Le reste R se décompose, d'après la condition II, en couches disjointes $H(k)$ déterminées par la condition:

$$W(\xi)/W(\eta) = k = \text{const}, \quad \text{où} \quad -\infty < k < \infty$$

(dans R on doit avoir $W(\eta) \neq 0$, puisque $W(\eta) = \eta^2$ et $\eta \neq 0$). Donc:

$$R = \sum_{-\infty<k<\infty} H(k). \tag{27}$$

La couche $H(0)$ se compose, d'après la condition III, de trois domaines de transitivité: D_7, D_8, D_9, correspondant respectivement aux cas où la partie symétrique ξ du point x appartient à D_1, D_2, D_3.

Chaque couche $H(k)$, où $k > 0$, se compose de deux domaines de transitivité $D_1(k)$ et $D_2(k)$ correspondant aux cas où la partie symétrique ξ appartient à D_4 et D_5.

Chaque couche $H(k)$, où $k < 0$, forme un domaine de transitivité $D_3(k)$.

Tout espace X_4 se décompose donc en neuf domaines de transitivité „exceptionnels": $D_1, D_2, \ldots, D_9$ et trois familles (dépendant du paramètre continu k) de domaines de transitivité $D_1(k), D_2(k), D_3(k)$, qui sont définis de la manière suivante:

$$\left.\begin{array}{ll}
D_1: & x_1^2+x_2^2+x_3^2+x_4^2 = 0, \\
D_2: & x_1 \geqslant 0, \quad x_4 \geqslant 0 \\
D_3: & x_1 \leqslant 0, \quad x_4 \leqslant 0 \quad \Big\} \; x_1^2+x_4^2 > 0, \quad W(\xi) = 0, \\
D_4: & x_1 > 0, \quad x_4 > 0 \\
D_5: & x_1 < 0, \quad x_4 < 0 \quad \Big\} \; W(\xi) > 0, \\
D_6: & W(\xi) < 0,
\end{array}\right\} \eta = 0,$$

$$\left.\begin{array}{ll}
D_7: & \xi \in D_1 \\
D_8: & \xi \in D_2 \quad \Big\} \; W(\xi) = 0 \\
D_9: & \xi \in D_3 \\
D_1(k): & \xi \in D_4 \\
D_2(k): & \xi \in D_5 \quad \Big\} \; \dfrac{W(\xi)}{W(\eta)} = k > 0 \\
D_3(k): & \dfrac{W(\xi)}{W(\eta)} = k < 0
\end{array}\right\} \eta \neq 0.$$

Travaux cités

[1] S. Gołąb et E. Siwek, *Sur les domaines de transitivité d'un groupe de trans formations*, ce volume, p. 209-216.

Reçu par la Rédaction le 27. 5. 1960

ANNALES
POLONICI MATHEMATICI
X (1961)

On the evaluation of the solutions of a system of ordinary differential equations with an analytical right-hand member

by Tsin-Hwa Shu (Kraków)

Introduction. We consider an ordinary differential equation

$$\frac{dw}{dz} = f(z, w) \tag{1}$$

with the initial condition $w(0) = 0$, z and w being complex variables and $f(z, w)$ being analytic in the domain

$$|z| < a, \quad |w| < b, \tag{2}$$

(a, b are positive constants).

Moreover, we consider the equation

$$\frac{ds}{dr} = F(r, s) \tag{3}$$

with the initial condition $s(0) = 0$, r and s being the real variables and $F(r, s)$ being a function continuous in the rectangle

$$0 \leqslant r < a, \quad 0 \leqslant s < b \tag{4}$$

(a, b are identical to those in formula (2)).

In the year 1956 A. Wintner proved [3] the following theorem:

Theorem of A. Winter. *We assume that the inequality*

$$|f(z, w)| \leqslant F(|z|, |w|) \tag{5}$$

holds if z and w both satisfy (2). We assume also that $F(r, s)$ does not decrease with respect to the variable s in rectangle (4). Suppose we have given an arbitrary solution of (3) which satisfies the initial condition $s(0) = 0$ and exists in an interval $[0, a)$ $(a \leqslant a)$. Then we have the following proposition:

