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On some properties of Borelian methods of the exponential type

by L. Włodarski (Łódź)

§ 1. Let A be a functional method of limitability defined by the sequence of real functions $\{a_n(t)\}$ for $t_0 \leqslant t < T$ $(T < +\infty)$. A sequence $x = \{\xi_n\}$ is said to be *limitable by the method* A to the number ξ if:

1° the series $\sum_{n=0}^{\infty} a_n(t)\,\xi_n$ is convergent for $t_0 \leqslant t < T$,

2° the limit $\lim_{t\to T-} \sum_{n=0}^{\infty} a_n(t)\,\xi_n = \xi$ exists.

The transform $A(t, x)$ of the sequence $x = \{\xi_n\}$ with respect to the method A is called the expression

$$A(t, x) = \sum_{n=0}^{\infty} a_n(t)\,\xi_n. \tag{1}$$

We say that the transform $A(t, x)$ exists when condition 1° is satisfied.

In this paper we shall deal with *Borelian methods* of the exponential type $B_{a\gamma}$ $(a > 0,\ \gamma$ real, otherwise arbitrary) defined by the functions

$$b_{a\gamma n}(t) = ae^{-t}\frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} \quad \text{for} \quad 1 \leqslant t < \infty \tag{2}$$

and we assume as usual

$$1/\Gamma(k) = 0 \quad \text{for} \quad k = 0, -1, -2, \ldots \tag{3}$$

Thus the transform $B_{a\gamma}(t, x)$ of the sequence $x = \{\xi_n\}$ with respect to the method $B_{a\gamma}$ is found to be

$$B_{a\gamma}(t, x) = ae^{-t}\sum_{n=0}^{\infty}\frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)}\,\xi_n. \tag{4}$$

If the above transform exists and

$$\lim_{t\to\infty} B_{a\gamma}(t, x) = \xi$$

then the sequence $x = \{\xi_n\}$ is said to be *limitable by the method* $B_{a\gamma}$ to the number ξ.

In my previous papers [12] and [13] I dealted with a subclass of the methods $B_{a\gamma}$ with $a = 2^k$ ($k = 0, \pm 1, \pm 2, \ldots$), and $\gamma = 0$. In paper [14] I gave the results concerning the case $a > 0$, $\gamma = 0$ (which was presented at a meeting of the Łódź section of the Polish Mathematical Association, October 10, 1957).

In the meantime these methods have been dealt with, independently of me, by D. Borwein, who has extended in [1] (p. 28, I) my previous consistency theorem for the methods $B_{a\gamma}$ to the case $a > 0$; $\gamma = 0$.

A functional method defined by the functions $\{a_n(t)\}$ for $t_0 \leqslant t < T$ is said to be *continuous* if:

I° the functions $a_n(t)$ are continuous for $t_0 \leqslant t < T$;

II° there exists such an increasing sequence of positive numbers $\{t_m\}$, $t_0 \leqslant t_m < T$ tending to T that for each $x = \{\xi_n\}$ the convergence of the series (1) for $t = t_m$ and $t = t_{m+1}$ implies the uniform convergence of the series (1) within the interval $t_m \leqslant t \leqslant t_{m+1}$.

The properties of continuous methods which I have found in [11] are as general as those given by S. Mazur and W. Orlicz [5] for the matrix-methods. Moreover, these methods include, in the sense of equivalency, the matrix-methods as well (this means that for every matrix-method there exists a corresponding continuous method which associates precisely the same sequences with the same limits). Typical continuous methods are the methods of Abel and Borel.

In general if $P(t) = \sum_{n=0}^{\infty} p_n t^n$ for $|t| < T$, $p_n \geqslant 0$, where there are infinitely many $p_n > 0$, $\lim_{t\to T-} P(t) = +\infty$, then, as is easy to verify, the method defined by the functions $a_n(t) = p_n t^n / P(t)$ for $0 \leqslant t < T$ is regular (permanent) and continuous.

THEOREM 1. *The methods* $B_{a\gamma}$ ($a > 0$) *defined by the functions* (2) *are continuous.*

Proof. It is evident that property I° takes place. In order to show that property II° also takes place, we shall write the transform $B_{a\gamma}(t, x)$ in the form

$$B_{a\gamma}(t, x) = ae^{-t} t^{\gamma} \varphi(t^a) \quad \text{for} \quad 1 \leqslant t < \infty, \tag{5}$$

where

$$\varphi(u) = \sum_{n=0}^{\infty} \frac{\xi_n}{\Gamma(an+\gamma+1)} u^n. \tag{6}$$

Let $\{t_m\}$ be an arbitrary increasing sequence ($t_m \geqslant 1$) convergent $+\infty$.

