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On an extremal function and domains of convergence of series of homogeneous polynomials

by J. Siciak (Kraków)

1. Introduction. Let C^n be a space of n complex variables $z_1, z_2, \ldots, z_n$, $n \geqslant 1$, where $z_k = x_k + iy_k$. Let $z = (z_1, \ldots, z_n)$ be a point of C^n. We shall consider the series of the form

$$\sum_{\nu=0}^{\infty} P_\nu(z), \tag{1.1}$$

where

$$P_\nu(z) = \sum_{k_1+k_2+\ldots+k_n=\nu} a_{k_1k_2\ldots k_n} z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}, \quad \nu = 0, 1, \ldots, \tag{1.2}$$

is a homogeneous polynomial of degree ν (polynomial of the form (1.2), for which all the coefficients $a_{k_1k_2\ldots k_n} = 0$, is also called the polynomial of degree ν).

Given a closed bounded point set E of the space C^n, let $G(E)$ be the largest set such that every series (1.1) for which

$$|P_\nu(z)| \leqslant M, \quad \nu = 0, 1, \ldots, \quad z \in E, \quad M = \text{const},$$

converges for $z \in G(E)$.

The aim of the paper is, first of all, to give a construction of the set $G(E)$ and to prove a sufficient condition for the set $G(E)$ to have interior points. For this purpose, using an interpolation formula for homogeneous polynomials (see § 2), we shall connect with the set E a function $t(z, E)$, absolutely homogeneous of order 1, which, in the case of the space C^2, is identical with that defined by F. Leja with the aid of the so called triangle distance (see for instance [3], [4]). The function $t(z, E)$ is a limit of some sequences of homogeneous extremal functions (see (4.3), (4.5) and (4.6)), attached to the set E. The methods of the proof of the existence of the limits (4.5) and (4.6) given by F. Leja used the fact that every homogeneous polynomial above C^2 is a product of homogeneous polynomials of degree 1. This fact no longer holds in C^n, $n \geqslant 3$.

In the last part of this paper we shall find the relations between the set $G(E)$ and the envelope of holomorphy of the circular domains; for this purpose we shall use fact that $t^*(z, E)$ (upper regularization of $t(z, E)$) is plurisubharmonic in C^n.

2. Interpolation formula for homogeneous polynomials of n complex variables. The polynomial (1.2) may be written in the form

(2.1) $$P_\nu(z) = \sum_{l=1}^{\nu_*} a_{k_{1l}k_{2l}\dots k_{nl}} z_1^{k_{1l}} z_2^{k_{2l}} \dots z_n^{k_{nl}},$$

where $\nu_* = \binom{\nu+n-1}{n-1}$ is a number of coefficients of $P_\nu(z)$ and $k_{1l}+k_{2l}+\dots+k_{nl} = \nu$ for $l = 1, 2, \dots, \nu_*$.

Let $p^{(\nu)} = \{p_1, p_2, \dots, p_{\nu_*}\}$ and $(z, p_i^{(\nu)}) = \{p_1, p_2, \dots, p_{i-1}, z, p_{i+1}, \dots, p_{\nu_*}\}$ be systems of ν_* points, where $p_i = (z_{1i}, z_{2i}, \dots, z_{ni})$, $i = 1, 2, \dots, n$, and $z = (z_1, z_2, \dots, z_n)$. Let $V(p^{(\nu)})$ be a determinant

(2.2) $$V(p^{(\nu)}) = \det(z_{1s}^{k_{1l}} z_{2s}^{k_{2l}} \dots z_{ns}^{k_{nl}}), \quad l, s = 1, 2, \dots, \nu_*,$$

and let $V(z, p_i^{(\nu)})$ be a determinant whose i-th row is $z_1^{k_{1l}} z_2^{k_{2l}} \dots z_n^{k_{nl}}$, $l = 1, 2, \dots, \nu_*$, and whose other rows are the same as in $V(p^{(\nu)})$. Let us assume that $V(p^{(\nu)}) \neq 0$ and define the homogeneous polynomials

(2.3) $$T^{(i)}(z, p^{(\nu)}) = \frac{V(z, p_i^{(\nu)})}{V(p^{(\nu)})}, \quad i = 1, 2, \dots, \nu_*.$$