Each function in the sequence of successive approximations of the solution of (1)

$$w_0(z) \equiv 0, \quad w_1(z), \quad w_2(z), \quad \ldots \tag{6}$$

where

$$w_{n+1} = \int_0^z f(\xi, w_n(\xi))\,d\xi, \qquad n = 0, 1, 2, \ldots,$$

exists, it is an analytic function and satisfies the equality

$$|w_n(z)| \leqslant s(|z|)$$

in the circle $|z| < a$ *for* $n = 0, 1, 2, \ldots$

Moreover, sequence (6) *converges uniformly in every circle* $|z| \leqslant a-\varepsilon$ $(0<\varepsilon<a)$ *and there exists a solution of* (1) *with the initial condition* $w(0) = 0$. *This solution is analytic at least in the circle* $|z| < a$ *and satisfies the inequality*:

$$|w(z)| \leqslant s(|z|) \quad \text{for} \quad |z| < a.$$

In the paper [4] (1956) A. Wintner has proved the following theorem:

Assume that $F(r, s)$ *is continuous in a certain rectangle* (4) *and that the functions* $f(z, w)$ *and* $F(r, s)$ *satisfy inequality* (5). *If* $s(r)$ *is a solution of equation* (3) *with the initial condition* $s(0) = 0$ *and if it is defined in the interval* $[0, a)$ $(a \leqslant a)$, *then the solution of* (1) $w(z)$ $(w(0) = 0)$ *is an analytic function in the circle* $|z| < a$ *and satisfies the inequality*

$$|w(z)| \leqslant s(|z|).$$

A similar theorem was proved by T. Ważewski [2] (1937) for the system

$$\frac{dw_i}{dz} = f_i(z, w_1, w_2, \ldots, w_n), \qquad i = 1, 2, 3, \ldots, n, \tag{7}$$

$$w_i(0) = 0, \qquad i = 1, 2, \ldots, n. \tag{8}$$

There z and w_i, $i = 1, 2, \ldots, n$, are complex variables and functions $f_i(z, w_1, \ldots, w_n)$, $i = 1, 2, \ldots, n$, are analytic in an $(n+1)$-circle domain,

$$0 \leqslant |z| < a, \qquad 0 \leqslant |w_i| < b_i, \qquad i = 1, 2, \ldots, n, \tag{9}$$

a, b_i, $i = 1, 2, \ldots, n$, being positive constants.

In the theorem of T. Ważewski (1937) the majorating system for (7) is

$$\frac{ds_i}{dr} = F_i(r, s_1, s_2, \ldots, s_n), \qquad i = 1, 2, \ldots, n, \tag{10}$$

$$s_i(0) = 0, \qquad i = 1, 2, \ldots, n \tag{11}$$

and it is assumed that the inequalities

$$|f_i(z, w_1, \ldots, w_n)| \leqslant F_i(|z|, |w_1|, \ldots, |w_n|), \qquad i = 1, 2, \ldots, n$$

are satisfied for $(z, w_1, \ldots, w_n) \in (9)$ and that F_i, $i = 1, 2, \ldots, n$, are defined in the set

$$0 \leqslant r < a, \qquad 0 \leqslant s_i < b_i, \qquad i = 1, 2, \ldots, n, \tag{12}$$

a and b_i, $i = 1, 2, \ldots, n$ being constants identical to those in (9).

Moreover, it is assumed that each F_i, $i = 1, 2, \ldots, n$, is continuous in set (12) and not decreasing with respect to any of the variables $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$, and that the integral of system (10) is the upper integral satisfying (11) and not an arbitrary integral as in the theorem of A. Wintner [4].

The present paper contains a generalization of the above results of A. Wintner and T. Ważewski. The point is that we assume only the continuity of the right-hand members of (10) and solution of (7) is majorated by an arbitrary integral of (10). It should be noticed that under our assumption it is not necessary that the majorating system (10) should have its upper integral satisfying (11), as has been assumed in the paper of T. Ważewski. We base ourselves here on the method of differential inequalities.

§ 1. DEFINITION 1. Let $U(r, s_1, \ldots, s_n)$ be a real function of the real variables $r, s_1, \ldots, s_n$ defined in a closed set of points $(r, s_1, \ldots, s_n)$. We say that $U(r, s_1, \ldots, s_n)$ satisfies the *condition of Lipschitz* with respect to the variables $s_1, s_2, \ldots, s_n$ if there exists a constant N such that for an arbitrary pair of points $(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)$, $(r, \bar{\bar{s}}_1, \bar{\bar{s}}_2, \ldots, \bar{\bar{s}}_n)$ of that set we have

$$|U(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) - U(r, \bar{\bar{s}}_1, \bar{\bar{s}}_2, \ldots, \bar{\bar{s}}_n)| \leqslant N \cdot \sum_{i=1}^{n} |\bar{s}_i - \bar{\bar{s}}_i|,$$

where N is a constant independent of the choice of points.

