sum_{j=0}^{k-i}\sum_{a_1,\dots,a_k}A^{ij}_{a_1\dots a_k}\frac{d^iq_r}{ds^i}\cdot\frac{\gamma_{rj}}{g_{r0}}\left(\frac{g_{r1}}{g_{r0}}\right)^{a_1}\dots\left(\frac{g_{rk}}{g_{r0}}\right)^{a_k}\end{gathered} \tag{28}$$

for $k=1,\dots,n$, the following equation will be obtained:

$$\frac{d^k\Psi_r(\sigma)}{d\sigma^k}=\int_0^L\varphi_{rk}(s,\sigma)ds \qquad (k=0,1,\dots,n). \tag{29}$$

To prove (26) it is sufficient to show that uniformly in σ, $a\leqslant\sigma\leqslant b$, we have

$$\lim_{\varrho\to 0}\left[\int_0^L\varphi_{\varrho k}(s,\sigma)ds-\int_0^L\varphi_{-\varrho k}(s,\sigma)ds\right]=0 \qquad (k=0,1,\dots,n). \tag{30}$$

The proof of condition (30) is based on lemma 3. Putting $r=0$ in (27) and (28) we define the functions $\gamma_{0j}(s,\sigma)$, $g_{0j}(s,\sigma)$ $(j=0,1,\dots,n)$ in the domain $-\infty<s<\infty$, $a\leqslant\sigma\leqslant b$, and the functions $\varphi_{0j}(s,\sigma)$ $(j=0,1,\dots,n)$ in the domain $-\infty<s<\infty$, $a\leqslant\sigma\leqslant b$, $s\neq\sigma+kL$ $(k=0,\pm1,\pm2,\dots)$.

It is evident that the function $\varphi_{rk}(s,\sigma)$ $(|r|\leqslant\varrho_0)$ satisfies the assumption (i) of lemma 3. In order to show that the second assumption of this lemma is satisfied by these functions also, we introduce the functions

$$\begin{gathered}M_0(s,\sigma)=M(s,\sigma),\qquad N_0(s,\sigma)=N(s,\sigma),\qquad P_0(s,\sigma)=P(s,\sigma),\\ R_0(s,\sigma)=R(s,\sigma),\qquad Q_0(s,\sigma)=Q(s,\sigma),\end{gathered}$$

$$\begin{gathered}M_j(s,\sigma)=\frac{\partial M_{j-1}}{\partial s}+\frac{\partial M_{j-1}}{\partial\sigma},\qquad N_j(s,\sigma)=\frac{\partial N_{j-1}}{\partial s}+\frac{\partial N_{j-1}}{\partial\sigma},\\ P_j(s,\sigma)=\frac{\partial P_{j-1}}{\partial s}+\frac{\partial P_{j-1}}{\partial\sigma},\qquad Q_j(s,\sigma)=\frac{\partial Q_{j-1}}{\partial s}+\frac{\partial Q_{j-1}}{\partial\sigma},\\ R_j(s,\sigma)=\frac{\partial R_{j-1}}{\partial s}+\frac{\partial R_{j-1}}{\partial\sigma}\qquad (j=1,\dots,n)\end{gathered} \tag{31}$$

and we denote by C their maximal value on the domain $0\leqslant s\leqslant L$, $a\leqslant\sigma\leqslant b$. Then from (23)-(25) will be obtained the following inequalities

$$\begin{gathered}\left|\frac{\gamma_{rj}(s,\sigma)}{g_{r0}(s,\sigma)}\right|\leqslant\frac{(s-\sigma)^2+|r||s-\sigma|+r^2}{A(s-\sigma)^2+r^2}C=\left(1+\sqrt{A}+A\right)C/A\\ (j=0,1,\dots,n),\\ \left|\frac{g_{rj}(s,\sigma)}{g_{r0}(s,\sigma)}\right|\leqslant\frac{(s-\sigma)^2(1+2\varrho_0)C}{A(s-\sigma)^2+r^2}\leqslant(1+2\varrho_0)C/A \qquad (j=1,\dots,n)\end{gathered} \tag{32}$$

that are valid for $0\leqslant s\leqslant L$, $a\leqslant\sigma\leqslant b$, $s\neq\sigma$, $|r|\leqslant\varrho_0$.

It follows from (32) that the functions $\varphi_{rk}(s,\sigma)$ $(|r|\leqslant\varrho_0,\ k=0,1,\dots,n)$ are uniformly bounded in the set $0\leqslant s\leqslant L$, $a\leqslant\sigma\leqslant b$, $s\neq\sigma$.

It is easy to see that

$$\lim_{r\to 0}\gamma_{rj}(s,\sigma)=\gamma_{0j}(s,\sigma),\qquad \lim_{r\to 0}g_{rj}(s,\sigma)=g_{0j}(s,\sigma)$$

uniformly in $0\leqslant s\leqslant L$, $a\leqslant\sigma\leqslant b$.

Thus in view of (25) it follows that for each δ, $0<\delta<L/2$, the following condition hold uniformly in $a\leqslant\sigma\leqslant b$, $0\leqslant s\leqslant L$, $|s-\sigma|\geqslant\delta$

$$\lim_{r\to 0}\frac{\gamma_{rj}(s,\sigma)}{g_{r0}(s,\sigma)}=\frac{\gamma_{0j}(s,\sigma)}{g_{00}(s,\sigma)},\qquad \lim_{r\to 0}\frac{g_{rj}(s,\sigma)}{g_{r0}(s,\sigma)}=\frac{g_{0j}(s,\sigma)}{g_{00}(s,\sigma)},$$

and hence, in virtue of (22)

$$\lim_{r\to 0}\varphi_{rk}(s,\sigma)=\varphi_{0k}(s,\sigma)\qquad (k=0,1,\ldots,n),$$

uniformly in $a\leqslant\sigma\leqslant b$, $0\leqslant s\leqslant L$, $|s-\sigma|\geqslant\delta$.