We have $T^{(i)}(p_j, p^{(\nu)}) = \delta_{ij}$. Now, it is easy to prove that *if $P(z)$ is an arbitrary homogeneous polynomial of degree ν, then the following interpolation formula holds*:

(2.4) $$P(z) = \sum_{i=1}^{\nu_*} P(p_i) T^{(i)}(z, p^{(\nu)}).$$

In fact, the function $W(z) = \sum_{i=1}^{\nu_*} P(p_i) T^{(i)}(z, p^{(\nu)})$ is a homogeneous polynomial of degree ν and at the points of the system $p^{(\nu)}$ it is equal to $P(z)$, $W(p_i) = P(p_i)$, $i = 1, 2, \dots, \nu_*$. Since the determinant $V(p^{(\nu)}) \neq 0$, there exists exactly one homogeneous polynomial of degree ν, assuming values given in advance.

3. Definition of the ν-th extremal point system of set E. Let E be a closed bounded point set of C^n and let there exist for any $\nu = 0, 1, \dots$ a system $p^{(\nu)} = \{p_1, p_2, \dots, p_{\nu_*}\}$ of ν_* points of E such that $V(p^{(\nu)}) \neq 0$. Let

(3.1) $$q^{(\nu)} = \{q_1, q_2, \dots, q_{\nu_*}\}, \quad \nu = 1, 2, \dots,$$

be such a system of ν_* points of E that for every $p^{(\nu)} \subset E$

(3.2) $$|V(q^{(\nu)})| \geqslant |V(p^{(\nu)})|, \quad \nu = 1, 2, \dots$$

Such a system (one at least) certainly exists, since $|V(p^{(\nu)})|$ is a real continuous function with respect to $p^{(\nu)} = \{p_1, p_2, \dots, p_{\nu_*}\} \subset E$ and E is compact.

System (3.1) will be called the *ν-th extremal point system* (or *ν-th system of the extremal points*) *of the set E*. In the case of C^2 system (3.1) is identical with the ν-th extremal system of E with respect to the triangle distance (compare [3], [4]).

4. Extremal function $t(z, E)$. The definition of the extremal system $q^{(\nu)}$ implies that the function $T^{(i)}(z, q^{(\nu)})$ given by (2.3) satisfies the inequalities

(4.1) $$|T^{(i)}(z, q^{(\nu)})| \leqslant 1, \quad z \in E, \quad i = 1, 2, \dots, \nu_*.$$

Let

(4.2) $$t_\nu(z, E) = \max_{(i)} |T^{(i)}(z, q^{(\nu)})|, \quad \nu = 1, 2, \dots$$

THEOREM 1. *At every point z of C^n there exists a limit $t(z, E)$ (finite or not) of the sequence $\{\sqrt[\nu]{t_\nu(z, E)}\}$*

(4.3) $$t(z, E) = \lim_{\nu\to\infty} \sqrt[\nu]{t_\nu(z, E)}.$$

Proof. It is obvious that $t_\nu(0, E) = 0$, whence $t(z, 0) = 0$. Let $(z_1^0, z_2^0, \dots, z_n^0) = z^0 \neq 0$ be a fixed point of C^n. We may assume, without loss of generality, that the first coordinate $z_1^0 \neq 0$. Let μ be a fixed natural number. For every $\nu = 0, 1, \dots$ there exists one and only one pair of non-negative integers k and r such that $\nu = k\mu + r$ and $0 \leqslant r < \mu$. For any fixed μ and given z^0 there exists an index i_0 such that $1 \leqslant i_0 \leqslant \nu_*$ and $t_\mu(z^0, E) = |T^{(i_0)}(z^0, q^{(\mu)})|$. Let us denote by $P(z)$ the following homogeneous polynomial of degree ν:

$$P(z) = [T^{(i_0)}(z, q^{(\mu)})]^k (z_1)^r.$$

By interpolation formula (2.4) and inequality (4.1) we have

$$t_\mu^k(z^0, E) |z_1^0|^r \leqslant \nu_* M t_\nu(z^0, E), \quad \nu = 1, 2, \dots,$$

where $M = \max_{z\in E} |z_1|$. The last inequality is equivalent to the following one:

$$t_\mu^{k/(k\mu+r)}(z^0, E) |z_1^0|^{r/\nu} \leqslant (\nu_* M)^{1/\nu} \sqrt[\nu]{t_\nu(z^0, E)}, \quad \mu = 1, 2, \dots,$$

whence, if $\nu \to \infty$, we have

(4.4) $$\sqrt[\mu]{t_\mu(z^0, E)} \leqslant \liminf_{\nu\to\infty} \sqrt[\nu]{t_\nu(z^0, E)}, \quad \mu = 1, 2, \dots,$$

and consequently

$$\limsup_{\nu\to\infty} \sqrt[\nu]{t_\nu(z^0, E)} \leqslant \liminf_{\nu\to\infty} \sqrt[\nu]{t_\nu(z^0, E)} .$$

Since z^0 is arbitrary, it follows that (4.3) is true.

The function $t(z, E)$ is, in the case of C^2, identical with that defined by F. Leja ([3], [4]). It is the consequence of the following

THEOREM 2. *The function $t(z, E)$ is also the limit of the following sequences*:

$$(4.5)\qquad t(z, E) = \lim_{\nu\to\infty} \sqrt[\nu]{\sum_{i=1}^{\nu_*} |T^{(i)}(z, q^{(\nu)})|},$$

$$(4.6)\qquad t(z, E) = \lim_{\nu\to\infty} \sqrt[\nu]{\inf_{p^{(\nu)}\subset E} \{\max_{(i)} |T^{(i)}(z, p^{(\nu)})|\}},$$

the infimum being taken over $p^{(\nu)} \subset E$ for which $V(p^{(\nu)}) \neq 0$.

Proof. It is evidently sufficient to prove only that the following inequalities are true:

$$(4.7)\qquad t_\nu(z, E) \leqslant \sum_{i=1}^{\nu_*} |T^{(i)}(z, q^{(\nu)})| \leqslant \nu_* t_\nu(z, E), \qquad \nu = 1, 2, \ldots,$$

and

$$(4.8)\qquad \frac{1}{\nu_*} t_\nu(z, E) \leqslant \inf_{p^{(\nu)}\subset E} \{\max_{(i)} |T^{(i)}(z, p^{(\nu)})|\} \leqslant t_\nu(z, E), \qquad \nu = 1, 2, \ldots$$

Inequalities (4.7) and the second one of (4.8) are obvious. The proof of the last one may be as follows: Let $p^{(\nu)}$ be a system of ν_* points $\{p_1, p_2, \ldots, p_{\nu_*}\} \subset E$ such that $V(p^{(\nu)}) \neq 0$. Then, owing to the interpolation formula (2.4), we have

$$|T^{(i)}(z, q^{(\nu)})| \leqslant \nu_* \max_{(j)} |T^{(j)}(z, p^{(\nu)})|, \qquad i = 1, 2, \ldots, \nu_*,$$

whence, since the system $p^{(\nu)}$ may be arbitrary, the required inequality follows.

5. Some properties of the extremal function $t(z, E)$. It is an immediate consequence of the definition of $t(z, E)$ that this function is absolutely homogeneous of order 1, i. e. that $t(\lambda z, E) = |\lambda| t(z, E)$ for any $z \in C^n$ and any complex λ. We also have

$$(5.1)\qquad t(z, E) \geqslant m(|z_1| + |z_2| + \ldots + |z_n|),$$

where $m > 0$ depends on E only. Indeed, it follows from the interpolation formula (2.4) that $|z_k|^\nu \leqslant \nu_* M^\nu t_\nu(z, E)$, $\nu = 1, 2, \ldots$, where $M = \max_{z\in E} |z_k|$ and $k = 1, 2, \ldots, n$. Therefore, to complete the proof of (5.1), it is sufficient only to put $m = 1/nM$.