Expression (6), as a power series, possesses the following property: the uniform convergence of this series in the interval $t_m^a \leqslant u \leqslant t_{m+1}^a$ follows from its ordinary convergence at the points $u = t_m$ and $u = t_{m+1}$. Taking into account formula (5) we see that the uniform convergence of series (4) in the interval $t_m \leqslant t \leqslant t_{m+1}$ follows from its ordinary convergence at the points $t = t_m$ and $t = t_{m+1}$, which was to be proved.

DEFINITION 1. If x stands for a sequence $\{\xi_0, \xi_1, \xi_2, \ldots\}$ then by x_p (p being a positive integer) we shall denote the sequence $\{\xi_p, \xi_{p+1}, \xi_{p+2}, \ldots\}$ and by x_{-p} the sequence $\{\xi_{-p}, \xi_{-p+1}, \xi_{-p+2}, \ldots, \xi_0, \xi_1, \xi_2, \ldots\}$, where $\xi_{-p}, \xi_{-p+1}, \ldots, \xi_{-1}$ are arbitrary numbers.

Remark 1. The limitability of the sequence $x = \{\xi_0, \xi_1, \xi_2, \ldots\}$ by the method $B_{a\gamma}$ is equivalent to the limitability of the sequence $x_p = \{\xi_p, \xi_{p+1}, \xi_{p+2}, \ldots\}$ by the method $B_{a,\gamma+pa}$.

Proof. Let us note that

$$B_{a\gamma}(t, x) = ae^{-t} \sum_{n=0}^{p-1} \frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} \xi_n + B_{a,\gamma+pa}(t, x_p). \tag{7}$$

In view of the fact that the first term on the right of (7) always tends to zero as $t \to \infty$, our remark is true.

DEFINITION 2. We shall write $f(t) \sim g(t)$ if $\lim_{t\to\infty} f(t)/g(t) = 1$.

Next we shall prove a lemma which is necessary for the proof of regularity of the methods $B_{a\gamma}$. This lemma has been given by Hardy ([3], p. 198) for the case $\gamma = 0$; the idea of my proof, in the general case, is different from that of Hardy.

LEMMA 1. *For an arbitrary* $a > 0$, *a real* γ *and an integer* k *the following formula is true*:

$$\lim_{t\to\infty} \left[e^{-t} \sum_{n=k}^{\infty} \frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} \right] = \frac{1}{a}. \tag{8}$$

Proof. Without reducing the generality of the proof we may assume

$$0 \leqslant \gamma < a \quad \text{and} \quad k = 0. \tag{9}$$

For, if these conditions are not satisfied, then let n_0 denote such an integer that

$$0 \leqslant \delta = an_0 + \gamma > a.$$

Then we have

$$e^{-t} \sum_{n=k}^{\infty} \frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} = e^{-t} \sum_{n=0}^{\infty} \frac{t^{an+\delta}}{\Gamma(an+\delta+1)} + \varepsilon e^{-t} \sum_{n=p}^{q} \frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} \tag{10}$$

where

for $n_0 > k$ we have $\varepsilon = +1$, $p = k$, $q = n_0 - 1$;

for $n_0 = k$ we have $\varepsilon = 0$;

for $n_0 < k$ we have $\varepsilon = -1$, $p = n_0$, $q = k-1$.

Since the second term on the right of (10) always tends to zero as $t \to \infty$, we see, indeed, that the case of an arbitrary γ and k has been reduced to case (9).

Thus in the sequel we shall assume that hypothesis (9) holds.

