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On some properties of analytic functions

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Introduction. We consider an analytic function $f(w_0, w_1, \ldots, w_n)$ of $n+1$ complex variables which is defined in the domain defined by

$$(1) \qquad 0 \leqslant |w_i| < b_i, \quad i = 0, 1, \ldots, n,$$

b_i being positive constants. We put

$$(2) \qquad H(s_0, s_1, s_2, \ldots, s_n) = \max_{|w_0|=s_0, |w_1|=s_1, \ldots, |w_n|=s_n} |f(w_0, w_1, \ldots, w_n)|.$$

The object of this paper is the demonstration of the following:

THEOREM. *If $s_0, s_1, \ldots, s_n$ are contained in a cube*

$$(3) \qquad 0 \leqslant s_i < b_i$$

(b_i being the same as in (1)) then the right-hand partial derivatives

$$\left(\frac{\partial H(s_0, s_1, \ldots, s_n}{\partial s_i}\right)_+, \quad i = 0, 1, 2, \ldots, n,$$

exist and for the arbitrarily chosen $s_0, s_1, \ldots, s_n$ belonging to cube (3) there exist points $\overline{w}_0^i, \overline{w}_1^i, \ldots, \overline{w}_n^i$ $(i = 0, 1, \ldots, n)$ such that

$$|\overline{w}_0^i| = s_0, \quad |\overline{w}_1^i| = s_1, \quad |\overline{w}_2^i| = s_2, \quad \ldots, \quad |\overline{w}_n^i| = s_n, \quad i = 0, 1, \ldots, n,$$

$$H(s_0, s_1, s_2, \ldots, s_n) = |f(\overline{w}_0^i, \overline{w}_1^i, \ldots, \overline{w}_n^i)|, \quad i = 0, 1, \ldots, n,$$

$$\left(\frac{\partial H(s_0, s_1, \ldots, s_n)}{\partial s_i}\right)_+ = |f'_{w_i}(\overline{w}_0^i, \overline{w}_1^i, \ldots, \overline{w}_n^i)|, \quad i = 0, 1, \ldots, n.$$

This theorem is useful for the evaluation of the solution $w_i(z)$ $(i = 1, 2, \ldots, n)$ of the system

$$\frac{dw_i}{dz} = f_i(z, w_1, w_2, \ldots, w_n), \quad i = 1, 2, \ldots, n,$$

with the initial condition:

$$w_i(0) = 0, \quad i = 1, 2, \ldots, n.$$

The functions $f_i(z, w_1, w_2, \ldots, w_n)$ $(i = 1, 2, \ldots, n)$ in the right-hand members are analytic in an $(n+1)$-circular domain (cf. [2]). Our theorem is also useful for the evalution of the radius of existence of the solution.

§ 1. LEMMA. *Assume that the complex function $f(w_0, w_1, \ldots, w_n)$ of the complex variables $w_0, w_1, \ldots, w_n$ is analytic in an $(n+1)$-circular domain* (1). *We put*

$$H(s_0, s_1, s_2, \ldots, s_n) = \max_{|w_0|=s_0, |w_1|=s_1, \ldots, |w_n|=s_n} |f(w_0, w_1, w_2, \ldots, w_n)| .$$

Then $H(s_0, s_1, \ldots, s_n)$ is a real function of real variables, it is continuous and does not decrease in cube (2).

There are no difficulties in the proof.

LEMMA 2. *Let $g(z_1, z_2, \ldots, z_n)$ be an analytic function in neighbourhood of $(a_1, a_2, \ldots, a_n)$. If we have two sequences of points $\{z_i^\nu\}$, $\{\xi_i^\nu\}$, $i = 1, 2, \ldots, n$, $\nu = 1, 2, \ldots$, such that $z_i^\nu \neq \xi_i^\nu$ and for every i we have*

$$\lim_{\nu\to\infty} z_i^\nu = \lim_{\nu\to\infty} \xi_i^\nu = a_i ,$$

then

$$(4)_i \qquad \frac{g(z_1^\nu, z_2^\nu, \ldots, z_{i-1}^\nu, z_i^\nu, z_{i+1}^\nu, \ldots, z_n^\nu) - g(z_1^\nu, \ldots, z_{i-1}^\nu, \xi_i^\nu, z_{i+1}^\nu, \ldots, z_n^\nu)}{z_i^\nu - \xi_i^\nu} \underset{\nu\to\infty}{\to} g'_{z_i}(a_1, a_2, \ldots, a_n) , \quad i = 1, 2, \ldots, n .$$