(5.2) *The function $t(z, E)$ is lower semicontinuous in C^n.*

By (4.4), (4.7) and theorem 1 we have

$$(5.3)\qquad \sqrt[\nu_*]{\frac{1}{\nu_*} \sum_{i=1}^{\nu_*} |T^{(i)}(z, q^{(\nu)})|} \leqslant t(z, E), \qquad z \in C^n, \qquad \nu = 1, 2, \ldots,$$

so that, owing to (4.5), the function $t(z, E)$ is the upper envelope of the continuous functions $\sqrt[\nu_*]{\frac{1}{\nu_*} \sum_{i=1}^{\nu_*} |T^{(i)}(z, q^{(\nu)})|}$, $\nu = 1, 2, \ldots$ Hence $t(z, E)$ is lower semicontinuous.

(5.4) *If $t(z, E)$ is bounded on the unit sphere $\{z \mid |z_1|^2 + \ldots + |z_n|^2 = 1\}$ and $t^*(z, E) = \limsup_{z'\to z} t(z', E)$, then $\operatorname{Log} t^*(z, E)$ is a plurisubharmonic function in C^n*

for the definition of plurisubharmonic function, see [1], [2]).

Indeed, under the hypothesis of the theorem, the function $t(z, E)$, being absolutely homogeneous, is bounded on every compact set of C^n and therefore the function $\operatorname{Log} t^*(z, E)$, being the upper envelope of plurisubharmonic functions $(1/\nu) \operatorname{Log} \{\max_{(i)} |T^{(i)}(z, q^{(\nu)})|\}$, is again a plurisubharmonic one in C^n (see [1]).

(5.5) *If $E \subset F$, then $t(z, E) \geqslant t(z, F)$.*

This follows from (4.6), since

$$\inf_{p^{(\nu)}\subset E} \{\max_{(i)} |T^{(i)}(z, p^{(\nu)})|\} \geqslant \inf_{p^{(\nu)}\subset F} \{\max_{(i)} |T^{(i)}(z, p^{(\nu)})|\}.$$

Let us denote by E_0 the subset of such points $z = (z_1, z_2, \ldots, z_n)$ of the set E that no point $sz = (sz_1, sz_2, \ldots, sz_n)$, where s is a real number > 1, belongs to E. Then

$$(5.6)\qquad t(z, E) = t(z, E_0).$$

This equality is a consequence of the obvious fact that every ν-th extremal point system of E is a ν-th extremal point system of E_0 and vice versa.

Let E_k, $k = 1, 2, \ldots, n$, be a closed bounded point set lying in the plane (z_k). Let $d(E_k)$ be its logarithmic capacity. Let E_1 contain only one point $z_1^0 \neq 0$. We shall prove the following

THEOREM 3. *If $d(E_k) > 0$, $k = 2, 3, \ldots, n$ and $E = \{z_1^0\} \times E_2 \times \ldots \times E_n$ (Cartesian product), then there exists a number $M > 0$ such that*

$$t(z, E) \leqslant M(|z_1| + |z_2| + \ldots + |z_n|), \qquad z \in C^n.$$

Remark. The set E is lying in the hyperplane $z_1 = z_1^0$. Obviously, we could take an analogous set in each hyperplane $z_k = z_k^0$, $2 \leqslant k \leqslant n$.

Proof. Let $\xi^{(\nu,k)} = \{\xi_0^k, \xi_1^k, \ldots, \xi_\nu^k\}$, $k = 2, 3, \ldots, n$, denote the ν-th extremal point system of E_k with respect to $|z_k - \zeta_k|$ (see [4], p. 260). Let $z' = (z_1^0, z_2, \ldots, z_n)$ and let $P(z)$ be an arbitrary polynomial (not necessarily homogeneous) of degree $\leqslant \nu$. Then, iterating $n-1$ times the Lagrange interpolation formula for one complex variable, we have

(5.7) $$P(z') = \sum_{k_2,k_3,\ldots,k_n=0}^{\nu} P(z_1^0, \xi_{k_2}^2, \xi_{k_3}^3, \ldots, \xi_{k_n}^n) L^{(k_2)}(z_2, E_2) L^{(k_3)}(z_3, E_3) \ldots L^{(k_n)}(z_n, E_n),$$

where

$$L^{(kj)}(z, E_j) = \prod_{s=0(s\neq k_j)}^{\nu} \frac{z_j - \xi_s^j}{\xi_{k_j}^j - \xi_s^j}.$$

If $P(z') = T^{(i)}(z', q^{(\nu)})$, then by (4.1) and (5.7)

