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A note on approximation of solutions of abstract differential equations

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We present in this paper some theorems concerning the approximation of solutions of the equation

$$(*)\qquad \frac{dx}{dt} = A(t)x + f(t, x), \quad 0 \leqslant t < a.$$

$A(t)$ is a closed linear operator and the domain and range of $A(t)$ are included in the Banach space E. The function $f(t, x)$ is defined on the Cartesian product $\langle 0, a) \times E$ and takes on values in E.

Our method is closely related to the Hille-Yosida theory of semigroups of linear bounded operators [3]. On the other hand, we apply in this paper a technique similar to that developed in [4]. For the terminology and notation we refer to [3].

In [2] and [7] it is assumed that $(*)$ implies a suitable intergal equation. In what follows we do not need any integral equations implied by $(*)$. The case of time-independent $A(t) = A = \text{const}$ corresponds to the abstract Cauchy problem (see for instance [3]) discussed by Hille. For other general results in this matter we refer to [9].

Let E be a real Banach space. The norm of an element $x \in E$ is denoted by $|x|$. The norm of a linear operator V is denoted by $|V|$. By small Greek letters we denote the real-valued functions. The right-hand upper Dini derivative $\overline{D}_+\varphi(t_0)$ is defined by

$$\overline{D}_+\varphi(t_0) = \limsup_{h\to 0+} \frac{\varphi(t_0+h)-\varphi(t_0)}{h}.$$

Let the function $x(t)$ be defined in the neighbourhood of t_0. The values of $x(t)$ belong to E. The right-hand derivative $D_+x(t_0)$ is defined by

$$D_+x(t_0) = \lim_{h\to 0+} \frac{x(t_0+h)-x(t_0)}{h}.$$

We shall make use of the following assumption:

(1) The non-negative function $\sigma(t, u)$ is continuous for $t \in \langle 0, a)$ and $u \geqslant 0$. The right-hand maximum solution $\omega(t, u_0)$ of the ordinary differential equation $u' = \sigma(t, u)$ satisfying the initial condition $\omega(0, u_0) = u_0$ exists in $\langle 0, a)$ for each $u_0 \geqslant 0$.

The following lemma will be needed for later use:

LEMMA 1. ([8]). *Let the function $\varphi(t) \geqslant 0$ be continuous on $\langle 0, a)$. Suppose that* (1) *holds. Let the inequality*

$$\overline{D}_+\varphi(t) \leqslant \sigma(t, \varphi(t))$$

be satisfied nearly everywhere on $\langle 0, a)$ (1). *Then $\varphi(t) \leqslant \omega(t, \varphi(0))$ in $\langle 0, a)$.*

We now prove the following

LEMMA 2. *Suppose that $\sigma(t, u)$ satisfies* (1), *and $\omega(t, 0) \equiv 0$ in $\langle 0, a)$. Let the functions $\varphi_n(t) \geqslant 0$ be continuous on $\langle 0, a)$ and suppose that the sequence $\psi_n(t) = \int_0^t \varphi_n(s)\,ds$ converges to zero almost uniformly on $\langle 0, a)$. Suppose that $\lim_{n\to\infty} u_n = 0$, $u_n \geqslant 0$. Our assumptions imply that the sequence $\omega_n(t)$ of the right-hand maximum solutions of equations $u' = \sigma(t, u) + \varphi_n(t)$ such that $\omega_n(0) = u_n$ is alsmot uniformly convergent to zero on $\langle 0, a)$.*

Proof. The function $\omega_n(t)$ satisfies the equation

$$\omega_n'(t) = \sigma(t, \omega_n(t)) + \varphi_n(t)$$

for $t \in \langle 0, a_n)$, $a_n \leqslant a$. We now define $\varrho_n(t) = \omega_n(t) - \int_0^t \varphi_n(s)\,ds = \omega_n(t) - \psi_n(t)$. Hence

$$\varrho_n'(t) = \sigma(t, \varrho_n(t) + \psi_n(t)) .$$

The sequence $\sigma_n(t, u) = \sigma(t, u + \psi_n(t))$ converges to $\sigma(t, u)$ almost uniformly for $(t, u) \in \langle 0, a) \times \langle 0, \infty)$. Consequently, the sequence $\eta_n(t)$ of right-hand maximum solutions of equations $u' = \sigma_n(t, u)$ such that $\eta_n(0) = u_n$ tends to $\omega(t, 0) \equiv 0$ almost uniformly on $\langle 0, a)$. On the other hand, $\varrho_n(t) \leqslant \eta_n(t)$. We conclude therefore that $\omega_n(t) \leqslant \eta_n(t) + \psi_n(t)$. This completes the proof.