Proof. Since $g(z_1, z_2, \ldots, z_n)$ is analytic, we have

$$g(z_1^\nu, z_2^\nu, \ldots, z_{i-1}^\nu, z_i^\nu, z_{i+1}^\nu, \ldots, z_n^\nu) - g(z_1^\nu, \ldots, z_{i-1}^\nu, \xi_i^\nu, z_{i+1}^\nu, \ldots, z_n^\nu) = \int_{\xi_i^\nu}^{z_i^\nu} g_{z_i^\nu}(z_1^\nu, \ldots, z_{i-1}^\nu, \eta, z_{i+1}^\nu, \ldots, z_n^\nu)\, d\eta$$

where the path of integration is the segment $[\xi_i^\nu, z_i^\nu]$. This equality yields our thesis in a simple manner.

LEMMA 3. *If $g(z_1, z_2, \ldots, z_n)$ is analytic in a neighbourhood of $(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$, then for every sequence of positive numbers $\{k_i^\nu\}$ which converges to 0 $(i = 1, 2, \ldots, n)$ there exists a sequence of points $\{z_i^\nu\}$ such that*

$$|z_i^\nu - \bar{z}_i| \leqslant k_i^\nu , \quad i = 1, 2, \ldots, n ,$$

and

$$\frac{|g(\bar{z}_1, \ldots, \bar{z}_{i-1}, z_i^\nu, \bar{z}_{i+1}, \ldots, \bar{z}_n)| - |g(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)|}{|z_i^\nu - \bar{z}_i|} \underset{\nu\to\infty}{\to} |g'_{z_i}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)| .$$

Proof. We put

$$(5)_1 \qquad \varphi_i(z_i) = g(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{i-1}, z_i, \bar{z}_{i+1}, \ldots, \bar{z}_n) , \quad i = 1, 2, \ldots, n ,$$

$$(5)_2 \qquad l_i(z_i) = \varphi_i(\bar{z}_i) + \varphi'_i(\bar{z}_i)\cdot(z_i - \bar{z}_i) , \quad i = 1, 2, \ldots, n .$$

Since $g(z_1, z_2, \ldots, z_n)$ is analytic in a neighbourhood of $(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$, therefore $\varphi_i(z_i)$ is analytic in a neighbourhood of $\bar{z}_i$. Then by means of the Taylor expansion we have

$$(6) \qquad \varphi_i(z_i) = l_i(z_i) + \varepsilon_i(z_i)\cdot(z_i - \bar{z}_i) , \quad i = 1, 2, \ldots, n ,$$

where $\varepsilon_i(z_i)$, $i = 1, 2, \ldots, n$, are continuous at $z_i = \bar{z}_i$ and $\varepsilon_i(\bar{z}_i) = 0$. In view of a theorem of T. Ważewski (cf. [1], p. 222, § 3, lemma 1) for every $k_i^\nu > 0$ there exists a point z_i^ν $(i = 1, 2, \ldots, n)$ such that

$$|z_i^\nu - \bar{z}_i| = k_i^\nu , \quad i = 1, 2, \ldots, n ,$$

$$|l_i(z_i^\nu)| - |l_i(\bar{z}_i)| = |l'_i(\bar{z}_i)|\cdot(z_i^\nu - \bar{z}_i) = |\varphi'_i(\bar{z}_i)|\cdot|z_i^\nu - \bar{z}_i| , \quad i = 1, 2, \ldots, n .$$

Now we show that z_i^ν is the desired sequence. By (6) we have

$$\varphi_i(z_i^\nu) = l_i(z_i^\nu) + \varepsilon_\nu(z_i^\nu)(z_i^\nu - \bar{z}_i) , \quad i = 1, 2, \ldots, n ,$$

and thus

$$|\varphi_i(z_i^\nu)| - |\varphi_i(\bar{z}_i)| = |l_i(z_i^\nu) + \varepsilon_i(z_i^\nu)(z_i^\nu - \bar{z}_i)| - |l_i(\bar{z}_i)| \geqslant |l_i(z_i^\nu)| - |l_i(\bar{z}_i)| - |\varepsilon_i(z_i^\nu)||z_i^\nu - \bar{z}_i| ,$$