(5.8) $$|T^{(i)}(z', q^{(\nu)})| \leqslant (\nu+1)^{n-1} \prod_{k=2}^{n} \max_{0\leqslant j\leqslant \nu} |L^{(j)}(z_k, E_k)|,$$
$$i = 1, 2, \ldots, \nu_*, \quad \nu = 1, 2, \ldots$$

But we know ([4], p. 265) that there exists a limit $L(z_k, E_k) = \lim_{\nu\to\infty} \sqrt[\nu]{\sum_{j=0}^{\nu} |L^{(j)}(z_k, E_k)|}$ and the function $L(z_k, E_k)$, $k = 2, 3, \ldots, n$, is bounded in any compact set contained in the plane (z_k). It is also known that $\max_{(j)} |L^{(j)}(z_k, E_k)| \leqslant L(z_k, E_k)$ for any z_k, $k = 2, 3, \ldots, n$. Therefore there exists a number $L > 1$ such that if $|z_k| \leqslant |z_k^0|$, $k = 2, 3, \ldots, n$, then

$$\prod_{k=2}^{n} \max_{0\leqslant j\leqslant \nu} |L^{(j)}(z_k, E_k)| \leqslant L^{(n-1)\nu},$$

whence by (5.8)

(5.9) $$|T^{(i)}(z', q^{(\nu)})| \leqslant [(\nu+1)L^\nu]^{n-1}.$$

The polynomial $T^{(i)}(z, q^{(\nu)})$ may be written in the form

(5.10) $$T^{(i)}(z, q^{(\nu)}) = \sum_{k_1+k_2+\ldots+k_n=\nu} b_{k_1k_2\ldots k_n} z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}.$$

By (5.9) and owing to the Cauchy inequalities, we have the inequality

$$|b_{k_1k_2\ldots k_n} z_1^{0\,k_1}| \leqslant [(\nu+1)L^\nu]^{n-1} / r^{k_2+k_3+\ldots+k_n},$$

which is equivalent to the following one:

(5.11) $$|b_{k_1k_2\ldots k_n}| \leqslant \frac{(\nu+1)^{n-1}}{\varrho^\nu}, \quad \varrho = r/L^{n-1},$$

whence by (5.10)

$$|T^{(i)}(z, q^{(\nu)})| \leqslant \nu_*(\nu+1)^{n-1}, \quad |z_k| \leqslant \varrho, \quad k = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, \nu_*.$$

Therefore

$$t_\nu(z, E) \leqslant \nu_*(\nu+1)^{n-1}, \quad |z_k| \leqslant \varrho, \quad k = 1, 2, \ldots, n, \quad \nu = 1, 2, \ldots,$$

or

$$\sqrt[\nu]{t_\nu(z, E)} \leqslant \sqrt[\nu]{\nu_*(\nu+1)^{n-1}}, \quad |z_k| \leqslant \varrho, \quad k = 1, 2, \ldots, n, \quad \nu = 1, 2, \ldots,$$

whence, if $\nu\to\infty$, we have

$$t(z, E) \leqslant 1, \quad |z_k| \leqslant \varrho, \quad k = 1, 2, \ldots, n.$$

Since $t(z, E)$ is absolutely homogeneous of order 1, we have

$$t(z, E) \leqslant \frac{1}{\varrho}(|z_1| + |z_2| + \ldots + |z_n|), \quad z \in C^n.$$

If we now put $M = 1/\varrho$, our theorem is proved.

COROLLARY. *If E contains interior points then $t(z, E)$ is bounded on any compact set of C^n.*

Now we shall prove an inequality which is entirely analogous to the inequality of Bernstein-Walsh for polynomials of one complex variable. Namely, *if $P(z)$ is a homogeneous polynomial of degree μ and $M = \max_{z\in E} |P(z)|$, then*

