ANNALES
POLONICI MATHEMATICI
XI (1961)

Concerning an expansion formula for a type of integrals

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The object of this paper is to investigate some conditions ensuring the validity of an expansion formula for integrals of the form

$$I(\lambda) = \int_0^c \Phi(\lambda t) f(t)\, dt,$$

where c may be finite or infinite, $\Phi(t)$ is a real piecewise continuous function defined on $[0, \infty)$, $f(t)$ is a function having derivatives of all orders, and λ represents a positive parameter whose value is often very large in practical problems. Finally, a certain approximation method for evaluating $I(\lambda)$ will be sketched.

1. In what follows we always assume that the Laplace transform $\Psi(s)$ of the function $\Phi(t)$ has an abscissa of convergence $s_c \leqslant 0$. Evidently this assumption is satisfied by many familiar functions of practical importance, e.g. $\Phi(t) = e^{-t}$, $\Phi(t) = e^{-t^2}$, $\Phi(t) = \cos t$, $\Phi(t) = \sin t$, $\Phi(t) = (\sin t)/t$ $(t > 0)$, $\Phi(t) = J_0(t)$, $\Phi(t) = J_1(t)$ (the Bessel functions), etc.

First let us prove the following:

Theorem 1. *Let $f(t) = \sum_0^\infty c_n t^n$ be an entire function such that both $\Phi(\lambda t) f(t)$ and $e^{-st}|\Phi(\lambda t)| \sum_0^\infty |c_n| t^n$ are integrable (in the sense of Riemann) over $[0, \infty)$, where $s > 0$ is arbitrary. Then*

$$\int_0^\infty \Phi(\lambda t) f(t)\, dt = \lim_{s\to 0+} \sum_0^\infty \frac{1}{n!} (-1)^n \Psi^{(n)}(s) f^{(n)}(0) \left(\frac{1}{\lambda}\right)^{n+1}. \tag{1}$$

In particular, if $|\Phi(\lambda t)| \sum_0^\infty |c_n| t^n$ is integrable over $[0, \infty)$, then

$$\int_0^\infty \Phi(\lambda t) f(t)\, dt = \sum_0^\infty \frac{1}{n!} (-1)^n \Psi^{(n)}(0) f^{(n)}(0) \left(\frac{1}{\lambda}\right)^{n+1}. \tag{2}$$

Proof. For each fixed $\lambda > 0$ and $s \geqslant 0$, it is easily verified that the series $\sum_0^\infty c_n \Phi(\lambda t) t^n \cdot e^{-st}$ converges uniformly for all values of t in any finite interval $[0, R]$. Moreover, the integrability condition imposed on $e^{-st}\sum_0^\infty |c_n||\Phi(\lambda t)|\cdot t^n$ ensures that the general theorem for term-by-term integration is applicable to the case of the following (with $s > 0$)

$$\int_0^\infty \Big\{\sum_0^\infty c_n \Phi(\lambda t) t^n \cdot e^{-st}\Big\}\, dt = \sum_0^\infty \int_0^\infty c_n \Phi(\lambda t) t^n \cdot e^{-st} dt. \tag{3}$$

By the analytic character of the Laplace integral $\Psi(s) = \int_0^\infty e^{-st}\Phi(t)\,dt$, we know that $\Psi(s)$ is analytic for $s > 0 \geqslant s_c$, and we have (with $s > 0$)

$$\Psi^{(n)}\left(\frac{s}{\lambda}\right) = \int_0^\infty \Phi(\lambda t)(-\lambda t)^n e^{-st} d(\lambda t). \tag{4}$$

Consequently we get, by making use of (3),

$$\lim_{s\to 0+} \sum_0^\infty \frac{1}{n!}(-1)^n f^{(n)}(0)\, \Psi^{(n)}\left(\frac{s}{\lambda}\right)\left(\frac{1}{\lambda}\right)^{n+1} = \lim_{s\to 0+} \sum_0^\infty \int_0^\infty c_n \Phi(\lambda t) t^n e^{-st} dt$$
$$= \lim_{s\to 0+} \int_0^\infty \Phi(\lambda t) f(t) e^{-st} dt = \int_0^\infty \Phi(\lambda t) f(t)\, dt.$$

Here the last equality is actually obtained by the analogue for integrals of Abel's theorem on power series. In fact, the integral $\int_0^\infty \Phi(\lambda t) f(t) e^{-st} dt$ is uniformly convergent in $0 \leqslant s \leqslant \delta$ $(\delta > 0)$ under the assumption that $\Phi(\lambda t) f(t)$ is integrable over $[0, \infty)$. Hence we have shown that the left-hand side of (1) can be deduced from the right-hand side.