$$\frac{|\varphi_i(z_i^\nu)| - |\varphi_i(\bar{z}_i)|}{|z_i^\nu - \bar{z}_i|} \geqslant \frac{|l_i(z_i^\nu)| - |l_i(\bar{z}_i)|}{|z_i^\nu - \bar{z}_i|} - |\varepsilon_i(z_i^\nu)| , \quad i = 1, 2, \ldots, n ,$$

and in the limit we obtain

$$\liminf \frac{|\varphi_i(z_i^\nu)| - |\varphi_i(\bar{z}_i)|}{|\bar{z}_i - z_i^\nu|} \geqslant |\varphi'_i(\bar{z}_i)| , \quad i = 1, 2, \ldots, n .$$

We find

$$\limsup \frac{|\varphi_i(z_i^\nu)| - |\varphi_i(\bar{z}_i)|}{|z_i^\nu - \bar{z}_i|} , \quad i = 1, 2, \ldots, n .$$

For every sequence $\{z_i^\nu\}$ $(i = 1, 2, \ldots, n)$, $z_i^\nu \neq \bar{z}_i$, $z_i^\nu \to \bar{z}_i$, $i = 1, 2, \ldots, n$, we have the inequality

$$\frac{|\varphi_i(z_i^\nu)| - |\varphi_i(\bar{z}_i)|}{|z_i^\nu - \bar{z}_i|} \leqslant \frac{|\varphi_i(z_i^\nu) - \varphi_i(\bar{z}_i)|}{|z_i^\nu - \bar{z}_i|} \to |\varphi'_i(\bar{z}_i)| , \quad i = 1, 2, \ldots, n .$$

Hence

$$\limsup \frac{|\varphi_i(z_i^\nu)| - |\varphi_i(\bar{z}_i)|}{|z_i^\nu - \bar{z}_i|} \leqslant |\varphi_i(\bar{z}_i)| , \quad i = 1, 2, \ldots, n ,$$

and consequently

$$\frac{|\varphi_i(z_i^\nu)|-|\varphi_i(\bar z_i)|}{|z_i^\nu-\bar z_i|}\underset{\nu\to\infty}{\to}|\varphi_i'(\bar z_i)|\,,\qquad i=1,2,\ldots,n\,.$$

In view of the definition of $\varphi_i(z_i)$ we obtain

$$\frac{|g(\bar z_1,\bar z_2,\ldots,\bar z_{i-1},z_i^\nu,\bar z_{i+1},\ldots,\bar z_n)|-|g(\bar z_1,\bar z_2,\ldots,\bar z_n)|}{|z_i^\nu-\bar z_i|}\underset{\nu\to\infty}{\to}|g'_{z_i}(\bar z_1,\bar z_2,\ldots,\bar z_n)|\,,\quad i=1,2,\ldots,n\,.$$

§ 2. Proof of the theorem. We denote by $C(s_0,s_1,\ldots,s_n)$ a set of points $(w_0,w_1,\ldots,w_n)$ defined by the inequalities:

$$|w_0|=s_0\,,\qquad |w_1|=s_1\,,\qquad |w_2|=s_2\,,\qquad \ldots\,,\qquad |w_n|=s_n\,.$$

We denote by $A(s_0,s_1,\ldots,s_n)$ a set of points $(w_0,w_1,\ldots,w_n)$ contained in $C(s_0,s_1,\ldots,s_n)$ and satisfying the relation

$$H(s_0,s_1,s_2,\ldots,s_n)=|f(w_0,w_1,\ldots,w_n)|\,.$$

The set $A(s_0,s_1,\ldots,s_n)$ is not empty. It is closed and bounded. This follows from the fact that $f(w_0,w_1,\ldots,w_n)$ is analytic and $|f(w_0,w_1,\ldots,w_n)|$ is continuous. We introduce the notation

$$L(s_0,s_1,\ldots,s_n)=\max\left|\frac{\partial f(w_0,w_1,\ldots,w_n)}{\partial w_0}\right|,\quad\text{for}\quad (w_0,w_1,\ldots,w_n)\in A(s_0,s_1,\ldots,s_n)\,.$$