DEFINITION 2. We say that $U(r, s_1, \ldots, s_n)$ satisfies the *condition of Lipschitz* with respect to $s_1, s_2, \ldots, s_n$ *in a cube* (12), if $U(r, s_1, s_2, \ldots, s_n)$ satisfies the condition of Lipschitz in every closed cube contained in the cube (12).

DEFINITION 3. Let the real function $F_i(r, s_1, \ldots, s_n)$ be defined in (12) and the function $f(z, w_1, \ldots, w_n)$ be the complex function of $n+1$ complex variables $z, w_1, \ldots, w_n$ defined in the $(n+1)$-circular domain (9). If the inequality:

$$|f(z, w_1, w_2, \ldots, w_n)| \leqslant F(|z|, |w_1|, |w_2|, \ldots, |w_n|)$$

holds in (9), then we say that $F_i(r, s_1, \ldots, s_n)$ is for the function $f(z, w_1, \ldots, w_n)$ a *majorating function* of the type $T(0, 0, \ldots, 0)$.

§ 2. We consider n analytic functions $f(z, w_1, \ldots, w_n)$, $i = 1, 2, \ldots, n$, which are defined in the $(n+1)$-circular domain (9).

We define the real functions $H_i(r, s_1, \ldots, s_n)$ of the real variables $r, s_1, \ldots, s_n$ in the following manner:

$$H_i(r, s_1, s_2, \ldots, s_n) = \max_{|z|=r,\ |w_1|=s_1,\ |w_2|=s_2, \ldots, |w_n|=s_n} |f_i(z, w_1, w_2, \ldots, w_n)|, \qquad i = 1, 2, \ldots, n. \tag{13}$$

The functions $H_i(r, s_1, \ldots, s_n)$ are defined in (12).

The following three propositions can easily be proved.

PROPOSITION 1. *$H_i(r, s_1, \ldots, s_n)$, $i = 1, 2, \ldots, n$, are continuous in* (12).

PROPOSITION 2. *$H_i(r, s_1, \ldots, s_n)$ is not decreasing with respect to the variables $r, s_1, s_2, \ldots, s_n$ in cube* (12) $(i = 1, 2, \ldots, n)$.

We have the following evident inequalities:

$$|f_i(z, w_1, w_2, \ldots, w_n)| \leqslant H_i(|z|, |w_1|, |w_2|, \ldots, |w_n|), \quad i = 1, 2, \ldots, n$$

in the $(n+1)$-circle domain (9). Then the function $H_i(r, s_1, \ldots, s_n)$ is for $f(z, w_1, \ldots, w_n)$ a majorating function of $T(0, 0, \ldots, 0)$ type in domain (9).

PROPOSITION 3. *The function $H_i(r, s_1, \ldots, s_n)$ $(i = 1, 2, \ldots, n)$ defined above is the smallest among all the majorating functions of the type $T(0, 0, \ldots, 0)$ for the function $f_i(z, w_1, \ldots, w_n)$ $(i = 1, 2, \ldots, n)$.*

LEMMA 1. *The functions $H_i(r, s_1, \ldots, s_n)$, $i = 1, 2, \ldots, n$, satisfy the condition of Lipschitz in* (12) *with respect to the variables $s_1, \ldots, s_n$.*

Proof. We consider an arbitrary but fixed closed cube

$$\bar{K}_1\colon \quad 0 \leqslant r \leqslant a^{(1)}, \ 0 \leqslant s_i \leqslant b_i^{(1)}, \ i = 1, 2, \ldots, n,$$

a and b_i being constants satisfying the inequalities

$$0 < a^{(1)} < a, \quad 0 < b_i^{(1)} < b_i, \quad i = 1, 2, \ldots, n.$$

In order to prove our lemma it will be sufficient to show that $H_j(r, s_1, \ldots, s_n)$ (j is arbitrarily fixed) satisfies the condition of Lipschitz with respect to the variables $s_1, s_2, \ldots, s_n$ in the cube $\bar{K}_1$.

We denote by $\bar{D}_1$ a closed $(n+1)$-circular domain contained in the $(n+1)$-circular domain (9) corresponding to the cube $\bar{K}_1$. Since $f_j(z, w_1, \ldots, w_n)$ is analytic in (9), its partial derivatives of arbitrary order are bounded in $\bar{D}_1$. We have

$$(14) \qquad |f^j_{w_l}(z, w_1, \ldots, w_n)| \leqslant M, \quad l = 1, 2, \ldots, n,$$

for $(z, w_1, \ldots, w_n) \in \bar{D}_1$.