It follows from lemma 3 that uniformly in $a\leqslant\sigma\leqslant b$

$$\lim_{r\to 0}\int_0^L\varphi_{rk}(s,\sigma)\,ds=\int_0^L\varphi_{0k}(s,\sigma)\,ds\qquad (k=0,1,\ldots,n). \tag{33}$$

In view of (33) we obtain the assertion (30) which completes the proof.

For further applications we generalise lemma 5 as it follows:

Lemma 5*. *Let us admit the assumptions of lemma 5 with $a>0$, $b<L$, and denote by $\Psi_0(\sigma)$ the function*

$$\Psi_0(\sigma)=\int_0^L q_0(s)\frac{M(s,\sigma)}{R(s,\sigma)}\,ds.$$

Thus the following conditions hold uniformly in $a\leqslant\sigma\leqslant b$

$$\lim_{r\to 0}\frac{d^k\Psi_r(\sigma)}{d\sigma^k}=\frac{d^k\Psi_0(\sigma)}{d\sigma^k}\qquad (k=0,1,\ldots,n).$$

Proof. It follows from (23), (24), (27) and (31) that

$$\begin{aligned}\gamma_{0j}(s,\sigma)&=M_j(s,\sigma)(s-\sigma)^2,\\ g_{0j}(s,\sigma)&=R_j(s,\sigma)(s-\sigma)^2,\\ R_0(s,\sigma)&=R(s,\sigma)\end{aligned}\qquad (j=0,1,\ldots,n), \tag{34}$$

in the domain $-\infty<s<\infty$, $a\leqslant\sigma\leqslant b$.

Let us observe that the functions $\gamma_{0j}(s,\sigma)$, $g_{0j}(s,\sigma)$, $j=0,1,\ldots,n$, are periodic in s with the period L. Hence

(35) $\gamma_{0j}(L,\sigma)=\gamma_{0j}(0,\sigma)$, $g_{0j}(L,\sigma)=g_{0j}(0,\sigma)$, $a\leqslant\sigma\leqslant b$, $j=0,1,\ldots,n$.

In view of the assumption $a>0$, $b<L$ it follows from (25), (34) and (35) that for each σ, $a\leqslant\sigma\leqslant b$

$$\frac{M_j(L,\sigma)}{R_0(L,\sigma)}=\frac{\gamma_{0j}(L,\sigma)}{g_{00}(L,\sigma)}=\frac{\gamma_{0j}(0,\sigma)}{g_{00}(0,\sigma)}=\frac{M_j(0,\sigma)}{R_0(0,\sigma)}\qquad (j=0,1,\ldots,n),$$

$$\frac{R_j(L,\sigma)}{R_0(L,\sigma)}=\frac{g_{0j}(L,\sigma)}{g_{00}(L,\sigma)}=\frac{g_{0j}(0,\sigma)}{g_{00}(0,\sigma)}=\frac{R_j(0,\sigma)}{R_0(0,\sigma)}\qquad (j=1,\ldots,n).$$

Thus all the assumptions of lemma 4 concerning the functions $q(s)$, $g(s,\sigma)$, $\gamma(s,\sigma)$ are satisfied also by the functions $q_0(s)$, $R(s,\sigma)$ and $M(s,\sigma)$ respectively.

Let $\varphi_{rk}(s,\sigma)$ $(k=0,1,\ldots,n)$ be the functions defined in the proof of lemma 5. It follows that (see (34))

$$\frac{d^k\Psi_0(\sigma)}{d\sigma^k}=\int_0^L\varphi_{0k}(s,\sigma)\,ds\qquad (k=0,1,\ldots,n),$$

and hence, in virtue of (29), we obtain from (33) the assertion of lemma 5*.

Definition. Let Σ be a simple closed curve of the class C^n (see part 1, Denotations). We say that $z=z(s)$ is a *parametric normal representation of the curve Σ* if $z(s)$ has the following properties: $z(s)$ is a function of the class C^n, it is a periodical function of period L, $|z'(s)|=1$ and $z(s_1)=z(s_2)$ for $0\leqslant s_1<s_2\leqslant L$ if and only if $s_1=0$ and $s_2=L$.

Lemma 6. *Let Σ denote a simple closed curve of the class C^n and $w_r(\zeta)$ a function defined for $\zeta\in\Sigma$ and $0\leqslant r\leqslant\varrho_0$ such that for every parametric normal representation $z=z(\sigma)$ of the curve*

$$\lim_{r\to 0}w_r[z(\sigma)]=w_0[z(\sigma)] \tag{36}$$

uniformly in the interval $L/5\leqslant\sigma\leqslant 4L/5$.

Then

$$\lim_{r\to 0}w_r(\zeta)=w_0(\zeta) \tag{37}$$

uniformly convergent as to ζ, $\zeta\in\Sigma$.

Proof. Supose $z=z(\sigma)$ is a certain parametric normal representation of the curve Σ. Let us denote by Σ_1 the arc described by the point

$$z=z(\sigma)\qquad\text{when}\qquad L/5\leqslant\sigma\leqslant 4L/5.$$

According to the hypothesis (36) we have

$$\lim_{r\to 0}w_r(\zeta)=w_0(\zeta) \tag{38}$$

and the convergence is uniform as to ζ, $\zeta\in\Sigma_1$.