(5.12) $$|P(z)| \leqslant M t^\mu(z, E), \quad z \in C^n.$$

In fact, if k is any integer and $q^{(k\mu)} = \{q_1, q_2, \ldots, q_{(k\mu)_*}\}$ is the $(k\mu)$-th extremal point system of E, then, in accordance with (2.4),

$$P^k(z) = \sum_{j=1}^{(k\mu)_*} P^k(q_j) T^{(j)}(z, q^{(k\mu)}),$$

whence

$$|P(z)|^k \leqslant M^k (k\mu)_* \max_{(j)} |T^{(j)}(z, q^{(k\mu)})|.$$

Therefore

$$|P(z)| \leqslant M \sqrt[k]{(k\mu)_*} \left[\sqrt[k\mu]{\max_{(j)} |T^{(j)}(z, q^{(k\mu)})|}\right]^\mu, \quad k = 1, 2, \ldots$$

and thus, by virtue of (4.2) and (4.3), the desired inequality is proved. Inequality (5.12) may be used for proving some theorems concerning the best approximation of functions of several complex variables defined in the circular sets by polynomials. It will be the subject of my other paper.

From (5.12) and from the definition of $t(z, E)$ it follows that $t(z, E)$ is an upper envelope of all functions $\sqrt[\nu]{|P_\nu(z)|}$, where $P_\nu(z)$ is a homogeneous polynomial of degree ν, $\nu = 1, 2, \ldots$, satisfying the inequality $|P_\nu(z)| \leqslant 1$, $z \in E$ (instead of the constant 1, one could take any sequence $\{M_\nu\}$ such that $\sqrt[\nu]{M_\nu} \to 1$). In particular, if for a bounded point set E there exist k homogeneous polynomials $Q_1(z), Q_2(z), \ldots, Q_n(z)$, of degree $\nu_1, \nu_2, \ldots, \nu_k$ respectively, such that $E = \{z \mid |Q_i(z)| = 1,\ i = 1, 2, \ldots, k\}$, then

$$t(z, E) = \max\{\sqrt[\nu_1]{|Q_1(z)|},\ \sqrt[\nu_2]{|Q_2(z)|}, \ldots, \sqrt[\nu_k]{|Q_k(z)|}\}.$$

E.g. if E is a polycylinder, $E = \{z \mid |z_k| = r_k,\ k = 1, 2, \ldots, n\}$, then $t(z, E) = \max\limits_{1 \leqslant k \leqslant n}\{|z_k|/r_k\}$.

6. Application of $t(z, E)$ to the estimation of the domains of convergence of series (1.1) whose terms are uniformly bounded on E. THEOREM 4. *Each series (1.1) whose terms are uniformly bounded on E converges absolutely at least in the set $G(E)$ given by*

$$G(E) = \{z \mid t(z, E) < 1\}. \tag{6.1}$$

Proof. Let the polynomials $P_\nu(z)$, $\nu = 0, 1, \ldots$, satisfy the inequalities

$$|P_\nu(z)| \leqslant K, \quad z \in E, \quad K = \text{const}.$$

By virtue of interpolation formula (2.4) we have

$$P_\nu(z) = \sum_{i=1}^{\nu_*} P_\nu(q_i) T^{(i)}(z, q^{(\nu)}), \quad \nu = 0, 1, \ldots,$$

whence, if $q^{(\nu)}$ is an extremal point system of E,

$$|P_\nu(z)| \leqslant \nu_* K t_\nu(z, E), \quad \nu = 1, 2, \ldots$$

Then, according to (4.3), we have the inequality

$$\limsup_{\nu \to \infty} \sqrt[\nu]{|P_\nu(z)|} \leqslant t(z, E), \tag{6.2}$$

which implies the theorem.

Let us add that (6.2) implies that a series whose terms $P_\nu(z)$ are uniformly bounded on E converges uniformly absolutely for z in $G_\varepsilon(E) = \{z \mid t(z, E) \leqslant 1 - \varepsilon\}$, $0 < \varepsilon < 1$.

If $t(z, E)$ is bounded in any compact set of C^n, then $G(E)$ contains interior points. The interior $\mathring{G}(E)$ of $G(E)$ is given by

$$\mathring{G}(E) = \{z \mid t^*(z, E) < 1\}, \tag{6.3}$$

where $t^*(z, E) = \limsup\limits_{z' \to z} t(z', E)$.