Now, for the proof, let us consider the function:

(11) $$f_t(u) = \frac{t^u}{\Gamma(u+1)} \quad \text{for} \quad u > 0 .$$

Let us observe that

(12) $$\frac{f_t(u)}{f_t(u-1)} = \frac{t}{u} \begin{cases} >1 & \text{for} \quad u < t, \\ =1 & \text{for} \quad u = t, \\ <1 & \text{for} \quad u > t \end{cases}$$

and that

(13) $$f_t'(u) = \frac{t^u}{\Gamma(u+1)} \left[\log t - \frac{\Gamma'(u+1)}{\Gamma(u+1)} \right].$$

For the logarithmic derivative of Γ-function we have the following formula

(14) $$\psi(u) = \frac{\Gamma'(u+1)}{\Gamma(u+1)} = C + \sum_{\nu=1}^{\infty} \left(\frac{1}{\nu} - \frac{1}{\nu+u} \right),$$

where C is Euler's constant

$$C = 0.577215665\ldots$$

(see e.g. [4], p. 18).

As can be seen from formula (14), the function $\psi(u)$ is an increasing one, and, when n is a positive integer, we have

(15) $$\psi(n) = -C + 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$

Now let us assume that

(16) $$t \geqslant a + 2 .$$

It follows from formulae (13), (14) and (15) that function (11) possesses exactly one maximum $M_t = f_t(u_t)$, and from formula (12) it follows that

(17) $$t - 1 < u_t < t .$$

It follows from Stirling's formula (see e.g. [8], p. 402)

(18) $$\Gamma(u+1) \sim \sqrt{2\pi}\, u^{u+1/2} e^{-u}$$

that

(19) $$\lim_{u \to \infty} f_t(u) = 0 .$$

Under hypothesis (16) the graph of function (11) has approximately the following shape:

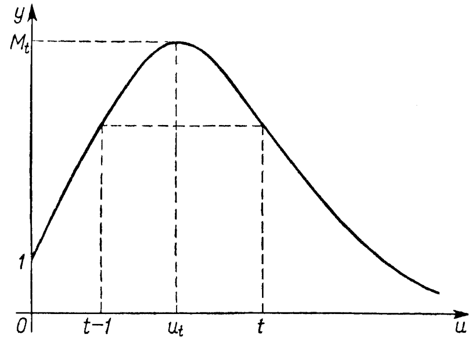

Fig. 1

Now we shall prove that for the maximum $M_t = f_t(u_t)$ of the function $f_t(u)$ the following asymptotical formula is true:

(20) $$\lim_{t \to \infty} [e^{-t} M_t] = 0 .$$

As we know,

$$M_t = f_t(u_t) = \frac{t^{u_t}}{\Gamma(u_t+1)} = \frac{(u_t+\delta_t)^{u_t}}{\Gamma(u_t+1)},$$

where

(21) $$0 < \delta_t = t - u_t < 1 .$$

Using Stirling's formula we have

(22) $$\frac{M_t}{e^t} \sim \frac{(u_t+\delta_t)^{u_t}}{\sqrt{2\pi}\, u_t^{u_t+1/2} e^{-u_t} e^t} = \left(1 + \frac{\delta_t}{u_t}\right)^{u_t} \frac{e^{\delta_t}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{u_t}}$$

which tends to 0 by (21).

Let us notice now that for a function of such type as the one in the above figure, i.e. a continuous function increasing for $0 \leqslant u \leqslant u_t$ and decreasing for $u \geqslant u_t$, we always have the following:

(a) the integral $\int_0^{\infty} f(u)\,du$ converges simultanously with the series $\sum_{n=0}^{\infty} f(n)$;

(b) $$\left|\int_0^\infty f(u)\,du-\sum_{n=0}^\infty f(n)\right| < M_f, \text{ where } M_f=\sup_{u\geq 0} f(u).$$

It follows from formula (20) and from property (b) that for function (11) we have

(23) $$e^{-t}\int_0^\infty \frac{t^u}{\Gamma(u+1)}\,du \sim e^{-t}\sum_{n=0}^\infty \frac{t^n}{\Gamma(n+1)} = 1\,.$$