Now we consider an arbitrarily fixed pair of points $(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)$ $(r, \bar{\bar{s}}_1, \bar{\bar{s}}_2, \ldots, \bar{\bar{s}}_n)$ which belong to the cube $\bar{K}_1$. Since $\bar{K}_1$ is a convex set, then the difference $H_j(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) - H_j(r, \bar{\bar{s}}_1, \bar{\bar{s}}_2, \ldots, \bar{\bar{s}}_n)$ may be represented as follows:

$$(15) \qquad H_j(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) - H_j(r, \bar{\bar{s}}_1, \bar{\bar{s}}_2, \ldots, \bar{\bar{s}}_n)$$
$$= H_j(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) - H_j(r, \bar{\bar{s}}_1, \bar{s}_2, \ldots, \bar{s}_n) +$$
$$+ H_j(r, \bar{\bar{s}}_1, \bar{s}_2, \ldots, \bar{s}_n) - H_j(r, \bar{\bar{s}}_1, \bar{\bar{s}}_2, \bar{s}_3, \ldots, \bar{s}_n) +$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ H_j(r, \bar{\bar{s}}_1, \bar{\bar{s}}_2, \ldots, \bar{\bar{s}}_{n-1}, \bar{s}_n) - H_j(r, \bar{\bar{s}}_1, \bar{\bar{s}}_2, \ldots, \bar{\bar{s}}_n).$$

We now evaluate the first difference which appears in the right-hand member of (15), i.e. $H_j(r, \bar{s}_1, \ldots, \bar{s}_n) - H_j(r, \bar{\bar{s}}_1, \bar{s}_2, \ldots, \bar{s}_n)$. In view of the definition of $H_j(r, s_1, s_2, \ldots, s_n)$ and by the continuity of the function $|f_j(z, w_1, \ldots, w_n)|$ there exist for points $(r, \bar{s}_1, \ldots, \bar{s}_n)$, $(r, \bar{\bar{s}}_1, \bar{s}_2, \ldots, \bar{s}_n)$ two another points $(\bar{z}, w_1, \bar{w}_2, \ldots, w_n)$, $(\bar{\bar{z}}, \bar{\bar{w}}_1, \ldots, \bar{\bar{w}}_1)$ such that

$$|\bar{z}| = r, \quad |\bar{w}_1| = \bar{s}_1, \quad \ldots, \quad |\bar{w}_n| = \bar{s}_n;$$
$$|\bar{\bar{z}}| = r, \quad |\bar{\bar{w}}_1| = \bar{\bar{s}}_1, \quad |\bar{\bar{w}}_2| = \bar{s}_2, \quad \ldots, \quad |\bar{\bar{w}}_n| = \bar{s}_n,$$
$$H_j(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) = |f_j(\bar{z}, \bar{w}_1, \ldots, \bar{w}_n)|,$$
$$H_j(r, \bar{\bar{s}}_1, \bar{s}_2, \ldots, \bar{s}_n) = |f_j(\bar{\bar{z}}, \bar{\bar{w}}_1, \bar{\bar{w}}_2, \ldots, \bar{\bar{w}}_n)|,$$

we then have

$$(16) \qquad H_j(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) - H(r, \bar{\bar{s}}_1, \bar{s}_2, \ldots, \bar{s}_n)$$
$$= |f_j(\bar{z}, \bar{w}_1, \ldots, \bar{w}_n)| - |f_j(\bar{\bar{z}}, \bar{\bar{w}}_1, \ldots, \bar{\bar{w}}_n)|.$$

For $\bar{w}_1$ we choose $\bar{\bar{\eta}}_1$ in such a way that (see Fig. 1)

$$(17) \qquad |\bar{\bar{\eta}}_1| = \bar{\bar{s}}_1, \quad |\bar{w}_1 - \bar{\bar{\eta}}_1| = |\bar{s}_1 - \bar{\bar{s}}_1|.$$