Moreover, if $|\Phi(\lambda t)|\cdot \sum_0^\infty |c_n| t^n$ is integrable, so is $\Phi(t)\cdot t^n$ for every $n \geqslant 0$, and consequently the relations (3) and (4) are valid for $s = 0$. Hence, using (3) and (4) with $s = 0$, we may again deduce the left-hand side of (2) from the right-hand side.

To see that the integrability conditions concerning $e^{-st}|\Phi(\lambda t)|\cdot \sum_0^\infty |c_n| t^n$ and $\Phi(\lambda t)\cdot f(t)$ do not imply each other one needs only to consider the examples

$$\Phi(\lambda t) = \sin(\lambda t), \qquad f(t) = e^{-t};$$
$$\Phi(\lambda t) = \sin(\lambda t), \qquad f(t) = 1.$$

The formula (2) was first derived formally by H. F. Willis [8] (cf. C. J. Tranter [7], § 5.3) without investigating convergence conditions. Various examples may also be found in [7] and [8]. As a simple example, we may take $\Phi(\lambda t) = e^{-\lambda^2 t^2}$, $f(t) = \cos t$ so that (2) gives

$$\int_0^\infty e^{-\lambda^2 t^2} \cos t\, dt = \frac{\sqrt{\pi}}{2\lambda} \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{-1}{4}\right)^k \left(\frac{1}{\lambda}\right)^{2k} = \frac{\sqrt{\pi}}{2\lambda} e^{-1/4\lambda^2},$$

which is known to be also obtainable by use of Cauchy's residue theorem.

Corollary. *Let the Laplace transform of $|\Phi(t)|$ have a non-positive abscissa of convergence. Then the formula (1) is valid for any entire function $f(t)$ of finite order $\varrho < 1$ such that $\Phi(\lambda t) f(t)$ is integrable over $[0, \infty)$.*

Proof. Since the entire function $f(t) = \sum_0^\infty c_n t^n$ is of finite order ϱ, so is the function $g(t) = \sum_0^\infty |c_n|\cdot t^n$ (see, e.g. Titchmarsh [6], § 8.3). Consequently we have $g(t) = O\big(\exp(t^{\varrho+\varepsilon})\big)$ $(t\to\infty)$ for every positive value of ε. Take ε so small that $\varrho + \varepsilon < 1$. Then $O\big(\exp(-st)\cdot \exp(t^{\varrho+\varepsilon})\big) = O(e^{-\sigma t})$ with $0 < \sigma < s$. Hence it follows that $e^{-st}|\Phi(\lambda t)| g(t) = O(e^{-\sigma t})\cdot |\Phi(\lambda t)|$ is integrable over $[0, \infty)$. The corollary is therefore implied by Theorem 1.

We have not yet known whether the condition $\varrho < 1$ of the corollary can be improved to $\varrho \leqslant 1$. The following result seems sharper, though it does not give a complete answer to the question just mentioned.

Theorem 2. *Let both the Laplace transforms of $\Phi(t)$ and $[\Phi(t)]^2$ have non-positive abscissae of convergence. Then the formula (1) is valid for any entire function $f(t) = \sum_0^\infty c_n t^n$ such that $\Phi(\lambda t) f(t)$ is integrable over $[0, \infty)$ and that $c_n = O\big((n\cdot\gamma_n)^{-n}\big)$ with γ_n increasing to $+\infty$ as $n\to\infty$.*

In the statement of Theorem 2 the number γ_n may tend to $+\infty$ very slowly with n, e.g. $\gamma_n = \log n$, $\gamma_n = \log\log n$.

Proof. It suffices to show that, for each fixed $s > 0$, we have

$$\int_0^\infty \Phi(\lambda t) f(t) e^{-st} dt = \sum_0^\infty \frac{1}{n!}(-1)^n \Psi^{(n)}\left(\frac{s}{\lambda}\right) f^{(n)}(0) \left(\frac{1}{\lambda}\right)^{n+1}. \tag{5}$$

By Abel's method or the second mean-value theorem we evidently have

$$\int_0^\infty \Phi(\lambda t) f(t) e^{-st} dt = \int_0^N \Phi(\lambda t) f(t) e^{-st} dt + e^{-sN}\cdot \xi_N, \tag{6}$$

where $N > 0$ and

$$\inf_{N\leqslant x<\infty} \int_N^x \Phi(\lambda t) f(t)\,dt \leqslant \xi_N \leqslant \sup_{N\leqslant x<\infty} \int_N^x \Phi(\lambda t) f(t)\,dt\,,$$

so that $\xi_N = o(1)$ $(N\to\infty)$.