Let us put

(39) $$\tau = \sigma - L/2, \quad z^*(\tau) = z(\tau + L/2).$$

It is easily seen that

$$z = z^*(\tau), \quad 0 \leqslant \tau \leqslant L,$$

is a parametric normal representation of the curve Σ also. It results therefore from our assumption that

(40) $$\lim_{r\to 0} w_r[z^*(\tau)] = w_0[z^*(\tau)]$$

and the convergence is uniform as to τ in the interval $L/5 \leqslant \tau \leqslant 4L/5$.

Let us denote by Σ_2 the arc of the curve Σ described by the point

$$z = z^*(\tau), \quad \text{when} \quad L/5 \leqslant \tau \leqslant 4L/5.$$

Now it turns out from (40) that

(41) $$\lim_{r\to 0} w_r(\zeta) = w_0(\zeta)$$

and the convergence is uniform as to ζ, $\zeta \in \Sigma_2$.

It follows, however, immediately from (39) that

$$\Sigma = \Sigma_1 \cup \Sigma_2.$$

Hence in view of (38) and (41) the relation (37) which was to be proved, follows easily.

The following condition will often be repeated in further lemmas.

CONDITION $H(h, k)$. We say that a function $p(z)$ and a curve Σ satisfy the assumption $H(h, k)$, where h, k are non-negative integers, if $p(z)$ is a function of the class C^h and Σ is a closed single curve of the class C^k.

LEMMA 7. *Let m be an arbitrary non-negative integer. We assume that a function $p(z)$ and a curve Σ satisfy $H(0, 2)$ if $m = 0$ and $H(m, m+1)$ if $m > 0$. By this assumption we have*

(42) $$\lim_{\varrho\to 0} \frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho} p(z)\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\log|z-\zeta|\,ds_{-\varrho}\Big] = 0$$

$$(k = 0, 1, \ldots, m)$$

the convergence being uniform with respect to $\zeta \in \Sigma$.

Proof. We shall prove (42) by induction. We shall show first that it holds for $m = 0$.

Let $\varrho_0 > 0$ and let $b > 0$ be a constant such that

(43) $$|p(z)| \leqslant b \quad \text{for} \quad z \in \Sigma_r, \quad |r| \leqslant \varrho_0.$$

We denote by $\Sigma_{\zeta,\delta}$ the arc of the curve Σ contained in the circle of the centre $\zeta \in \Sigma$ and the radius δ, and we denote by $\Sigma_{r,\zeta,\delta}$ the part of the curve Σ_r obtained from the translation of $\Sigma_{\zeta,\delta}$ in the normal direction.

It is easy to show that for each $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$, such that for arbitrary $\zeta \in \Sigma$ the following inequality

$$\Big|\int_{\Sigma_{r,\zeta,\delta_\varepsilon}} \log|z-\zeta|\,ds_r\Big| \leqslant \varepsilon/3b \quad (|r| \leqslant \varrho_0)$$

holds. Hence in view of (43) we obtain

(44) $$\Big|\int_{\Sigma_{r,\zeta,\delta_\varepsilon}} p(z)\log|z-\zeta|\,ds_r\Big| \leqslant \varepsilon/3 \quad (|r| \leqslant \varrho_0).$$

We note further that

$$\lim_{r\to 0} |z_r - \zeta| = |z-\zeta| \quad \text{uniformly with respect to } z \in \Sigma, \zeta \in \Sigma,$$

and

$$|z_r - \zeta| \geqslant |z-\zeta| - r \geqslant \delta_\varepsilon/2 \quad \text{for} \quad |r| \leqslant \delta_\varepsilon/2, \quad z \in \Sigma - \Sigma_{\zeta,\delta_\varepsilon}.$$

It follows that

$$\lim_{r\to 0} \log|z_r - \zeta| = \log|z-\zeta| \quad \text{uniformly in } z \in \Sigma - \Sigma_{\zeta,\delta_\varepsilon}, \zeta \in \Sigma$$

and since (see (1) and (2))

$$\lim_{r\to 0} p(z_r) = p(z), \quad \lim_{r\to 0} \frac{ds_r}{ds} = 1 \quad \text{uniformly in } z \in \Sigma$$

there exists a number ϱ_1, $0 < \varrho_1 \leqslant \varrho_0$ such that

(45) $$\Big|\int_{\Sigma_r - \Sigma_{r,\zeta,\delta_\varepsilon}} p(z)\log|z-\zeta|\,ds_r - \int_{\Sigma - \Sigma_{\zeta,\delta_\varepsilon}} p(z)\log|z-\zeta|\,ds\Big| \leqslant \varepsilon/3$$

for each $\zeta \in \Sigma$ and $|r| \leqslant \varrho_1$.

It follows from (44) and (45) that for $\zeta \in \Sigma$ and $|r| \leqslant \varrho_1$ we have the following inequality:

$$\Big|\int_{\Sigma_r} p(z)\log|z-\zeta|\,ds_r - \int_{\Sigma} p(z)\log|z-\zeta|\,ds\Big| \leqslant \varepsilon$$

and so lemma 7 has been proved in the case $m = 0$.