In fact, if for a given point z^0 we have $t^*(z, E) < 1$, then by the upper semicontinuity of $t^*(z, E)$, the inequality $t^*(z, E) < 1$ must hold in a neighbourhood of z^0. Naturally $t(z, E) < 1$ holds in that neighbourhood. Therefore z^0 is an interior point of $G(E)$.

Conversely, if z^0 is an interior point of $G(E)$, then in a neighbourhood of z^0 the inequality $t(z, E) < 1$ holds, which implies that $t^*(z, E) \leqslant 1$ in that neighbourhood. Since $t^*(z, E)$ is homogeneous of order 1, then $t^*(z^0, E) < 1$, whence, because of the upper semicontinuity of $t^*(z, E)$, it follows that $t^*(z, E) < 1$ in a neighbourhood of z^0. Thus (6.3) is proved.

Each series (1.1) whose terms are uniformly bounded on E converges almost uniformly absolutely on $\mathring{G}(E)$.

Now we shall prove that

There exists a series of homogeneous polynomials uniformly bounded on E whose domain of almost uniform convergence is exactly equal to $\mathring{G}(E)$.

Indeed, let $\{p_k\}$ be a sequence of points each of which is repeated in that sequence infinitely many times. Moreover, let the set P of points of this sequence be contained in $E_1 = \{z \mid t(z, E) = 1\}$ and let P be dense in E_1. Put $P_\nu(z) = T^{(i_\nu)}(z, q^{(\nu)})$, where $q^{(\nu)}$ is the ν-th system of extremal points of E and $|T^{(i_\nu)}(p_\nu, q^{(\nu)})| = \max\limits_{1 \leqslant i \leqslant \nu_*} |T^{(i)}(p_\nu, q^{(\nu)})|$. By (4.1) we have $|P_\nu(z)| \leqslant 1$, $z \in E$. Then the series $\sum\limits_{\nu=0}^{\infty} P_\nu(z)$ converges almost uniformly absolutely in $\mathring{G}(E)$. Since for a given $p \in \{p_k\}$

$$\limsup_{\nu \to \infty} \sqrt[\nu]{|P_\nu(p)|} = \limsup_{\nu \to \infty} \sqrt[\nu]{|T^{(i_\nu)}(p, q^{(\nu)})|} = 1,$$

the series diverges in the set $\{z \mid z = \lambda p,\ p \in \{p_k\},\ |\lambda| > 1\}$, which is dense in the complement of $G(E)$. Therefore the series constructed cannot converge in any domain containing $\mathring{G}(E)$.

7. Envelope of holomorphy of starlike circular domains. Let D be a circular domain starlike with respect to 0, i. e. such a domain that, if $z^0 \in D$, then $\{z \mid z = tz^0,\ |t| \leqslant 1\} \subset D$ (t being a complex number). Let $E = D^\cdot$, $D^\cdot$ denoting the boundary of D. Let $H(D)$ denote the envelope of holomorphy of D. $H(D)$ is the interior of the intersection of all domains of existence of functions holomorphic in D (see [2]). Each function holomorphic in D is analytically continuable on $H(D)$. I shall prove that

The set $\mathring{G}(E)$ *defined by* (6.3) *is in this case identical with envelope of holomorphy of* D,

$$\mathring{G}(E) = H(D). \tag{7.1}$$

Indeed, since the domain D is starlike, then it does not possess a Nebenhülle (see [2]). Therefore $H(D) = H(\bar{D})$, i. e. $H(D)$ is the interior of the intersection of all domains of existence of functions holomorphic in $\bar{D}$. But each function holomorphic in $\bar{D}$ may be expressed as a series of homogeneous polynomials converging uniformly on $\bar{D}$ (see [2], p. 114). The terms of the series must be uniformly bounded on $E = D^{\cdot}$. Therefore the series converges uniformly on $\mathring{G}(E)$ at least, whence $\mathring{G}(E) \subset H(D)$.

On the other hand, since $t(z, E) = t(z, \bar{D})$, each series (1.1) whose terms are uniformly bounded on E converges almost uniformly in D at least, whence it follows that it converges almost uniformly in $H(D)$ at least (see [2], p. 276). Therefore $H(D) \subset \mathring{G}(E)$, and (7.1) is completely proved.

We have thus proved that in order to construct the envelope of holomorphy of a given circular domain, it is sufficient to construct $t(z, D)$.