This maximum exists because $|\partial f/\partial w_0|$ is continuous and $A(s_0,s_1,\ldots,s_n)$ is a compact set. Denote by $B(s_0,s_1,\ldots,s_n)$ the set of points $(w_0,w_1,\ldots,w_n)$ satisfying the condition

$$(w_0,w_1,\ldots,w_n)\in A(s_0,s_1,\ldots,s_n)\,,\qquad L(s_0,s_1,\ldots,s_n)=\left|\frac{\partial f(w_0,w_1,\ldots,w_n)}{\partial w_0}\right|.$$

In order to prove our theorem it is sufficient to prove the equality

$$\left(\frac{\partial H(s_0,s_1,\ldots,s_n)}{\partial s_0}\right)_+=L(s_0,s_1,\ldots,s_n)\,.$$

It will be proved if we show that the following inequalities hold:

$$(7)_1\quad \limsup_{h\to0+}\frac{H(s_0+h,s_1,s_2,\ldots,s_n)-H(s_0,s_1,\ldots,s_n)}{h}\leqslant L(s_0,s_1,\ldots,s_n)\,,$$

$$(7)_2\quad \liminf_{h\to0+}\frac{H(s_0+h,s_1,s_2,\ldots,s_n)-H(s_0,s_1,\ldots,s_n)}{h}\geqslant L(s_0,s_1,\ldots,s_n)\,.$$

At first we prove $(7)_1$. We choose a sequence $\{s_i^\nu\}$ such that

$$s_0^\nu>s_0\,,\qquad (s_0^\nu,s_1,\ldots,s_n)\to(s_0,s_1,\ldots,s_n)$$

and

$$(8)\quad \lim_{\nu\to+\infty}\frac{H(s_0^\nu,s_1,\ldots,s_n)-H(s_0,s_1,\ldots,s_n)}{s_0^\nu-s_0}=\limsup_{h\to0+}\frac{H(s_0+h,s_1,\ldots,s_n)-H(s_0,s_1,\ldots,s_n)}{h}\,.$$

Since $f(w_0,w_1,\ldots,w_n)$ is an analytic function in the domain (1), for every point $(w_1,w_2,\ldots,w_n)$ satisfying relations $|w_i|=b_i$, $i=1,2,\ldots,n$, $f(w_0,\ldots,w_n)$ is analytic in w_0 in the circle $|w_0|<b_0$. Then for every $(s_0^\nu,s_1,\ldots,s_n)$ there exists $(w_0^\nu,w_1^\nu,\ldots,w_n^\nu)$ such that (see Fig. 1)

$$(9)\quad \begin{gathered}|w_0^\nu|=s_0^\nu\,,\qquad |w_i^\nu|=s_i\,,\qquad i=1,2,\ldots,n\,,\\ H(s_0^\nu,s_1,\ldots,s_n)=|f(w_0^\nu,w_1^\nu,\ldots,w_n^\nu)|\,.\end{gathered}$$

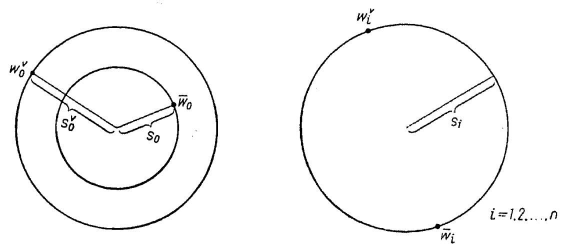

Fig. 1

It may be assumed that the sequences $\{w_0^\nu\}$, $\{w_i^\nu\}$, $i=1,2,\ldots,n$ converge to $\bar w_0$ and $\bar w_i$ respectively, $i=1,2,\ldots,n$, because we can always take a convergent subsequence. By the continuity of $H(s_0,s_1,\ldots,s_n)$ and $|f(w_0,w_1,\ldots,w_n)|$ we infer from (9) the equality

$$(10)\quad H(s_0,s_1,\ldots,s_n)=|f(\bar w_0,\bar w_1,\ldots,\bar w_n)|\,.$$

Hence we see that $(\bar w_0,\bar w_1,\ldots,\bar w_n)\in A(s_0,s_1,\ldots,s_n)$.