Applying to the function

$$g_t(u)=f_t(au+\gamma)=\frac{t^{au+\gamma}}{\Gamma(au+\gamma+1)} \qquad (a>0,\ 0\leqslant\gamma<a)$$

an argument similar to that applied before to function (11), we receive

(24) $$e^{-t}\sum_{n=0}^\infty \frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} \sim e^{-t}\int_0^\infty \frac{t^{au+\gamma}}{\Gamma(au+\gamma+1)}\,du = \frac{e^{-t}}{a}\int_\gamma^\infty \frac{t^\nu}{\Gamma(\nu+1)}\,d\nu$$

for $\max_{u\geq 0} g_t(u)=\max_{u\geq 0} f_t(u)$. In view of (9), (16) and (17) is also $\max_{u\geq 0} g_t(u) = \max_{u\geq 0} f_t(u)$ and consequently

$$\lim_{t\to\infty}[e^{-t}\max_{u\geq 0} g_t(u)]=0\,.$$

Let us notice that, taking into consideration (9) and (16)

$$\left|\int_0^\infty \frac{t^\nu}{\Gamma(\nu+1)}\,d\nu-\int_\gamma^\infty \frac{t^\nu}{\Gamma(\nu+1)}\,d\nu\right| = \int_0^\gamma \frac{t^\nu}{\Gamma(\nu+1)}\,d\nu < \int_0^a \frac{t^\nu}{\Gamma(\nu+1)}\,d\nu < a\frac{t^a}{\Gamma(a+1)}\,.$$

Thus, taking into account equality (23), we have

$$\frac{e^{-t}}{a}\int_\gamma^\infty \frac{t^\nu}{\Gamma(\nu+1)}\,d\nu \sim \frac{e^{-t}}{a}\int_0^\infty \frac{t^\nu}{\Gamma(\nu+1)}\,d\nu = \frac{1}{a}$$

in view of which it follows from equality (24) that

$$e^{-t}\sum_{n=0}^\infty \frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} \sim \frac{1}{a},$$

which proves our lemma.

THEOREM 2. *The methods $B_{a\gamma}$ $(a>0,\ \gamma$ — any real number) are regular (permanent).*

Proof. It is known (see e.g. [11], p. 163, Th. III c) that a continuous method A defined by a function sequence $a_n(t)$, $t_0\leqslant t<T$ is regular (permanent) if and only if the following three conditions are satisfied:

1° $\lim_{t\to T-} a_n(t)=0$ $(n=0,1,2,\ldots)$,

2° $\lim_{t\to T-}\sum_{n=0}^\infty a_n(t)=1$,

3° $\sup_{t_0\leqslant t<T}\sum_{n=0}^\infty |a_n(t)|<\infty$.

Condition 1° is obviously satisfied for functions (2). Condition 2° is satisfied for function (2) because of lemma 1, and consequently, because of $a_n(t)\geqslant 0$, condition 3 is also satisfied.

DEFINITION 3. The fact that the sequence $x=\{\xi_n\}$ is limitable by a method A to the number ξ will be denoted in symbols as $A\text{-}\lim\xi_n=\xi$ or $A(x)=\xi$.

DEFINITION 4. The *convolution* of the two functions $f(t)$ and $g(t)$ is called the function $h(t)$ defined by the formula $h(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$. Or in symbols

$$h(t)=f(t)\ast g(t)\,.$$

LEMMA 2 (for the proof see [13], p. 146, lemma 1). *Let $f(t)$ be a complex function of a real variable defined and continuous for $t\geqslant 0$. If in addition $\lim_{t\to\infty} f(t)=m$ (m being a finite number or $+\infty$) then for the function*

$$g(t)=f(t)\ast e^{-t}\frac{t^a}{\Gamma(a+1)} \qquad (a>1)$$

we have also $\lim_{t\to\infty} g(t)=m$.

THEOREM 3. *If a sequence $x=\{\xi_n\}$ is limitable by a method $B_{a\gamma}$ ($a>0$ γ real, otherwise arbitrary) to the number ξ, it is limitable to the same number by every method $B_{a\delta}$ provided $\delta>\gamma$, which means that the methods $B_{a\delta}$ are more general and consistent with $B_{a\gamma}$, for all $\delta>\gamma$.*

Proof. It is enough to prove the theorem in the case $\gamma\geqslant 0$. For, let us assume that the theorem is true for all $\gamma\geqslant 0$ and we want to prove it for a certain $\gamma<0$. Let us take into consideration an arbitrary method $B_{a\gamma}$ and an arbitrary sequence $x=\{\xi_n\}$ limitable by the method $B_{a\gamma}$ to the number ξ. We shall prove that the sequence x is also limitable by a method $B_{a\delta}$ to the number ξ provided $\delta>\gamma$. Now let p be such a positive integer that