Notice that $\Phi(\lambda t) f(t) = \sum_0^\infty c_n t^n \Phi(\lambda t)$ is uniformly convergent in any finite interval $[0, N]$. Thus we have

$$\int_0^N \Phi(\lambda t) f(t) e^{-st}dt = \sum_0^\infty \int_0^N c_n t^n \Phi(\lambda t) e^{-st} dt \tag{7}$$
$$= \sum_0^\infty c_n \cdot \int_0^\infty e^{-st} t^n \Phi(\lambda t)\,dt + \sum_0^\infty c_n \cdot \delta_n\,,$$

where the convergence of $\int_0^\infty e^{-st} t^n \Phi(\lambda t)\,dt$ is ensured by the analyticity of the Laplace transform (cf. (4)), and δ_n is defined by

$$\delta_n = -\int_N^\infty e^{-st} t^n \Phi(\lambda t)\,dt \qquad (n = 0, 1, 2, \ldots).$$

We now proceed to prove $\sum_0^\infty c_n \cdot \delta_n = o(1)$ $(N\to\infty)$. By use of Buniakowski's inequality and Stirling's formula we may estimate $|\delta_n|$ as follows

$$|\delta_n| \leqslant \left(\int_N^\infty e^{-st} t^{2n} dt\right)^{1/2} \left(\int_N^\infty e^{-st} [\Phi(\lambda t)]^2 dt\right)^{1/2}$$
$$\leqslant \left(\frac{(2n)!}{s^{2n+1}}\right)^{1/2} \cdot \left(\frac{1}{\lambda} \int_{\lambda N}^\infty e^{-su/\lambda} [\Phi(u)]^2 du\right)^{1/2}$$
$$\leqslant \frac{1}{s^n \cdot \sqrt{s}} \left(\frac{2n}{e}\right)^n (4\pi n)^{1/4} \left(1 + \frac{1}{n}\right) \cdot o(1) \qquad (N\to\infty)$$

where the last inequality holds for all sufficiently large n, and the factor $o(1)$ (independent of n) is implied by the assumption that $[\Phi(t)]^2$ has a Laplace transform with a non-positive convergence-abscissa. Consequently we obtain

$$\left|\sum_0^\infty c_n \cdot \delta_n\right| \leqslant \sum_0^\infty |c_n| \cdot \left(\frac{2n}{s\cdot e}\right)^n \cdot n^{1/4} \cdot o(1)$$
$$= o(1) \cdot \sum_0^\infty \left(\frac{2}{s e \gamma_n}\right)^n \cdot n^{1/4} = o(1) \qquad (N\to\infty)$$

in view of the fact that $\sum_0^\infty (2/s e \gamma_n)^n \cdot n^{1/4} < +\infty$.

Finally, comparing (6) with (7), we obtain (3) by letting $N\to\infty$. Since (3) is equivalent to (5) our proof is complete.

2. For the case $I(\lambda)$ being a definite integral (i.e. $0 < c < +\infty$), the formula (2) is valid under much weaker hypotheses. In fact we have

THEOREM 3. *Let $\Psi(s)$ be the Laplace transform of $\Phi(t)$, where $\Phi(t) = 0$ for $t \geqslant K > 0$. Then for any function $f(z)$ which is analytic in a region containing $|z| \leqslant c$ we have*

$$\int_0^c \Phi(\lambda t) f(t)\,dt = \sum_0^\infty \frac{1}{n!} (-1)^n \Psi^{(n)}(0) f^{(n)}(0) \left(\frac{1}{\lambda}\right)^{n+1}, \tag{8}$$

provided that $\lambda c \geqslant K$.

Proof. Since $\Phi(t) = 0$ for $t \geqslant K$ we see that the convergence-abscissa for $\Psi(s)$ is $s_c = -\infty$. Moreover, the series $\sum_0^\infty \frac{1}{n!} f^{(n)}(0) t^n \Phi(\lambda t)$ is uniformly convergent for all values of t in $0 \leqslant t \leqslant c$. Hence by the term-by-term integration we have

$$\int_0^c \Phi(\lambda t) f(t)\,dt = \sum_0^\infty \frac{1}{n!} f^{(n)}(0) \int_0^c \Phi(\lambda t) t^n dt$$

which is precisely equivalent to (8) in view of the fact that (cf. (4))

$$(-1)^n \int_0^c \Phi(\lambda t) t^n dt = \left(\frac{1}{\lambda}\right)^{n+1} \int_0^\infty \Phi(u)(-u)^n du = \left(\frac{1}{\lambda}\right)^{n+1} \cdot \Psi^{(n)}(0)\,.$$

3. It is known that some approximation methods for evaluating integrals of rapidly oscillating functions of the form $\Phi(\lambda t) f(t)$ have already been investigated by Filon [2], Erugin-Sobolev [1], Krylov [4], Longman [5] and the author himself [3], etc., respectively. Here, basing upon the formula (2) or (8), we may propose another approximation method for evaluating the integral $I(\lambda)$ (λ being a large parameter).