8. Final remarks. 8.1. A remark on circular domains. With any circular domain D, starlike with respect to 0, one may connect a homogeneous function $t_D(z)$ in the following manner. We define the value of $t_D(z)$ at a point $z^1 \in C^n$ as equal to $\|z^1\|/\|z^2\|$, where $\|z^i\| = \sqrt{\sum_{k=1}^{n} |z_k^i|^2}$ and z^2 is such a point of the domain D that the segment $\{z \mid z = tz^2,\ 0 \leqslant t \leqslant 1\}$ belongs to D and lies on the half straight-line going from 0 through z^1. The function $t_D(z)$ is absolutely homogeneous of order 1, upper semicontinuous and such that $D = \{z \mid t_D(z) < 1\}$.

THEOREM. *The circular domain D, starlike with respect to 0, is a domain of holomorphy* (i. e. $D = H(D)$) *if and only if the function* $\log t_D(z)$ *is plurisubharmonic in* C^n.

Proof. Sufficiency. It is known (see [1]) that if a domain G can be expressed in the form $G = \{z \mid V(z) < 0\}$, $V(z)$ being a plurisubharmonic function in a neighbourhood of $\bar{G}$, then G is a domain of holomorphy. Since $\log t_D(z)$ is plurisubharmonic in C^n and $D = \{z \mid \log t_D(z) < 0\}$, then, in accordance with the theorem cited, the domain D must be a domain of holomorphy.

The necessity of the condition follows from (7.1) and theorem cited, since, putting $E = D^{\cdot}$, we have $\mathring{G}(E) = D$, whence $t_D(z) = t^*(z, E)$. But we know by (5.4) that $\log t^*(z, E)$ is plurisubharmonic in C^n.

8.2. A remark on n-circular domains. We say that D is a complete n-circular domain (a Reinhardt domain) if together with $z^0 \in D$ we have $\{z \mid |z_k| = |z_k^0|,\ k = 1, 2, \ldots, n\} \subset D$. It is obvious that every complete n-circular domain is a circular domain starlike with respect to 0. It is easy to prove that the real function $f(z) = f(z_1, \ldots, z_n)$, upper semicontinuous in an n-circular domain D, depending only on absolute values of the coordinates of z (i.e. $f(z_1, \ldots, z_n) = f(|z_1^0|, \ldots, |z_n^0|)$, if $|z_k| = |z_k^0|$, $k = 1, 2, \ldots, n$), is plurisubharmonic in D if and only if the function $F(\xi_1, \xi_2, \ldots, \xi_n) = f(e^{\xi_1}, e^{\xi_2}, \ldots, e^{\xi_n})$, $\xi_k = \log|z_k|$, $(z_1, z_2, \ldots, z_n) \in D$, is convex with respect to $(\xi_1, \xi_2, \ldots, \xi_n)$. Using this theorem one may prove that *the complete n-circular domain is a domain of holomorphy if and only if the function* $\log t_D(z)$ *is convex in* C^n *with respect to* $(\log|z_1|, \ldots, \log|z_n|)$.

8.3. Examples of $t(z, E)$ for some sets E. If $E = \{z \mid \|z\| = 1\}$, i.e. if E is a sphere with radius 1 and centre 0, then $t(z, E) = \|z\|/r$. This follows from the fact that the sphere is a starlike domain of holomorphy and the unique function absolutely homogeneous of order 1 and equal to 1 on the sphere is the function $\|z\|/r$.

If $E = \{z \mid z_n = z_n^0,\ |z_k| = r_k,\ k = 1, 2, \ldots, n-1\}$, $|z_n^0| = r_n \neq 0$, $r_k \neq 0$, then $t(z, E) = \max\{|z_1|/r_1, |z_2|/r_2, \ldots, |z_n|/r_n\}$. This follows from the fact that every series (1.1) whose terms are uniformly bounded on E converges at least in the domain $\{z \mid \max_k |z_k|/r_k < 1\}$ (it is a consequence of the Cauchy inequalities for coefficients of expansion in the power series; compare the reasoning of [5], p. 21) and, hence, that this domain—as a polycylinder—is a domain of holomorphy.

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