Let us assume that it is valid for $m = h \geqslant 0$ and assume that $p(z)$ and Σ satisfy the condition $H(h+1, h+2)$. It is easy to prove that if

$\partial\varphi/\partial s_r$ denotes the derivative of φ in the direction of the tangent to the curve Σ_r we have for $\varrho > 0$

$$\frac{\partial}{\partial s_\zeta}\Big[\int_{\Sigma_\varrho} p(z)\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\log|z-\zeta|\,ds_{-\varrho}\Big]$$
$$= \int_{\Sigma_\varrho} p(z)\Big[\frac{\partial}{\partial s_\zeta}\log|z-\zeta| + \frac{\partial}{\partial s_\varrho}\log|z-\zeta|\Big]ds_\varrho -$$
$$- \int_{\Sigma_{-\varrho}} p(z)\Big[\frac{\partial}{\partial s_\zeta}\log|z-\zeta| + \frac{\partial}{\partial s_{-\varrho}}\log|z-\zeta|\Big]ds_{-\varrho} +$$
$$+ \int_{\Sigma_\varrho} \frac{\partial p}{\partial s_\varrho}\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}} \frac{\partial p}{\partial s_{-\varrho}}\log|z-\zeta|\,ds_{-\varrho}.$$

Putting (see (1) and (2))

(46) $$q_r(s) = p\left[z(s)+r\nu\left(z(s)\right)\right]\frac{ds_r}{ds},$$

$$f_r(s,\sigma) = [x'(s)-x'(\sigma)][x(s)-ry'(s)-x(\sigma)] + $$
$$+ [y'(s)-y'(\sigma)][y(s)+rx'(s)-y(\sigma)],$$

(47) $$g_r(s,\sigma) = [x(s)-ry'(s)-x(\sigma)]^2 + [y(s)+rx'(s)-y(\sigma)]^2,$$

$$J_r(\zeta) = \int_{\Sigma_r} \frac{\partial p}{\partial s_r}\log|z-\zeta|\,ds_r,$$

where $\sigma = s_\zeta$ denotes the value of the parameter corresponding to the point ζ of the curve Σ, we can write this equation in the following form

$$\frac{\partial}{\partial s_\zeta}\Big\{\int_{\Sigma_\varrho} p(z)\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\log|z-\zeta|\,ds_{-\varrho}\Big\}$$
$$= \int_0^L q_\varrho(s)\frac{f_\varrho(s,\sigma)}{g_\varrho(s,\sigma)}\,ds - \int_0^L q_{-\varrho}(s)\frac{f_{-\varrho}(s,\sigma)}{g_{-\varrho}(s,\sigma)}\,ds + J_\varrho(\zeta) - J_{-\varrho}(\zeta).$$

Hence it follows that

(48) $$\frac{\partial^{h+1}}{\partial s_\zeta^{h+1}}\Big[\int_{\Sigma_\varrho} p(z)\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\log|z-\zeta|\,ds_{-\varrho}\Big]$$
$$= \frac{d^h}{d\sigma^h}\Big[\int_0^L q_\varrho(s)\frac{f_\varrho(s,\sigma)}{g_\varrho(s,\sigma)}\,ds - \int_0^L q_{-\varrho}(s)\frac{f_{-\varrho}(s,\sigma)}{g_{-\varrho}(s,\sigma)}\,ds\Big] +$$
$$+ \frac{\partial^h}{\partial s_\zeta^h}[J_\varrho(\zeta) - J_{-\varrho}(\zeta)].$$

Since the function $\partial p/\partial s_r$ and the curve Σ satisfy $H(h, h+2)$, in view of our inductive assumption it follows that uniformly with respect to $\zeta \in \Sigma$

(49) $$\lim_{\varrho\to 0}\frac{\partial^k}{\partial s_\zeta^k}[J_\varrho(\zeta) - J_{-\varrho}(\zeta)] = 0 \qquad (k = 0, 1, \ldots, h).$$

Let

(50) $$\Psi_r(\sigma) = \int_0^L q_r(s)\frac{f_r(s,\sigma)}{g_r(s,\sigma)}\,ds.$$

It will be proved that for each $k = 0, 1, \ldots, h$

(51) $$\lim_{\varrho\to 0}\frac{d^k}{d\sigma^k}[\Psi_\varrho(\sigma) - \Psi_{-\varrho}(\sigma)] = 0$$

uniformly with respect to σ, $L/5 \leqslant \sigma \leqslant 4L/5$.

The proof of (51) will be based on lemma 5. To this aim we note that $q_r(s)$ is a function of the class C^h which satisfies (22) with $n = h$ and is periodic with the period L, and the functions $f_r(s,\sigma)$, $g_r(s,\sigma)$

1. are of the class C^h in the set $-\infty < s < \infty$, $-\infty < \sigma < \infty$ and periodic as to s of period L;

2. they can be expressed in the form

$$f_r(s,\sigma) = M(s,\sigma)(s-\sigma)^2 + N(s,\sigma)(s-\sigma)r,$$
$$g_r(s,\sigma) = [R(s,\sigma) + 2rQ(s,\sigma)](s-\sigma)^2 + r^2,$$

where $M(s,\sigma)$, $N(s,\sigma)$, $R(s,\sigma)$, $Q(s,\sigma)$ are the functions of the class C^h in the domain $-\infty < s < \infty$, $-\infty < \sigma < \infty$, defined respectively by

$$M(s,\sigma) = \int_0^1 x''[\sigma+\tau(s-\sigma)]\,d\tau\cdot\int_0^1 x'[\sigma+\tau(s-\sigma)]\,d\tau +$$
$$+ \int_0^1 y''[\sigma+\tau(s-\sigma)]\,d\tau\cdot\int_0^1 y'[\sigma+\tau(s-\sigma)]\,d\tau,$$

$$N(s,\sigma) = x'(s)\int_0^1 y''[\sigma+\tau(s-\sigma)]\,d\tau - y'(s)\int_0^1 x''[\sigma+\tau(s-\sigma)]\,d\tau,$$

$$R(s,\sigma) = \Big\{\int_0^1 x'[\sigma+\tau(s-\sigma)]\,d\tau\Big\}^2 + \Big\{\int_0^1 y'[\sigma+\tau(s-\sigma)]\,d\tau\Big\}^2,$$

$$Q(s,\sigma) =$$
$$\int_0^1(\tau-1)\Big\{x'(s)\int_0^1 y''[s+u(s-\sigma)(\tau-1)]\,du - y'(s)\int_0^1 x''[s+u(s-\sigma)(\tau-1)]\,du\Big\}\,d\tau.$$