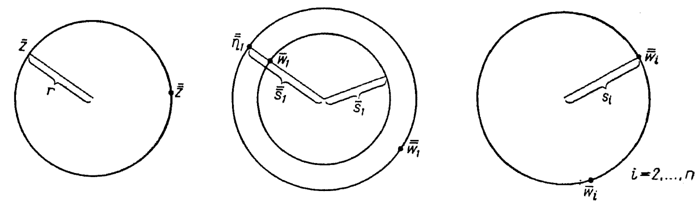

Fig. 1

It follows by the definition of the function $H_j(r, s_1, s_2, \ldots, s_n)$ that

$$(18) \qquad |f_j(\bar{z}, \bar{\bar{\eta}}_1, \bar{w}_1, \ldots, \bar{w}_n)| \leqslant H_j(r, \bar{\bar{s}}_1, \bar{s}_2, \ldots, \bar{s}_n) = |f_j(\bar{\bar{z}}, \bar{\bar{w}}_1, \ldots, \bar{\bar{w}}_n)|.$$

By formulas (16) and (18) we obtain

$$H_j(r, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) - H_j(r, \bar{\bar{s}}_1, \bar{s}_2, \ldots, \bar{s}_n)$$
$$\leqslant |f_j(\bar{z}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)| - |f_j(\bar{z}, \bar{\bar{\eta}}_1, \bar{w}_2, \ldots, \bar{w}_n)|$$
$$\leqslant |f_j(\bar{z}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n) - f_j(\bar{z}, \bar{\bar{\eta}}_1, \bar{w}_2, \ldots, \bar{w}_n)|$$
$$= \Big|\int_{\bar{\bar{\eta}}_1}^{\bar{w}_1} f^j_{w_1}(\bar{z}, \eta_1, \bar{w}_2, \ldots, \bar{w}_n)\, d\eta_1\Big|$$
$$\leqslant \Big|\int_{\bar{\bar{\eta}}_1}^{\bar{w}_1} |f^j_{w_1}(\bar{z}, \eta_1, \bar{w}_2, \ldots, \bar{w}_n)|\, d\eta_1\Big|$$

and hence (14) and (17) yield

$$H_j(r,\bar{s}_1,\bar{s}_2,\dots,\bar{s}_n)-H_j(r,\bar{\bar{s}}_1,\bar{s}_2,\dots,\bar{s}_n)\leqslant M\cdot|\bar{w}_1-\bar{\eta}_1| = M\cdot|\bar{s}_1-\bar{\bar{s}}_1|.$$

In a similar way we obtain

$$H_j(r,\bar{\bar{s}}_1,\bar{s}_2,\dots,\bar{s}_n)-H_j(r,\bar{s}_1,\bar{s}_2,\dots,\bar{s}_n)\leqslant M\cdot|\bar{s}_1-\bar{\bar{s}}_1|,$$

which yields

(19)$_1$ $$|H_j(r,\bar{s}_1,\dots,\bar{s}_n)-H_j(r,\bar{\bar{s}}_1,\bar{s}_2,\dots,\bar{s}_n)|\leqslant M\cdot|\bar{s}_1-\bar{\bar{s}}_1|.$$

By the same method we obtain

(19)$_2$ $$|H_j(r,\bar{\bar{s}}_1,\bar{s}_2,\dots,\bar{s}_n)-H_j(r,\bar{\bar{s}}_1,\bar{\bar{s}}_2,\bar{s}_3,\dots,\bar{s}_n)|\leqslant M\cdot|\bar{s}_2-\bar{\bar{s}}_2|,$$

. . . .

(19)$_n$ $$|H_j(r,\bar{\bar{s}}_1,\bar{\bar{s}}_2,\dots,\bar{\bar{s}}_{n-1},\bar{s}_n)-H_j(r,\bar{\bar{s}}_1,\bar{\bar{s}}_2,\dots,\bar{\bar{s}}_n)|\leqslant M\cdot|\bar{s}_n-\bar{\bar{s}}_n|.$$

Now by the formulas (15), (19)$_1$, ..., (19)$_n$ follows the inequality

$$|H_j(r,\bar{s}_1,\bar{s}_2,\dots,\bar{s}_n)-H_j(r,\bar{\bar{s}}_1,\bar{\bar{s}}_2,\dots,\bar{\bar{s}}_n)|\leqslant M\cdot\sum_{i=1}^{n}|\bar{s}_i-\bar{\bar{s}}_i|$$

q.e.d.

§ 3. Hypothesis K. We say that the real functions of the real variables $U_i(r,s_1,s_2,\dots,s_n)$, $i=1,2,\dots,n$, satisfy hypothesis K if

1) $U_i(r,s_1,s_2,\dots,s_n)$, $i=1,2,\dots,n$, are continuous in cube (12);

2) $U_i(r,s_1,s_2,\dots,s_n)$ does not decrease with respect to any of the variables $s_1,\dots,s_{i-1},s_{i+1},\dots,s_n$ in cube (12) $(i=1,2,\dots,n)$.