The last lemma is the following one (see [4], lemma 2):

LEMMA 3. *Suppose we are given two linear operators A_1 and A_2. We assume that $(\lambda I - A_1)^{-1} = \widetilde{R}(\lambda, A_1)$, $(\lambda I - A_2)^{-1} = \widetilde{R}(\lambda, A_2)$ exist for λ sufficiently large. The functions $x_1(t)$, $x_2(t)$ are defined for $t \in \langle \xi, \xi + \delta)$ $(\delta > 0)$, $x_i(\xi) \in D[A_i]$ and*

$$D_+x_i(\xi) = A_i x_i(\xi) + y_i \quad (i = 1, 2) .$$

(1) This means that the set of those t for which the inequality is not satisfied is at most denumerable.

Suppose that

$$\lim \lambda \widetilde{R}(\lambda, A_i) A_i x_i(\xi) = A_i x_i(\xi) \quad (i = 1, 2)$$

and let the inequality

$$\left| \lambda \widetilde{R}(\lambda, A_1) x_1(\xi) - \lambda \widetilde{R}(\lambda, A_2) x_2(\xi) + \frac{1}{\lambda}[y_1 - y_2] \right| \leqslant \frac{1}{\lambda} p + |x_1(\xi) - x_2(\xi)|$$

be satisfied for λ sufficiently large. Then $\overline{D}_+|x_1(\xi) - x_2(\xi)| \leqslant p$.

We now prove the following

THEOREM 1. *Suppose that $U_1(t)$ is a closed linear operator and $|\lambda R(\lambda, U_1(t))| \leqslant 1$ for $\lambda > 0$, $t \in \langle 0, a)$ and $D[U_1(t)]$ is dense in E for each $t \in \langle 0, a)$. Let $U_2(t)$ be a family of linear operators such that $D[U_2(t)] \subset D[U_1(t)]$ for $t \in \langle 0, a)$. Let the functions $z_i(t)$ $(i = 1, 2)$ be continuous on $\langle 0, a)$ and suppose that*

$$D_+ z_i(t) = U_i(t) z_i(t) + y_i(t) \quad (i = 1, 2)$$

for $t \in Z \subset \langle 0, a)$. Our assumptions imply that the inequality

$$\overline{D}_+|z_1(t) - z_2(t)| \leqslant |[U_1(t) - U_2(t)] z_2(t)| + |y_1(t) - y_2(t)|$$

holds for $t \in Z \subset \langle 0, a)$.

Proof. Suppose that $\xi \in Z$. It is a simple matter to verify that the operators $A_1 = U_1(\xi)$, $A_2 = \theta$ and $y_1 = [U_1(\xi) - U_2(\xi)] z_2(\xi) + [y_1(\xi) - y_2(\xi)]$, $y_2 = \theta$, $z_1(t) - z_2(t) = x_1(t)$, $x_2(t) = \theta$ satisfy the assumptions of lemma 3. Furthermore, $p = |[U_1(\xi) - U_2(\xi)] z_2(\xi)| + |y_1(\xi) - y_2(\xi)|$. The assertion of our theorem now follows from lemma 3.

Suppose we are given a sequence $\{A_n(t)\}$ of closed linear operators with domains $D[A_n(t)]$ dense in E. According to the Hille-Yosida theorem [3] $A_n(t)$ are infinitesimal generators of semi-groups of contraction operators if and only if

(2) $\quad |\lambda R(\lambda, A_n(t))| \leqslant 1 \quad$ for $\quad \lambda > 0$, $\quad t \in \langle 0, a)$, $\quad n = 1, 2, \ldots$

Let $A(t)$ be for each $t \in \langle 0, a)$ a linear operator and suppose that

(3) $\quad D[A(t)] \subset D[A_n(t)]$, $\quad t \in \langle 0, a)$, $\quad n = 1, 2, \ldots$

We now introduce some assumptions concerning the approximation of a non-linear member of $(*)$.