Let us note that $R(\sigma,\sigma)=1$ and that (see (47))

$$R(s,\sigma)(s-\sigma)^2=g_0(s,\sigma)=|z(s)-z(\sigma)|^2 .$$

It follows

$$R(s,\sigma)>0 \quad \text{for} \quad 0\leqslant s\leqslant L, \quad L/5\leqslant\sigma\leqslant 4L/5$$

and because $Q(s,\sigma)$ is bounded in the domain $0\leqslant s\leqslant L$, $0\leqslant\sigma\leqslant L$, it is easy to prove the existence of the constants $\varrho_0>0$ and $A>0$ such that

(52) $$R(s,\sigma)+2rQ(s,\sigma)\geqslant A \quad \text{for} \quad |r|\leqslant\varrho_0, \quad 0\leqslant s\leqslant L, \quad L/5\leqslant\sigma\leqslant 4L/5.$$

From these properties of the functions $g_r(s)$, $f_r(s,\sigma)$, $g_r(s,\sigma)$ and in virtue of lemma 5 with $n=h$, $a=L/5$, $b=4L/5$, and $\gamma_r(s,\sigma)=f_r(s,\sigma)$ $(P(s,\sigma)=0)$ we obtain the equations (51).

Let

(53) $$w_\varrho(\zeta)=\frac{\partial^{h+1}}{\partial s_\zeta^{h+1}}\Big[\int_{\Sigma_\varrho} p(z)\log|z-\zeta|\,ds_\varrho-\int_{\Sigma_{-\varrho}} p(z)\log|z-\zeta|\,ds_{-\varrho}\Big]-\\ -\frac{\partial^h}{\partial s_\zeta^h}[J_\varrho(\zeta)-J_{-\varrho}(\zeta)].$$

We find from (48), (50) and (53) that

$$w_\varrho[z(\sigma)]=\frac{d^h}{d\sigma^h}[\Psi_\varrho(\sigma)-\Psi_{-\varrho}(\sigma)]$$

and since (51)

(54) $$\lim_{\varrho\to 0} w_\varrho[z(\sigma)]=0$$

uniformly in the interval $L/5\leqslant\sigma\leqslant 4L/5$.

As the parametric normal representation $z=z(\sigma)$, $0\leqslant\sigma\leqslant L$ of the curve Σ was chosen arbitrarily, it follows by the relation (54) in virtue of lemma 6 that

(55) $$\lim_{\varrho\to 0} w_\varrho(\zeta)=0$$

and the convergence is uniform in ζ, $\zeta\in\Sigma$.

From (55), (49) and (53) it follows at once that

$$\lim_{\varrho\to 0}\frac{\partial^{h+1}}{\partial s_\zeta^{h+1}}\Big[\int_{\Sigma_\varrho} p(z)\log|z-\zeta|\,ds_\varrho-\int_{\Sigma_{-\varrho}} p(z)\log|z-\zeta|\,ds_{-\varrho}\Big]=0$$

uniformly as to $\zeta\in\Sigma$. This accomplishes the proof of lemma 7.

Lemma 8. *Let m be an arbitrary non-negative integer, and let us assume that if $m=0$ a function $p(z)$ and a curve Σ satisfy $H(0,1)$, and if $m\geqslant 1$ they satisfy $H(m-1,m)$.*

Then uniformly in $\zeta\in\Sigma$ we have for $k=0,1,\ldots,m$

(56) $$\lim_{\varrho\to 0}\frac{\partial^k}{\partial s_\zeta^k}\Big[\iint_{\Omega_\varrho} p(z)\log|z-\zeta|\,d\tau_z-\iint_{\Omega_{-\varrho}} p(z)\log|z-\zeta|\,d\tau_z\Big]=0 .$$

Proof. It is easy to observe that lemma 8 is true for $m=0$, $m=1$. Let us assume that it holds for $m=\mu$, $\mu\geqslant 1$, and let us assume $H(\mu,\mu+1)$. It is now sufficient to prove (56) for $k=\mu+1$ so the assertion of lemma 8 is proved for all m.

In virtue of Green's theorem we have

$$\frac{\partial}{\partial\xi}\iint_{\Omega_r} p(z)\log|z-\zeta|\,d\tau_z\\ =-\iint_{\Omega_r}\frac{\partial}{\partial x}[p(z)\log|z-\zeta|]\,d\tau_z+\iint_{\Omega_r}\frac{\partial p(z)}{\partial x}\log|z-\zeta|\,d\tau_z\\ =\int_{\Sigma_r} p(z)\cos(n_z,x)\log|z-\zeta|\,ds_r+\iint_{\Omega_r}\frac{\partial p(z)}{\partial x}\log|z-\zeta|\,d\tau_z,$$

$$\frac{\partial}{\partial\eta}\iint_{\Omega_r} p(z)\log|z-\zeta|\,d\tau_z\\ =\int_{\Sigma_r} p(z)\cos(n_z,y)\log|z-\zeta|\,ds_r+\iint_{\Omega_r}\frac{\partial p(z)}{\partial y}\log|z-\zeta|\,d\tau_z$$

and so it follows that

(57) $$\frac{\partial^{\mu+1}}{\partial s_\zeta^{\mu+1}}\Big[\iint_{\Omega_\varrho} p(z)\log|z-\zeta|\,d\tau_z-\iint_{\Omega_{-\varrho}} p(z)\log|z-\zeta|\,d\tau_z\Big]\\ =\frac{\partial^\mu}{\partial s_\zeta^\mu}\Phi_\varrho(\zeta)+\frac{\partial^\mu}{\partial s_\zeta^\mu}\Psi_\varrho(\zeta),$$