For every point $(w_0^\nu,w_1^\nu,\ldots,w_n^\nu)$ we choose a point $(\xi_0^\nu,w_1^\nu,w_2^\nu,\ldots,w_n^\nu)$ such that (see Fig. 2)

$$(11)\quad |\xi_0^\nu|=s_0\,,\qquad |\xi_0^\nu-w_0^\nu|=s_0^\nu-s_0\,.$$

Evidently $\xi_0^\nu\to w_0$ because of $|\xi_0^\nu-\bar w_0|\leqslant|\xi_0^\nu-w_0^\nu|+|w_0^\nu-\bar w_0|$. Then by (10) we obtain

$$(12)\quad |f(\xi_0^\nu,w_1^\nu,\ldots,w_n^\nu)|\leqslant H(s_0,s_1,\ldots,s_n)=|f(\bar w_0,\bar w_1,\ldots,\bar w_n)|\,.$$

Now we find the evaluation from above for

$$\frac{H(s_0^\nu, s_1, \dots, s_n) - H(s_0, s_1, \dots, s_n)}{s_0^\nu - s_0}.$$

Equalities (9) and (10) yield

$$\frac{H(s_0^\nu, s_1, \dots, s_n) - H(s_0, s_1, \dots, s_n)}{s_0^\nu - s_0} = \frac{|f(w_0^\nu, w_1^\nu, \dots, w_n^\nu)| - |f(\overline{w}_0, \overline{w}_1, \dots, \overline{w}_n)|}{s_0^\nu - s_0}.$$

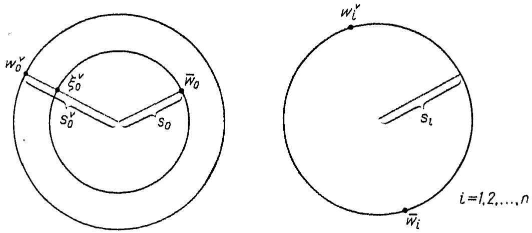

Fig. 2

Then in view of (11) and (12) we may write

$$\frac{|f(w_0^\nu, w_1^\nu, \dots, w_n^\nu)| - |f(\overline{w}_0, \overline{w}_1, \dots, \overline{w}_n)|}{s_0^\nu - s_0} \leqslant \frac{|f(w_0^\nu, w_1^\nu, \dots, w_n^\nu)| - |f(\xi_0^\nu, \overline{w}_1, \dots, \overline{w}_n)|}{|w_0^\nu - \xi_0^\nu|}$$
$$\leqslant \frac{|f(w_0^\nu, w_1^\nu, \dots, w_n^\nu) - f(\xi_0^\nu, w_1^\nu, \dots, w_n^\nu)|}{|w_0^\nu - \xi_0^\nu|}.$$

Consequently we obtain

$$(13)\qquad \frac{H(s_0^\nu, s_1, \dots, s_n) - H(s_0, s_1, \dots, s_n)}{s_0^\nu - s_0} \leqslant \frac{|f(w_0^\nu, w_1^\nu, \dots, w_n^\nu) - f(\xi_0^\nu, w_1^\nu, \dots, w_n^\nu)|}{|w_0^\nu - \xi_0^\nu|}.$$

By lemma 2 the right-hand member of (13) tends to $|f'_{w_0}(\overline{w}_0, \overline{w}_1, \dots, \overline{w}_n)|$; therefore

$$(13)_1\qquad \lim_{\nu\to\infty} \frac{H(s_0^\nu, s_1, \dots, s_n) - H(s_0, s_1, \dots, s_n)}{s_0^\nu - s_0} \leqslant |f'_{w_0}(\overline{w}_0, \overline{w}_1, \dots, \overline{w}_n)|.$$

Since $(\overline{w}_0, \overline{w}_1, \dots, \overline{w}_n) \in A(s_0, s_1, \dots, s_n)$, by the definition of $L(s_0, s_1, \dots, s_n)$ we have

$$|f'_{w_0}(\overline{w}_0, \overline{w}_1, \dots, \overline{w}_n)| \leqslant L(s_0, s_1, \dots, s_n),$$

which in view of (13) and (8) gives

$$\limsup_{h\to 0+} \frac{H(s_0+h, s_1, \dots, s_n) - H(s_0, s_1, \dots, s_n)}{h} \leqslant L(s_0, s_1, \dots, s_n).$$

Therefore equality $(7)_1$ holds.