(4) For every $\varrho > 0$ the sequence $f_n(t, x)$ is uniformly convergent to $f(t, x)$ for $t \in \langle 0, a - \eta\rangle$ $(\eta > 0)$, $|x| \leqslant \varrho$.

(5) The function $\sigma(t, u)$ satisfies (1) and $|f(t, x)-f(t, y)| \leqslant \sigma(t, |x-y|)$ for $t \in \langle 0, a)$ and $x, y \in E$. Furthermore, $\omega(t, 0) \equiv 0$ on $\langle 0, a)$.

THEOREM 2. *Let conditions* (2)-(5) *be satisfied. Suppose that*

(6) $$D_+x(t) = A(t)x(t)+f(t, x(t))$$

nearly everywhere on $\langle 0, a)$. *Let the function* $x_n(t)$ $(n = 1, 2, \ldots)$ *satisfy nearly everywhere the equation*

(7) $$D_+x_n(t) = A_n(t)x_n(t)+f_n(t, x_n(t)) .$$

We assume that $x(t), x_n(t)$ *are strongly continuous on* $\langle 0, a)$. *Suppose that* $x_n(t)$ *is uniformly bounded on each subinterval of* $\langle 0, a)$. *We assume that* $\lim\limits_{n\to\infty} x_n(0) = x(0)$ *and that*

(8) $$|[A_n(t)-A(t)]x(t)| \leqslant \gamma_n(t)$$

nearly everywhere for each n. *It is supposed that* $\gamma_n(t)$ *are continuous on* $\langle 0, a)$ *and*

(9) $$\beta_n(t) = \int_0^t \gamma_n(s)\,ds \to 0$$

almost uniformly on $\langle 0, a)$.

Our assumptions imply that $x_n(t) \to x(t)$ *almost uniformly on* $\langle 0, a)$.

Proof. It follows from theorem 1 $(U_1 = A_n,\ U_2 = A)$ that the inequality

$$\overline{D}_+|x_n(t)-x(t)| \leqslant \gamma_n(t)+|f_n(t, x_n(t))-f(t, x(t))|$$

holds nearly everywhere on $\langle 0, a)$. By (5)

$$|f_n(t, x_n(t))-f(t, x(t))| \leqslant \sigma(t, |x_n(t)-x(t)|)+|f_n(t, x_n(t))-f(t, x_n(t))| .$$

It follows from (4) that

$$\varepsilon_n = \sup_{\langle 0, a-\eta\rangle} |f_n(t, x_n(t))-f(t, x_n(t))| \to 0 .$$

Hence

$$\overline{D}_+|x_n(t)-x(t)| \leqslant \sigma(t, |x_n(t)-x(t)|)+\varphi_n(t)$$

where $\varphi_n(t) = \gamma_n(t)+\varepsilon_n$. The last inequality holds nearly everywhere on $\langle 0, a-\eta\rangle$. Making use of lemma 1 we conclude that

(10) $$|x_n(t)-x(t)| \leqslant \omega_n(t) \quad \text{for} \quad t \in \langle 0, a-\eta\rangle$$

for n sufficiently large: $\omega_n(t)$ denotes here the right-hand maximum solution of equation $u' = \sigma(t, u)+\varphi_n(t)$ such that $\omega_n(0) = |x_n(0)-x(0)|$.

By lemma 2 $\omega_n(t) \to 0$ almost uniformly on $\langle 0, a)$. The assertion of our theorem follows from (10).

Remark 1. It is a simple matter to formulate theorems similar to theorem 2 in the case where (6) and (7) are satisfied almost everywhere. Lemma 1 may be then replaced by a suitable theorem concerning differential inequalities with right-hand members satisfying the Carathéodory assumptions (see [1] and [5]). The real-valued functions $|x_n(t)-x(t)|$ may then satisfy any of the conditions (I), (II), (III) or (IV) of [5]. In order to apply the technique used in the proof of theorem 2 one must introduce instead of (2)-(5) some suitable conditions formulated in the language of the Lebesgue-Bochner integral theory.