where

$$\Phi_\varrho(\zeta)\\ =x'(\zeta)\Big[\int_{\Sigma_\varrho} p(z)\cos(n_z,x)\log|z-\zeta|\,ds_\varrho-\int_{\Sigma_{-\varrho}} p(z)\cos(n_z,x)\log|z-\zeta|\,ds_{-\varrho}\Big]+\\ +y'(\zeta)\Big[\int_{\Sigma_\varrho} p(z)\cos(n_z,y)\log|z-\zeta|\,ds_\varrho-\int_{\Sigma_{-\varrho}} p(z)\cos(n_z,y)\log|z-\zeta|\,ds_{-\varrho}\Big],$$

$$\Psi_\varrho(\zeta)=x'(\zeta)\Big[\iint_{\Omega_\varrho}\frac{\partial p}{\partial x}\log|z-\zeta|\,d\tau_z-\iint_{\Omega_{-\varrho}}\frac{\partial p}{\partial x}\log|z-\zeta|\,d\tau_z\Big]+\\ +y'(\zeta)\Big[\iint_{\Omega_\varrho}\frac{\partial p}{\partial y}\log|z-\zeta|\,d\tau_z-\iint_{\Omega_{-\varrho}}\frac{\partial p}{\partial y}\log|z-\zeta|\,d\tau_z\Big].$$

It is easy to see that the pair $(p(z)\cos(n_z,x),\Sigma)$ and the pair $(p(z)\cos(n_z,y),\Sigma)$ both satisfy $H(\mu,\mu+1)$. Hence we conclude by lemma 7 that by $\zeta\in\Sigma$ we have

(58) $$\lim_{\varrho\to 0}\frac{\partial^\mu}{\partial s_\zeta^\mu}\Phi_\varrho(\zeta)=0$$

and this convergence is uniform for $\zeta\in\Sigma$.

Since the pairs $(\partial p/\partial x, \Sigma)$ and $(\partial p/\partial y, \Sigma)$ both satisfy $H(\mu-1, \mu+1)$, it follows from our inductive assumption that

$$\lim_{\varrho\to 0}\frac{\partial^\mu}{\partial s_\zeta^\mu}\Psi_\varrho(\zeta) = 0 \tag{59}$$

the convergence being uniform by $\zeta \in \Sigma$.

From (57)-(59) it follows that (56) is valid for $k = \mu+1$, which completes the proof of lemma 8.

Lemma 9. *Let a function $p(z)$ and the curve Σ satisfy the condition $H(m+1, m+1)$, whenever $m > 0$, and $H(2, 2)$, in the case $m = 0$.*

Then we have

$$\lim_{\varrho\to 0}\frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|\,ds_{-\varrho}\Big] = 2\pi\frac{\partial^k p(\zeta)}{\partial s_\zeta^k} \qquad (k = 0, 1, \ldots, m),$$

the convergence being uniform by $\zeta \in \Sigma$.

Proof. In virtue of Green's theorem we have

$$\begin{aligned}(60)\quad &\frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|\,ds_{-\varrho}\Big]\\ &= \frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho}\frac{\partial p}{\partial n_z}\log|z-\zeta|\,ds_\varrho + \iint_{\Omega_\varrho}\Delta p\log|z-\zeta|\,d\tau_z\Big] -\\ &\quad -\frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_{-\varrho}}\frac{\partial p}{\partial n_z}\log|z-\zeta|\,ds_{-\varrho} - 2\pi p(\zeta) + \iint_{\Omega_{-\varrho}}\Delta p\log|z-\zeta|\,d\tau_z\Big]\\ &= \frac{\partial^k}{\partial s_\zeta^k}K_\varrho(\zeta) + \frac{\partial^k}{\partial s_\zeta^k}L_\varrho(\zeta) + 2\pi\frac{\partial^k p(\zeta)}{\partial s_\zeta^k},\end{aligned}$$

where

$$K_\varrho(\zeta) = \int_{\Sigma_\varrho}\frac{\partial p}{\partial n_z}\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}}\frac{\partial p}{\partial n_z}\log|z-\zeta|\,ds_{-\varrho},$$

$$L_\varrho(\zeta) = \iint_{\Omega_\varrho}\Delta p\log|z-\zeta|\,d\tau_z - \iint_{\Omega_{-\varrho}}\Delta p\log|z-\zeta|\,d\tau_z.$$

It follows from lemma 7 applied to $\partial p/\partial n_z$ that for every k, $0 \leqslant k \leqslant m$ we have

$$\lim_{\varrho\to 0}\frac{\partial^k}{\partial s_\zeta^k}K_\varrho(\zeta) = 0 \tag{61}$$

the convergence being uniform for $\zeta \in \Sigma$.

Similarly, lemma 8 applied to Δp and Σ yields

$$\lim_{\varrho\to 0}\frac{\partial^k}{\partial s_\zeta^k}L_\varrho(\zeta) = 0 \tag{62}$$

which holds uniformly for $\zeta \in \Sigma$ and $k = 0, 1, \ldots, m$. Then lemma 9 follows directly by the formulae (60)-(62).