In the following we shall make use of the following result of T. Ważewski (cf. [1], p. 124, theorem 2).

Lemma 2. *We consider the system*

(20) $$\frac{dy_i}{dx}=U_i(x,y_1,y_2,\dots,y_n),\qquad i=1,2,\dots,n.$$

We assume that $U_1,\dots,U_n$ satisfy hypothesis K. *By that hypothesis the upper integral of system* (20) *with the initial condition $y_i(0)=0$, $i=1,2,\dots,n$, exists in an interval $[0,a)$ $(a\leqslant\alpha)$. We denote this integral by $y_i=\tau_i(x)$, $i=1,2,\dots,n$.*

We now choose a curve $y_i=\varphi_i(x)$, $i=1,2,\dots,n$, which is continuous in $[0,a)$ and is contained in cube (12).

Moreover, we assume that

$$\varphi_i(0)=0,\quad \underline{D}_{+}\varphi_i(x)\leqslant U_i\big(x,\varphi_1(x),\dots,\varphi_n(x)\big),\qquad i=1,2,\dots,n,$$

for $0<x<a$.

Under these assumptions the following inequalities hold:

$$\varphi_i(x)\leqslant\tau_i(x),\qquad i=1,2,\dots,n\quad \textit{for}\quad 0\leqslant x<a.$$

Also the following lemma will be used in the sequel (cf. [1], p. 143, theorem II):

Lemma 3. *We consider two systems of equations,*

(21) $$\frac{dy_i}{dx}=U_i(x,y_1,y_2,\dots,y_n),\qquad i=1,2,\dots,n,$$

(22) $$\frac{dy_i}{dx}=G_i(x,y_1,y_2,\dots,y_n),\qquad i=1,2,\dots,n.$$

We assume that $U_1,\dots,U_n$ satisfy hypothesis K *and $G_1,G_2,\dots,G_n$ are continuous in cube* (12). *Moreover, we assume that the inequalities*

$$U_i\leqslant G_i,\qquad i=1,2,\dots,n,$$

hold in cube (12). *Let $y_i=\varphi_i(x)$ be the lower integral of system* (21) *containing the point* $(0,0,\dots,0)$ *and defined in an interval $[0,a)$ $(a\leqslant\alpha)$. Let $y_i=\psi_i(x)$ be an arbitrary integral curve of* (22) *defined in $[0,a)$ and satisfying the initial condition $\psi_i(0)=0$, $i=1,2,\dots,n$. Under the above assumptions we have the inequalities:*

$$\varphi_i(x)\leqslant\psi_i(x),\qquad i=1,2,\dots,n\quad \textit{for}\quad 0\leqslant x<a.$$

§ 4. Theorem 1. *Consider the system of equations*

(23)$_1$ $$\frac{dw_i}{dz}=f_i(z,w_1,w_2,\dots,w_n),\qquad i=1,2,\dots,n,$$

with the initial condition

(23)$_2$ $$w_i(0)=0,\qquad i=1,2,\dots,n,$$

z and w being complex variables. We assume that $f_1,f_2,\dots,f_n$ are analytic functions in the $(n+1)$-circular domain (9).

Simultaneously with system (23) *we consider also the system*

(24)$_1$ $$\frac{ds_i}{dr}=F_i(r,s_1,s_2,\dots,s_n),\qquad i=1,2,\dots,n,$$

with the initial condition

(24)$_2$ $$s_i(0)=0,\qquad i=1,2,\dots,n.$$

We assume that $F_i(r,s_1,\dots,s_n)$, $i=1,2,\dots,n$, are continuous real functions of real variables. We also assume that

$$|f_i(z,w_1,\dots,w_n)|\leqslant F_i(|z|,|w_1|,\dots,|w_n|),\qquad i=1,2,\dots,n,$$

in the $(n+1)$-circular domain (9). We denote by $s_i = s_i(r)$ an arbitrary solution of $(24)_1$ satisfying the initial condition $(24)_2$ and we assume that this solution exists in an interval $[0, a)$ $(a \leqslant a)$. We denote by $w_i(z)$ the solution of $(23)_1$, which satisfies the initial condition $(23)_2$.