Remark 2. Suppose that $f_n(t, x) = f(t, x) = \theta$. Let $A_n(t)$ be bounded operators and suppose that

(1) $$|[A_n(t)-A(t)]x(t)| \to 0$$

almost uniformly on $\langle 0, a)$. Assume that $|\lambda R(\lambda, A_n(t))| \leqslant 1$ for $\lambda > 0$ and $t \in \langle 0, a)$, $n = 1, 2, \ldots$ We can then apply our theorem. Condition (11) is the consistency condition appearing in Lax's stability theorem (see [6]). If for each t $A(t)$ is the infinitesimal generator of a semi-group of contraction operators, then we can take $A_n(t) = nA(t)R(n, A(t))$ or $A_n(t) = n[T(1/n, A(t))-I]$. Then the convergence in (11) is a pointwise one: in order to ensure the uniform convergence one must introduce some additional conditions. The simplest situation is that where $A(t)$ does not depend on t—$A(t) = A = \text{const}$. We may then put $A_n = n[T(1/n, A)-I]$, $A_n = nAR(n, A)$, $A_n = \frac{n}{t}AR\left(\frac{n}{t}, A\right)$ and thus obtain by theorem 2 the formulas of Dunford, Yosida and Hille respectively.

Remark 3. Suppose that (2) holds and let $A(t)$ be for each t the infinitesimal generator of a semi-group of contraction operators. Let the functions $\varphi_n(t) = |[A_n(t)-A(t)]x(t)|$ be continuous on $\langle 0, a)$. Observe now that $(A_n(t)-A(t))x(t) = (A_n(t)-A(t))\cdot R(1, A(t))(I-A(t))x(t)$. We may apply theorem 2 if the following conditions are satisfied:

(α) $[A_n(t)-A(t)]x \to 0$ for $t \in \langle 0, a)$, $x \in D[A(t)]$,

(β) $|[A_n(t)-A(t)]R(1, A(t))| \leqslant M$ on $\langle 0, a)$,

(γ) $|x'(t)| = |A(t)x(t)|$ is summable over $\langle 0, a-\eta\rangle$ $(\eta > 0)$.

Indeed, (α) implies that $\varphi_n(t) \to 0$. By (β) and (γ) $\varphi_n(t) \leqslant M[|x'(t)|+|x(t)|] = \psi(t)$ and $\psi(t)$ is summable over $\langle 0, a-\eta\rangle$. It follows from the Lebesgue theorem that $\int_0^t \varphi_n(s)\,ds \to 0$ almost uniformly on $\langle 0, a-\eta\rangle$, which was to be proved.

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Investigation of some measures and sequences related to the extreme points

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I. Introduction. Let X be a Hausdorff space. We shall denote the points of this space by $a, x, y, \ldots$ Let Φ be a function defined on $X\times X$ and satisfying the conditions

1. $\Phi(x, y) = \Phi(y, x)$.
2. $\Phi(x, x) = +\infty$.
3. $\Phi(x, y)$ is continuous on $X\times X$.

We shall name this function Φ a *kernel*. The function $\omega(x, y) = \exp\big(-\Phi(x, y)\big)$ (where by $\exp(-\infty)$ we mean 0) will be named the *generating function*.

We fix in X a compact set E. We choose on E $n+1$ points $x_0, \ldots, x_n$ and we seek

$$\inf_{\{x_i\}\subset E} \sum_{\substack{i\neq j\\ 0\leqslant i,j\leqslant n}} \Phi(x_i, x_j)$$

or, which is the same,

$$\sup_{\{x_i\}\subset E} \prod_{\substack{i\neq j\\ 0\leqslant i<j\leqslant n}} \omega(x_i, x_j).$$

By the above conditions on Φ and E there exists at least one system of points $\{\eta_0^n, \ldots, \eta_n^n\}$ such that

$$\min_{\{x_i\}\subset E} \sum_{i\neq j} \Phi(x_i, x_j) = \sum_{i\neq j} \Phi(\eta_i^n, \eta_j^n).$$

This system $\{\eta_0^n, \ldots, \eta_n^n\}$ will be named the n-th *extreme system of E with respect to the kernel Φ*. The object of this paper is the investigation of some measures and sequences obtained by the extreme points. In the classical extreme points theory, formed by M. Fekete and developed by G. Polyà, G. Szegö and F. Leja, the extreme points have been used for the construction of some polynomials, X being the complex plane. Some sequences related to those polynomials converge to some functions which give the solution of the Dirichlet problem or are useful in the theory of double power series.