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On a property of the upper envelope of plurisubharmonic functions

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One of the most important problems in the theory of analytic functions of several complex variables is the construction of the envelope of holomorphy of a given domain. In [1] and [3] a new construction of the envelope of holomorphy $H(D)$ of a schlicht domain D is given provided that $H(D)$ is also schlicht. The main tool in this construction is the upper envelope $\Phi(p)$ of all plurisubharmonic functions in D which are equal or smaller that the given function $f(p)$ in D.

The aim of this note is to prove formula (3), which states that the upper limit of the function $\Phi(p)$ equals $f(p)$ on some set G contained in D and defined by the mass-distributions connected with the so called extremal systems of points of the domains D. This note is a continuation of a previous paper (see [2]) and we shall use the same notation.

Let D be a bounded domain in the space of two complex variables and $\bar{D}$ its closure. We put [1] $E = \bar{D}$, $\lambda = 1$. Let $f(p)$ be a continuous function defined in D (e.g. $f(p) = -\log d_D(p)$, where $d_D(p)$ denotes the Euclidean distance of a point $p \in D$ from the boundary of D). Let $h(p, q)$ be a function which satisfies the following conditions: (i) $h(p, q)$ is a continuous function of the points p and q defined in a domain $B \supset \bar{D}$, (ii) $|h(p, q)| = |h(q, p)| = 0$, (iii) for fixed q, $h(p, q)$ is an analytic function of p.

Using the method of extremal points indicated in [2] for the set $E = \bar{D}$ and the function $f(p)$ bounded from below we obtain the plurisubharmonic function

$$u_h(p) = \int_D \log|h(p, q)|\, d\mu_h(q) ,$$

which possesses the following properties:

(1) $$u_h(p) \leqslant \gamma_h + f(p)$$

(1) Compare the notation used in [2].

in $\bar{D}$ except for a set of capacity (1) 0;

$$u_h(p) = \gamma_h + f(p) \tag{2}$$

in $\bar{D}_{\mu_h}$ except for a set of capacity 0 contained in $\bar{D}_{\mu_h}$. At exceptional points $u_h(p) \geqslant \gamma_h + f(p)$. Here

$$\gamma_h = \log d(\bar{D}, |h|, f) + \int_D f(q)\, d\mu_h(q)$$

and $\bar{D}_{\mu_h}$ denotes the kernel of mass distribution μ_h.

In the case under consideration we can obtain stronger results than (1) and (2). Let $K(p, r)$ be an arbitrary hypersphere with a centre at a point $p \in D$ and radius $r > 0$ contained in D. Since $u_h(p)$ is subharmonic in D, from (1) follows

$$u_h(p) \leqslant \frac{1}{\operatorname{vol} K(p, r(\varepsilon))} \int_{K(p, r(\varepsilon))} u_h(q)\, d\omega_q \leqslant \gamma_h + f(p) + \varepsilon$$

for every point $p \in D$ whose distance from the boundary $D^{\cdot}$ of D is smaller than $r(\varepsilon)$. Since $\varepsilon > 0$ is an arbitrary number, inequality (1) holds in the whole domain D. Similarly, equality (2) holds in $\bar{D}_{\mu_h} - D^{\cdot}$ without any exception.

In [2] we considered the function

$$w_1(p) = \overline{\lim_{q \to p}} \{\sup_h [u_h(q) - \gamma_h]\},$$

which is plurisubharmonic in D and satisfies the inequality

$$w_1(p) \leqslant f(p).$$

We shall prove that

$$\overline{\lim_{p \to p_0 \in G}} w_1(p) = \overline{\lim_{p \to p_0 \in G}} \Phi(p) = f(p_0), \quad G = \sum_h \bar{D}_{\mu_h} - D^{\cdot}, \tag{3}$$

where $\Phi(p)$ denotes the upper envelope of all plurisubharmonic functions in D which are $\leqslant f(p)$.

In fact, we have

$$u_h(q) - \gamma_h \leqslant f(q), \quad q \in D$$

and

$$\overline{\lim_{q \to p}} [u_h(q) - \gamma_h] = f(p), \quad p \in \bar{D}_{\mu_h} - D^{\cdot}.$$

According to the definition of $\Phi(p)$ we have $u_h(q) - \gamma_h \leqslant \Phi(q)$ for every function $h(p, q)$ which satisfies (i), (ii) and (iii).

(1) For the definition of the generalized capacity $d(\bar{D}, |h|, f)$ see [2].

Let p_0 be an arbitrary fixed point in $G = \sum_h \bar{D}_{\mu_h} - D^{\cdot}$. Suppose that p_0 belongs to $\bar{D}_{\mu_{h^*}} - D^{\cdot}$. Therefore

$$u_{h^*}(q) - \gamma_{h^*} \leqslant \sup_h [u_h(q) - \gamma_h] \leqslant \Phi(q)$$

and

$$f(p_0) = \overline{\lim_{q \to p_0}} [u_{h^*}(q) - \gamma_{h^*}] \leqslant \overline{\lim_{q \to p_0}} \{\sup_h [u_h(q) - \gamma_h]\} \leqslant \overline{\lim_{q \to p_0}} \Phi(q) \leqslant f(p_0). \tag{4}$$

From (4) follows

$$\overline{\lim_{q \to p_0}} w_1(q) = \overline{\lim_{q \to p_0}} \Phi(q) = f(p_0).$$

A similar result can be obtained if $\lambda = -1$ and if we consider the function $w_{-1}(p)$ and the lower envelope $\varphi(p)$.

References

[1] H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Silov boundaries*, Trans. Amer. Math. Soc. 91 (1959), p. 246-276.

[2] J. Górski, *The method of extremal points and Dirichlet's problem in the space of two complex variables*, Arch. Rat. Mech. Anal., Vol. 4, No 5 (1960), p. 412-427.

[3] J. G. Taylor, *A theorem of continuation for functions of several complex variables*, Proc. Cambr. Phil. Soc., Vol. 54, 3 (1958), p. 377-382.

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