Now it must be proved that $(7)_2$ holds. We take into consideration an arbitrary point $(\beta_0, \beta_1, \dots, \beta_n) \in B(s_0, s_1, \dots, s_n)$. By the definition of $B(s_0, s_1, \dots, s_n)$ we have

$$(14)\qquad H(s_0, s_1, \dots, s_n) = |f(\beta_0, \beta_1, \dots, \beta_n)|,$$

$$(15)\qquad L(s_0, s_1, \dots, s_n) = |f'_{w_0}(\beta_0, \beta_1, \dots, \beta_n)|.$$

Then we take a sequence $(t_0^\nu, s_1, \dots, s_n)$, $t_0^\nu > s_0$, $t_0^\nu \to s_0$, such that

$$(16)\qquad \lim_{\nu\to\infty} \frac{H(t_0^\nu, s_1, \dots, s_n) - H(s_0, s_1, \dots, s_n)}{t_0^\nu - s_0} = \liminf_{h\to 0+} \frac{H(s_0+h, s_1, \dots, s_n) - H(s_0, s_1, \dots, s_n)}{h}.$$

Since $|f(w_0, w_1, \dots, w_n)|$ is continuous, we choose for every $(t_0^\nu, s_1, \dots, s_n)$ a point $(\beta_0^\nu, \beta_1^\nu, \dots, \beta_n^\nu)$ such that (see Fig. 3)

$$(17)\qquad |\beta_0^\nu| = t_0^\nu, \quad |\beta_i^\nu| = s_i, \quad i = 1, 2, \dots, n,$$
$$H(t_0^\nu, s_1, \dots, s_n) = |f(\beta_0^\nu, \beta_1^\nu, \dots, \beta_n^\nu)|.$$

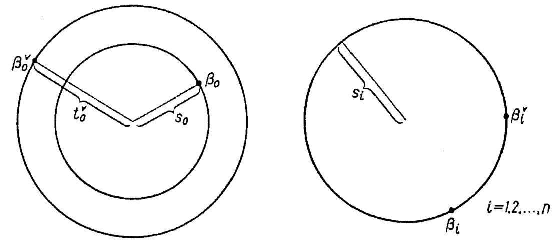

Fig. 3

For every sequence $\{t_0^\nu - s_0\}$, $t_0^\nu - s_0 > 0$, $t_0^\nu - s_0 \to 0$, there exists by lemma 3 a sequence of points $\{\alpha_0^\nu\}$ such that (see Fig. 4)

$$(18)_1\qquad |\alpha_0^\nu - \beta_0| = t_0^\nu - s_0,$$

$$(18)_2\qquad \frac{|f(\alpha_0^\nu, \beta_1, \dots, \beta_n)| - |f(\beta_0, \beta_1, \dots, \beta_n)|}{|\alpha_0^\nu - \beta_0|} \underset{\nu\to\infty}{\to} |f'_{w_0}(\beta_0, \beta_1, \dots, \beta_n)|.$$

By means of the maximum principle for analytic functions we have

$$(19)_1\qquad |f(\alpha_0^\nu, \beta_1, \beta_2, \dots, \beta_n)| \leqslant H(t_0^\nu, s_1, s_2, \dots, s_n) = |f(\beta_0^\nu, \beta_1^\nu, \dots, \beta_n^\nu)|.$$

Now we shall find the lower evaluation of

$$\frac{H(t_0^\nu, s_1, \dots, s_n) - H(s_0, s_1, \dots, s_n)}{t_0^\nu - s_0}.$$

We deduce from (17) and (14) the equality

$$(19)\qquad \frac{H(t_0^{\nu}, s_1, \ldots, s_n)-H(s_0, s_1, \ldots, s_n)}{t_0^{\nu}-s_0} = \frac{|f(\beta_0^{\nu}, \beta_1^{\nu}, \ldots, \beta_n^{\nu})|-|f(\beta_0, \beta_1, \ldots, \beta_n)|}{t_0^{\nu}-s_0}$$

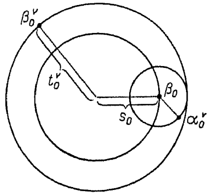

Fig. 4

and from $(19)_1$ and $(18)_1$ we have

$$(20)\qquad \frac{|f(\beta_0^{\nu}, \beta_1^{\nu}, \ldots, \beta_n^{\nu})|-|f(\beta_0, \beta_1, \ldots, \beta_n)|}{t_0^{\nu}-s_0} \geqslant \frac{|f(a_0^{\nu}, \beta_1, \ldots, \beta_n)|-|f(\beta_0, \beta_1, \ldots, \beta_n)|}{|a_0^{\nu}-\beta_0|}.$$

From (20), $(18)_2$, (19), (16) and (15) we obtain

$$\liminf_{h\to 0+} \frac{H(s_0+h, s_1, \ldots, s_n)-H(s_0, s_1, \ldots, s_n)}{h} \geqslant L(s_0, s_1, \ldots, s_n)$$

and we have proved $(7)_2$.