Lemma 10. *Assume that the pair $(p(z), \Sigma)$ satisfies $H(m+1, m+2)$ if $m > 0$ or $H(2, 2)$ if $m = 0$. Then we have*

$$\lim_{\varrho\to 0}\frac{\partial^k}{\partial s_\zeta^k}\Big[\int_{\Sigma_\varrho} p(z)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\varrho - \int_{\Sigma_{-\varrho}} p(z)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_{-\varrho}\Big] = -2\pi\frac{\partial^k}{\partial s_\zeta^k}p(\zeta) \qquad (k = 0, 1, \ldots, m), \tag{63}$$

the convergence being uniform for $\zeta \in \Sigma$.

Proof. Let $0 \leqslant k \leqslant m$, $0 < \varrho \leqslant \varrho_0$ and let us put

$$\begin{aligned} w_{\varrho k}(\zeta) = \frac{\partial^k}{\partial s_\zeta^k}\Big\{&\int_{\Sigma_\varrho} p(z)\Big[\frac{\partial}{\partial n_z}\log|z-\zeta| + \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\Big]ds_\varrho -\\ &- \int_{\Sigma_{-\varrho}} p(z)\Big[\frac{\partial}{\partial n_z}\log|z-\zeta| + \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\Big]ds_{-\varrho}\Big\}.\end{aligned}$$

On account of lemma 9, in order to prove lemma 10 it needs only be shown that for each k, $0 \leqslant k \leqslant m$,

$$\lim_{\varrho\to 0} w_{\varrho k}(\zeta) = 0 \tag{64}$$

the convergence being uniform for $\zeta \in \Sigma$.

It is easy to see that

$$\int_{\Sigma_r} p(z)\Big[\frac{\partial}{\partial n_z}\log|z-\zeta| + \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\Big]ds_r = \int_0^L q_r(s)\frac{h_r(s, \sigma)}{g_r(s, \sigma)}ds, \tag{65}$$

where the function $q_r(s)$ is given by (46), $g_r(s, \sigma)$ by (47), and $h_r(s, \sigma)$ is defined by the formula

$$\begin{aligned} h_r(s, \sigma) = {} & [y'(\sigma) - y'(s)]\cdot[x(s) - ry'(s) - x(\sigma)] +\\ & + [x'(s) - x'(\sigma)][y(s) + rx'(s) - y(\sigma)].\end{aligned}$$

We note further that

$$h_r(s, \sigma) = [(s-\sigma)^2 U(s, \sigma) + r(s-\sigma)V(s, \sigma)],$$

where

$$U(s,\sigma)=\int_0^1 x''[\sigma+\tau(s-\sigma)]d\tau\cdot\int_0^1 y'[\sigma+\tau(s-\sigma)]d\tau-$$
$$-\int_0^1 y''[\sigma+\tau(s-\sigma)]d\tau\cdot\int_0^1 x'[\sigma+\tau(s-\sigma)]d\tau,$$

$$V(s,\sigma)=y'(s)\int_0^1 y''[\sigma+\tau(s-\sigma)]d\tau+x'(s)\int_0^1 x''[\sigma+\tau(s-\sigma)]d\tau.$$

It follows by lemma 5 with $n=m$, $a=L/5$, $b=4L/5$ and $\gamma_r(s,\sigma)=h_r(s,\sigma)$ $\big(M(s,\sigma)=U(s,\sigma), N(s,\sigma)=V(s,\sigma), P(s,\sigma)=0\big)$ that we have for $k=0,1,\dots,m$

$$\lim_{\varrho\to0}\frac{\partial^k}{\partial\sigma^k}\int_0^L\left[q_\varrho(s)\frac{h_\varrho(s,\sigma)}{g_\varrho(s,\sigma)}-q_{-\varrho}(s)\frac{h_{-\varrho}(s,\sigma)}{g_{-\varrho}(s,\sigma)}\right]ds=0 \tag{66}$$

uniformly in the interval $L/5\leqslant\sigma\leqslant4L/5$ (see (52)).

It follows from (65) and (66) that for any k, $0\leqslant k\leqslant m$

$$\lim_{\varrho\to0}w_{\varrho k}[z(\sigma)]=0$$

uniformly as to σ in the interval $L/5\leqslant\sigma\leqslant4L/5$. The representation $z=z(\sigma)$, $0\leqslant\sigma\leqslant L$ of the curve Σ having been chosen arbitrarily, the relation (64) follows by lemma 6 and from it lemma 10.

LEMMA 11. *Assume that the pair (p,Σ) satisfies $H(m+2,m+2)$ if $m>0$, or resp. $H(3,2)$ if $m=0$. Then we have*

$$\lim_{\varrho\to0}\frac{\partial^k}{\partial s_\zeta^k}\left[\int_{\Sigma_\varrho}p(z)\frac{\partial}{\partial n_\zeta}\frac{\partial\log|z-\zeta|}{\partial n_z}ds_\varrho-\int_{\Sigma_{-\varrho}}p(z)\frac{\partial}{\partial n_\zeta}\frac{\partial\log|z-\zeta|}{\partial n_z}ds_{-\varrho}\right]=0$$
$$(k=0,1,\dots,m)$$

the convergence being uniform for $\zeta\in\Sigma$.