Suppose that we want to construct an approximation formula without using the derivatives $f^{(n)}(0)$. Naturally we have to replace $f^{(n)}(0)$ by their approximate values on using certain numerical differentiation formulas. Denoting $\Delta f(x) = f(x+h) - f(x)$, $\Delta^{n+1} = \Delta\Delta^n$, we know that there is a useful formula due to Markoff, viz.

$$h^n f^{(n)}(x) = \sum_{k=n}^m \frac{n}{(k-n)!\,k} B_{k-n}^{(k)} \cdot \Delta^k f(x) + \varepsilon_m\,,$$

where $\varepsilon_m = O(h^{m+1})$ in case $f^{(m+1)}(x)$ exists and is continuous, and $B_\nu^{(k)}$ are Bernoulli's numbers of order k given by the generating function

$$\frac{t^k}{(e^k-1)^k} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(k)}.$$

Thus, if the parameter λ is large, then we may take, for instance, $h = 1/\lambda$, and construct an approximation formula as follows

$$\int_0^c \Phi(\lambda t) f(t)\,dt \approx \frac{1}{\lambda}\sum_{n=0}^{m}\frac{1}{n!}(-1)^n \Psi^{(n)}(0)\cdot A_n, \tag{9}$$

where the numbers A_n and $\Psi^{(n)}(0)$ are given by $(A_0 = f(0))$

$$A_n = \sum_{k=n}^{m+r} \frac{n}{(k-n)!\,k} B_{k-n}^{(k)}\cdot \Delta^k f(0) \qquad (n = 1, 2, \ldots, m), \tag{10}$$

$$\Psi^{(n)}(0) = \int_0^{\lambda c} \Phi(t)(-t)^n\,dt \qquad (n = 0, 1, \ldots, m) \tag{11}$$

respectively, the number r being a non-negative integer chosen to be fixed.

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Reçu par la Rédaction le 27. 6. 1960

ANNALES
POLONICI MATHEMATICI
XI (1961)

О функциях $\varphi_2(n)$, $\mu_2(n)$, $\zeta_2(s)$

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§ 1. Рассмотрим следующие обобщения числовых функций Эйлера и Мёбиуса.

Пусть функция $\varphi_2(n)$ выражает число пар натуральных чисел a_1, a_2, с условиями $a_2 - a_1 = 2$, $(a_1, n) = 1$, $(a_2, a_n) = 1$, $a_1 \leqslant n$. Легко доказать, что $\varphi_2(n)$ мультипликативная функция и что при n нечётном:

$$\varphi_2(n) = n \prod_{p|n}\left(1-\frac{2}{p}\right), \tag{1}$$

где $p > 2$ простое число. Если n чётное, то

$$\varphi_2(n) = \frac{1}{2} n \prod_{p|n}\left(1-\frac{2}{p}\right), \qquad p > 2 \quad \text{простое}. \tag{2}$$

Введём функцию $\mu_2(n)$, определяемую равенствами:

$$\mu_2(n) = \begin{cases} (-1)^{k+1}\cdot 2^k, & \text{если} \quad n = 2p_1p_2\ldots p_k, \quad p > 2, \\ (-2)^k, & \text{если} \quad n = p_1p_2\ldots p_k, \quad p > 2, \\ \mu(n) & \text{для остальных натуральных } n. \end{cases} \tag{3}$$

Это определение можно получить, рассматривая функцию типа $\zeta(s)$. Пусть:

$$\zeta_2(s) = \left(1-\frac{1}{2^s}\right)^{-1} \prod_{p>2}\left(1-\frac{2}{p^s}\right)^{-1}, \tag{4}$$

где $s = \sigma + it$, произведение распространяется на все простые $p > 2$.

Запишем $\zeta_2(s)$ в виде ряда Дирихле, для чего введём ещё функцию $\Delta(n)$:

$$\Delta(n) = \begin{cases} 0, & \text{если} \quad n = 1, \\ \alpha+\beta+\ldots+\lambda, & \text{если} \quad n = 2^{\delta}p_1^{\alpha}p_2^{\beta}\ldots p_k^{\lambda}, \quad p_i > 2. \end{cases} \tag{5}$$

Тогда

$$\zeta_2(s) = \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\ldots\right)\prod_{p>2}\left(1+\frac{2}{p^s}+\frac{2^2}{p^{2s}}+\ldots\right) = \sum_{n=1}^{\infty}\frac{2^{\Delta(n)}}{n^s}.$$