It follows by $(7)_1$ and $(7)_2$ that we have

$$\limsup_{h\to 0+} \frac{H(s_0+h, s_1, \ldots, s_n)-H(s_0, s_1, \ldots, s_n)}{h} \leqslant L(s_0, s_1, \ldots, s_n) \leqslant \liminf_{h\to 0+} \frac{H(s_0+h, s_1, \ldots, s_n)-H(s_0, s_1, \ldots, s_n)}{h}.$$

Hence we deduce

$$\left(\frac{\partial H(s_0, s_1, \ldots, s_n)}{\partial s_0}\right)_+ = L(s_0, s_1, \ldots, s_n).$$

Thus for the right-hand derivative we have the relation

$$\left(\frac{\partial H(s_0, s_1, \ldots, s_n)}{\partial s_0}\right)_+ = |f'_{w_0}(\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_n)|$$

where $(\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_n)$ is a point which satisfies

$$|\hat{w}_0| = s_0, \quad |\hat{w}_1| = s_1, \quad \ldots, \quad |\hat{w}_n| = s_n,$$
$$H(s_0, s_1, \ldots, s_n) = |f(\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_n)|.$$

Remark 1. We can prove in the same way the relations

$$\left(\frac{\partial H(s_0, s_1, \ldots, s_n)}{\partial s_i}\right)_+ = |f'_{w_i}(w_0^{*i}, w_1^{*i}, \ldots, w_n^{*i})|, \qquad i = 1, 2, \ldots, n,$$

where $(w_0^{*i}, w_1^{*i}, \ldots, w_n^{*i})$, $i = 1, 2, \ldots, n$, is a point satisfying

$$|w_0^{*i}| = s_0, \quad |w_1^{*i}| = s_1, \quad \ldots, \quad |w_n^{*i}| = s_n, \qquad i = 1, 2, \ldots, n,$$
$$H(s_0, s_1, \ldots, s_n) = |f(w_0^{*i}, w_1^{*i}, \ldots, w_n^{*i})|, \qquad i = 1, 2, \ldots, n.$$

Remark 2. We can prove by a similar method an analogical theorem on the left-hand derivatives $(\partial H/\partial s_0)_-$, $(\partial H/\partial s_i)_-$, $i = 1, 2, \ldots, n$.

Remark 3. If f is an analytic function of one variable z, we obtain from our theorem the following corollary:

Assume that $f(z)$ is an analytic function in a circle $k_a = |z| < a$. We put

$$M(r) = \max_{|z|=r} |f(z)|.$$

Then we state that:

(i) $M(r)$ is a continuous and not decreasing function in $[0, a)$;

(ii) the derivatives $M'_+(r)$ and $M'_-(r)$ exist for every $r \in [0, a)$ and for every r there exist points ξ and η contained in k_a such that

$$|\xi| = |\eta| = r, \qquad M(r) = |f(\xi)| = |f(\eta)|,$$
$$M'_+(r) = |f'(\xi)|, \qquad M'_-(r) = |f'(\eta)|;$$

$M(r)$ is not differentiable in general, of course.

As an example we may take a function

$$f(z) = z^3+4z^2-z+1.$$

We write it in the form

$$f(z) = (z^2-1)\cdot(z+4)+5.$$

We shall show that for the corresponding function $M(r)$ we have

$$M(1) = |f(-1)| = |f(1)|.$$

The points on $|z| = 1$ will be represented by

$$z = e^{i\theta}, \qquad \theta \in [0, 2\pi).$$

We put

$$u(\theta) = |f(e^{i\theta})|^2.$$

This may be written in the form

$$u(\theta) = f(e^{i\theta})\cdot f(e^{-i\theta}).$$

Then we have

$$u'(\theta) = f(e^{i\theta})\cdot[-2ie^{-2i\theta}(e^{-i\theta}+4)-ie^{-i\theta}(e^{-2i\theta}-1)] + f(e^{-i\theta})\cdot[2ie^{2i\theta}(e^{i\theta}+4)+ie^{i\theta}(e^{2i\theta}-1)],$$