Proof. Let P_ϱ be the domain whose boundary consists of both curves Σ_ϱ and $\Sigma_{-\varrho}$, i.e.

$$P_\varrho=\Omega_{-\varrho}-\Omega_\varrho.$$

Assuming that the function $v(z)$ and the curve Σ satisfy the condition $H(m,m+1)$ we shall prove that we have the uniform convergence

$$\lim_{\varrho\to0}\frac{\partial^k}{\partial s_\zeta^k}\iint_{P_\varrho}v(z)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|d\tau_z=0\qquad(k=0,1,\dots,m). \tag{67}$$

In the case $m=0$ the validity of (67) is evident. If $m>0$ it follows from lemmas 7 and 8 and from the following formula

$$\iint_{P_\varrho}v(z)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|d\tau_z$$
$$=\iint_{P_\varrho}v(z)\left[-y'(\zeta)\frac{\partial}{\partial\xi}\log|z-\zeta|+x'(\zeta)\frac{\partial}{\partial\eta}\log|z-\zeta|\right]d\tau_z$$
$$=\iint_{P_\varrho}v(z)\left[y'(\zeta)\frac{\partial}{\partial x}\log|z-\zeta|-x'(\zeta)\frac{\partial}{\partial y}\log|z-\zeta|\right]d\tau_z$$
$$=-y'(\zeta)\left[\int_{\Sigma_{-\varrho}}p_1(z)\log|z-\zeta|ds_{-\varrho}-\int_{\Sigma_\varrho}p_1(z)\log|z-\zeta|ds_\varrho\right]-$$
$$-y'(\zeta)\iint_{P_\varrho}w_1(z)\log|z-\zeta|d\tau_z+x'(\zeta)\iint_{P_\varrho}w_2(z)\log|z-\zeta|d\tau_z+$$
$$+x'(\zeta)\left[\int_{\Sigma_{-\varrho}}p_2(z)\log|z-\zeta|ds_{-\varrho}-\int_{\Sigma_\varrho}p_2(z)\log|z-\zeta|ds_\varrho\right],$$

where

$$p_1(z)=v(z)\cos(n_z,x),\qquad p_2(z)=v(z)\cos(n_z,y),$$
$$w_1(z)=\frac{\partial v(z)}{\partial x},\qquad w_2(z)=\frac{\partial v(z)}{\partial y}.$$

In order to prove lemma 11 we first note that due to Green's theorem we have

$$\int_{\Sigma_\varrho}p(z)\frac{\partial}{\partial n_\zeta}\frac{\partial}{\partial n_z}\log|z-\zeta|ds_\varrho-\int_{\Sigma_{-\varrho}}p(z)\frac{\partial}{\partial n_\zeta}\frac{\partial}{\partial n_z}\log|z-\zeta|ds_{-\varrho}$$
$$=\frac{\partial}{\partial n_\zeta}\left\{\int_{\Sigma_\varrho}p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|ds_\varrho-\int_{\Sigma_{-\varrho}}p(z)\frac{\partial}{\partial n_z}\log|z-\zeta|ds_{-\varrho}\right\}$$
$$=\frac{\partial}{\partial n_\zeta}\left\{\int_{\Sigma_\varrho}\frac{\partial p}{\partial n_z}\log|z-\zeta|ds_\varrho-\right.$$
$$\left.-\int_{\Sigma_{-\varrho}}\frac{\partial p}{\partial n_z}\log|z-\zeta|ds_{-\varrho}+2\pi p(\zeta)-\iint_{P_\varrho}\Delta p(z)\log|z-\zeta|d\tau_z\right\}$$
$$=\int_{\Sigma_\varrho}\frac{\partial p}{\partial n_z}\cdot\frac{\partial}{\partial n_\zeta}\log|z-\zeta|ds_\varrho-\int_{\Sigma_{-\varrho}}\frac{\partial p}{\partial n_z}\cdot\frac{\partial}{\partial n_\zeta}\log|z-\zeta|ds_{-\varrho}+$$
$$+2\pi\frac{\partial p(\zeta)}{\partial n_\zeta}-\iint_{P_\varrho}\Delta p\frac{\partial}{\partial n_\zeta}\log|z-\zeta|d\tau_z.$$

Thus in virtue of lemma 10 and (67) with $v = \Delta p$, lemma 11 immediately follows.

Our main theorem is a direct consequence of the above proved lemmas. Indeed, from lemma 2 it follows immediately that this theorem is a consequence of (11), (12), (13), and (14), and these, in turn, follow from lemmas 7, 10, 9, and 11 respectively.

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On the asymptotic coincidence of sets filled up by integrals of two systems of ordinary differential equations

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Introduction. In many papers concerning the asymptotic behaviour of solutions of ordinary differential equations the following problem has been considered.

One has a system of differential equations

$$dy/dt = F(y,t) + \varepsilon(y,t) \tag{0,1}$$

(y is a vector $y_1, \ldots, y_n$, t is real variable, $F(y,t)$ and $\varepsilon(y,t)$ are vector-functions) which arose from the perturbation of the system

$$dx/dt = F(x,t)\,. \tag{0,2}$$

The behaviour of solutions of (0,2) is supposed to be known by some means (often system (0,2) is a linear one) and the perturbation $\varepsilon(y,t)$ becomes small as $t \to +\infty$. The problem consists in establishing, under the appropriate assumptions concerning the perturbation, asymptotic relations between the solutions of (0,1) and those of (0,2). More exactly, one wishes to establish that for every solution $x(t)$ of (0,2) there is a solution $y(t)$ of (0,1) which is, what we may call "asymptotically near" to $x(t)$ (as $t \to +\infty$). Of course the term "asymptotically near" has different meanings according to the aims we have in particular considerations.

For instance, we may say that $x(t)$ is asymptotically near to $y(t)$ if their characteristic numbers are equal, i.e. if

$$\limsup_{t\to+\infty} (\ln|y(t)|/t) = \limsup_{t\to+\infty} (\ln|x(t)|/t) \tag{0,3}$$

(see [2] and [4]), or if the following condition is satisfied

$$y(t) = x(t) + \eta(t) \quad \text{where} \quad |\eta(t)| = o(|x(t)|) \tag{0,4}$$

(see [9]) or, in the case (0,2) is a linear system, $|\eta(t)| = o(t^{\mu} e^{\omega t})$ where μ and ω are constants determined by $x(t)$ (see [3]).