We state that under these conditions

1) $w_i(z)$ is analytic in the circle $|z| < a$;

2) in the circle $|z| < a$ we have the inequalities

$$|w_i(z)| \leqslant s_i(|z|), \quad i = 1, 2, \ldots, n.$$

Proof. We take into consideration an auxiliary system

$$\frac{ds_i}{dr} = H_i(r, s_1, s_2, \ldots, s_n), \quad i = 1, 2, \ldots, n, \tag{25}$$

with the initial condition

$$s_i(0) = 0, \quad i = 1, 2, \ldots, n, \tag{26}$$

$H_i(r, s_1, s_2, \ldots, s_n)$, $i = 1, 2, \ldots, n$, being defined by formulas (13). In view of lemma 1 and proposition 1 there exists exactly one solution of (25), (26). We denote it by $s_i = \varrho_i(r)$. By the propositions of § 2 we have in cube (12) the inequalities

$$H_i(r, s_1, s_2, \ldots, s_n) \leqslant F_i(r, s_1, s_2, \ldots, s_n), \quad i = 1, 2, \ldots, n.$$

It follows by lemma 3 of § 3 that in the common interval of the existence of the integral curves $s_i(r)$ and $\varrho_i(r)$ we have

$$\varrho_i(r) \leqslant s_i(r), \quad i = 1, 2, \ldots, n. \tag{27}$$

By the theorem on the continuation of the integral curves it follows that the integral curve $s_i = \varrho_i(r)$, $i = 1, 2, \ldots, n$, exists at least in the interval $[0, a)$. Then in the whole interval $[0, a)$ we have the inequalities

$$\varrho_i(r) \leqslant s_i(r), \quad i = 1, 2, \ldots, n. \tag{28}$$

By the theorem of Cauchy-Kovalevska $w_i(z)$ is analytic in some circle. We denote by β the maximal radius of the circle of analyticity of $w_i(z)$, $i = 1, 2, \ldots, n$. We put

$$M^i(r) = \max_{|z|=r} |w_i(z)|, \quad i = 1, 2, \ldots, n, \quad \text{for} \quad |z| < \beta.$$

By the theorem which has been proved in [5], p. 1, $M^{i'}_+(r)$, $i = 1, 2, \ldots, n$, exist and there exist points ξ^i, $i = 1, 2, \ldots, n$, such that

$$M^{i'}_+(r) = |w_i'(\xi_i)|, \quad i = 1, 2, \ldots, n.$$

For the points ξ_i the following relations hold:

$$|\xi^i| = r, \quad M^i(\xi^i) = |w_i(\xi^i)|, \quad i = 1, 2, \ldots, n, \tag{29$_1$}$$

$$|w_j(\xi^i)| \leqslant M^j(r), \quad i \neq j. \tag{29$_2$}$$

Since $w_i(z)$ satisfy system $(23)_1$ we have

$$M^{i'}_+(r) = |w_i'(\xi^i)| = |f_i(\xi^i, w_1(\xi^i), \ldots, w_n(\xi^i))|, \quad i = 1, 2, \ldots, n. \tag{30}$$

By the definition of $H_i(r, s_1, s_2, \ldots, s_n)$, the formulae $(29)_1$, $(29)_2$ and by proposition 2 from § 2 we have

$$\begin{aligned} |f_i(\xi^i, w_1(\xi^i), w_2(\xi^i), \ldots, w_n(\xi^i))| &\leqslant H_i(|\xi^i|, |w_1(\xi^i)|, |w_2(\xi^i)|, \ldots, |w_n(\xi^i)|) \\ &\leqslant H_i(r, M^1(r), M^2(r), \ldots, M^n(r)), \quad i = 1, 2, \ldots, n. \end{aligned} \tag{31}$$

Relations (30), (31) yield

$$M^{i'}_+(r) \leqslant H_i(r, M^1(r), M^2(r), \ldots, M^n(r)), \quad i = 1, 2, \ldots, n, \quad \text{for} \quad 0 < r < \beta. \tag{32}$$

Evidently we have

$$M^i(0) = |w_i(0)| = \varrho_i(0) = 0, \quad i = 1, 2, \ldots, n. \tag{33}$$

It follows by the definition of $M^i(r)$ and by proposition 1 of § 2 that $M^i(r)$, $i = 1, 2, \ldots, n$, are continuous in the interval $0 \leqslant r < \beta$ and that their graph is contained in cube (12). Hence, by lemma 2 of § 3 we obtain

$$M^i(r) \leqslant \varrho_i(r), \quad i = 1, 2, \ldots, n, \quad \text{for} \quad 0 \leqslant r < \beta, \tag{34}$$