$$u'(0) = 5(-10i)+5(10i) = 0, \qquad u'(\pi) = 5(-10i)+5(10i) = 0,$$

$$u''(\theta) = f(e^{i\theta})[-4e^{-2i}(e^{-i\theta}+4)-2e^{-3i\theta}-e^{-i\theta}(e^{-2i\theta}-1)-2e^{-3i\theta}] + [2ie^{2i\theta}(e^{i\theta}+4)+ie^{i\theta}(e^{2i\theta}-1)]\cdot[-2ie^{-2i\theta}(e^{-i\theta}+4)-ie^{-i\theta}(e^{-2i\theta}-1)] + f(e^{-i\theta})[-4e^{2i\theta}(e^{i\theta}+4)-2e^{3i\theta}-e^{i\theta}(e^{2i\theta}-1)-2e^{3i\theta}] + [-2ie^{-2i\theta}(e^{-i\theta}+4)-ie^{-i\theta}(e^{-2i\theta}-1)]\cdot[2ie^{2i\theta}(e^{i\theta}+4)+ie^{i\theta}(e^{2i\theta}-1)],$$

$$u''(0) = 5(-24)+(10i)(-10i)+5(-24)+(-10i)(10i) = -240+200 = -40 < 0,$$

$$u''(\pi) = 5(-8)+(6i)(-6i)+5(-8)+(-6i)(6i) = -80+72 = -8 < 0$$

and consequently

$$M(1) = |f(-1)| = |f(1)| = 5 .$$

It may easily be shown that the set $A(1)$ consisting of the roots of the equation $u'(\theta) = 0$ contains only 0 and π. We find

$$f'(z) = 3z^2+8z-1 , \quad f'(-1) = 6 , \quad f'(1) = 10 .$$

In view of remark 3 we have

$$M'_+(1) = \max_{z \in A(1)} |f'(z)| , \quad M'_-(1) = \min_{z \in A(1)} |f'(z)| .$$

Hence

$$M_+(1) = 10 \quad \text{and} \quad M'_-(1) = 6 .$$

References

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Remarks on the extremal functions of a certain class of analytical functions

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Let us study the class T of the analytic functions satisfying in the ring $0 < |z| < 1$ the differential equation

$$\frac{zf'(z)}{f(z)} = \alpha p(z) + \beta \tag{1}$$

where α and β are any complex numbers and $p(z) = 1 + a_1 z + \ldots$ satisfies the inequality $\operatorname{re} p(z) > 0$. We can easily see that the form of the functions of the class T is as follows:

$$f(z) = Cz^{\alpha+\beta}\exp\left\{\alpha \int_0^z \frac{p(s)-1}{s}\,ds\right\} = Cz^{\alpha+\beta}\left\{1 + \sum_{k=1}^{\infty} a_k z^k\right\}, \quad C = \text{const} . \tag{2}$$

Let $T_{\alpha,\beta}$ be a subclass of the class T obtained by fixing the numbers α and β and putting $C = 1$. In particular $T_{1,0}$ will be identical with the class of all regular starlike schlicht functions, and $T_{-1,0}$ will be identical with the class of all meromorphic starlike schlicht functions. If we put $\alpha = \varrho = 1/(1-ai)$, $\beta = 1-\varrho$ (real a) we obtain the class of Špaček spiral schlicht functions [2] and putting $\alpha = -\varrho$, $\beta = -1+\varrho$ we obtain the class of spiral meromorphic schlicht functions. Class $T_{p\varrho,p(1-\varrho)}$ is a certain subclass (p integral) of the class of p-valent functions.

In my previous paper [3] it was proved that the following estimations are true:

(I) if $\operatorname{re}\alpha \geqslant 0$ then $|a_n| \leqslant \frac{1}{n!}\prod_{k=0}^{n-1}|2\alpha+k|$;

(II) if $-|\alpha|^2 < \operatorname{re}\alpha < 0$ then

(i) $|a_n| \leqslant \frac{1}{n!}\prod_{k=0}^{n-1}|2\alpha+k|$ for $n = 1, \ldots, N+1$,