$s_i = \varrho_i(r)$, $i = 1, 2, \ldots, n$, being the unique integral curve of (25), (26). It follows by (28) and (34) that we have

$$M^i(r) \leqslant s_i(r), \quad i = 1, 2, \ldots, n, \quad \text{for} \quad 0 \leqslant r < \min(a, \beta).$$

The definition of $M^i(r)$, $i = 1, 2, \ldots, n$, yields the relation

$$|w_i(z)| \leqslant s_i(|z|), \quad i = 1, 2, \ldots, n, \quad \text{for} \quad 0 \leqslant |z| < \min(a, \beta). \tag{35}$$

Now we shall prove that $\min(a, \beta) = a$. In fact, if we suppose $\beta < a$, then we have

$$|w_i(z)| \leqslant s_i(|z|), \quad i = 1, 2, \ldots, n, \quad \text{for} \quad 0 \leqslant |z| < \beta.$$

Since $F_i \geqslant 0$ in cube (12) then $s_i(r)$ does not decrease in $[0, a)$. Since $s_i(0) = 0$, we may conclude that

$$0 \leqslant s_i(r) < b_i, \quad i = 1, 2, \ldots, n, \quad \text{for} \quad 0 \leqslant r < a.$$

Hence

$$0 \leqslant s_i(r) \leqslant s_i(\beta) < b_i, \quad i = 1, 2, \ldots, n, \quad \text{for} \quad 0 \leqslant r < \beta.$$

Then we deduce the inequalities:

$$|w_i(z)| < c_i < b_i, \quad i = 1, 2, \ldots, n, \quad \text{for} \quad 0 \leqslant |z| < \beta \ (< \alpha),$$

where c_i, $i = 1, 2, \ldots, n$, are constants.

Now we make use of a certain lemma, which has been proved by T. Ważewski (cf. [2], p. 99, lemma 2). By that lemma we may conclude that there exists a number δ with the properties:

1) $\beta < \delta < \alpha$;

2) $w_i(z)$, $i = 1, 2, \ldots, n$, are analytic in the circle $|z| < \delta$.

This contradicts the assumption that β is the maximal radius of the circle in which $w_i(z)$ $(i = 1, 2, \ldots, n)$ is analytic. Hence we conclude that

1) $w_i(z)$, $i = 1, 2, \ldots, n$, are analytic in the circle $|z| < \alpha$;

2) in the circle $|z| < \alpha$ the following inequalities hold:

$$|w_i(z)| \leqslant s_i(|z|), \quad i = 1, 2, \ldots, n, \quad \text{for} \quad |z| < \alpha.$$

This completes the proof of theorem 1.

PROPOSITION 4. *We define the sequence of successive approximations of the solution of system* (23) *in the following manner:*

$$(36)_0 \qquad w_i^{(0)}(z) \equiv 0, \quad i = 1, 2, \ldots, n,$$

$$(36)_{m+1} \quad w_i^{(m+1)}(z) = \int_0^z f_i\big(\xi, w_1^{(m)}(\xi), w_2^{(m)}(\xi), \ldots, w_n^{(m)}(\xi)\big)\, d\xi, \quad i = 1, 2, \ldots, n,$$

$$\textit{for} \quad m = 0, 1, 2, \ldots,$$

the curve of integration in $\int_0^z$ *being the straight line segment* $\overline{0z}$. *Then all the functions* $w_i^{(m)}(z)$, $i = 1, 2, \ldots, n$, *are well-defined, analytic and satisfying the inequalities*:

$$|w_i^{(m)}(z)| \leqslant s_i(|z|), \quad i = 1, 2, \ldots, n, \quad m = 0, 1, 2, \ldots,$$

in the circle $|z| < \alpha$ *at least.* $s_i(r)$ *denotes here the solution of* $(24)_1$, $(24)_2$ *defined in* $[0, \alpha)$.

Proposition 4 can be proved by the method of mathematical induction and the lemma of Zygmund.

PROPOSITION 5. *From the foregoing proposition* 4 *and the Lipschitz condition for the functions* f_i, $i = 1, 2, \ldots, n$, *it follows directly that sequence* (36) *converges uniformly in the circle* $|z| < \alpha$ *to the solution of* $(23)_1$, $(23)_2$.

I wish to express my great thanks to Prof. T. Ważewski, and dr Cz. Olech for their valuable adwices and kind help during preparing this paper.

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Reçu par la Rédaction le 23. 3. 1960