(25) $$pa+\gamma\geqslant 0\,.$$

Because of remark 1 we infer that

(26) $$B_{a\gamma}(x)=\xi \quad \text{is equivalent to} \quad B_{a,\gamma+pa}(x_p)=\xi$$

(see def. 1 and 3). But in view of (25) we infer that

(27) $$B_{a,\gamma+pa}(x_p) = \xi \quad \text{implies} \quad B_{a,\delta+pa}(x_p) = \xi$$

since we have assumed that the theorem is true for every $\gamma_1 = \gamma+pa \geqslant 0$ and evidently $\delta_1 = \delta+pa > \gamma_1 = \gamma+pa$. Because of remark 1 we again infer that

(28) $$B_{a,\delta+pa}(x_p) = \xi \quad \text{is equivalent} \quad B_{a\delta}(x) = \xi\,.$$

From (26), (27) and (28) it follows that $B_{a\gamma}(x) = \xi$ implies $B_{a\delta}(x) = \xi$ provided $\delta > \gamma$, which means that our theorem is true for an arbitrary γ if it is true for $\gamma \geqslant 0$. Thus without reducing generality we may assume

(29) $$\gamma \geqslant 0\,.$$

In the sequel we shall use the idea of convolution (see def. 4). The convolution-product is, as we know, commutative, associative and distributive with respect to addition (see e.g. [7], p. 2-4). The following formula is also known ([7], p. 105, formulae (55.3) and (55.4)):

(30) $$e^{ct}\frac{t^a}{\Gamma(a+1)} * e^{ct}\frac{t^b}{\Gamma(b+1)} = e^{ct}\frac{t^{a+b+1}}{\Gamma(a+b+1)}$$

for $a, b > -1$. In particular, we have

(31) $$e^{-t}\frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} * e^{-t}\frac{t^{\delta-\gamma-1}}{\Gamma(\delta-\gamma-1)} = e^{-t}\frac{t^{an+\delta}}{\Gamma(an+\delta+1)}\,.$$

We state that for a sequence $x = \{\xi_n\}$ limitable by the method $B_{a\gamma}$ the following relation is true:

(32) $$B_{a\gamma}(t, x) * e^{-t}\frac{t^{\delta-\gamma-1}}{\Gamma(\delta-\gamma-1)} = B_{a\delta}(t, x)\,.$$

In view of formulae (4) and (31) in passing from the left to the right in (32) we only have to change the order of the signs $\sum\limits_{n=0}^{\infty}$ and $\int\limits_0^t$. Such a change is permissible for it follows from that limitability of the sequence $x = \{\xi_n\}$ that, in particular, series (4) under assumption (29) is uniformly convergent in every finite interval $0 \leqslant t \leqslant t_0 < \infty$. Thus the correctness of formula (32) has been proved.

It follows from the assumption $B_{a\gamma}\text{-}\lim \xi_n = \xi$ (see def. 3) that $\lim\limits_{t\to\infty} B_{a\gamma}(t, x) = \xi$, whence on the grounds of formula (32) and lemma 2 we have $\lim\limits_{t\to\infty} B_{a\delta}(t, x) = \xi$ or $B_{a\delta}\text{-}\lim \xi_n = \xi$, which was to be proved. Thus the proof of theorem 3 has been given.

THEOREM 4. *The methods $B_{a\gamma}$ ($a > 0$, γ — real, otherwise arbitrary) are capable of displacement to the right, i.e. the limitability of a sequence $x = \{\xi_0, \xi_1, \xi_2, \ldots\}$ by the method $B_{a\gamma}$ to the number ξ implies the limitability of the sequence $x_{-1} = \{\xi_{-1}, \xi_0, \xi_1, \ldots\}$ to the number ξ.*

Proof. It follows from formula (4) that

(33) $$B_{\nu\gamma}(t, x_{-1}) = a\xi_{-1}e^{-t}\frac{t^{\gamma}}{\Gamma(\gamma+1)} + B_{a,\gamma+a}(t, x)\,.$$

In virtue of the assumption

$$\lim_{t\to\infty} B_{a\gamma}(t, x) = \xi\,.$$

It follows from theorem 3 that also

$$\lim_{t\to\infty} B_{a,\gamma+a}(t, x) = \xi\,,$$

whence in virtue of formula (33)

$$\lim_{t\to\infty} B_{a\gamma}(t, x_{-1}) = \xi\,,$$

which proves our theorem.

§ 2. DEFINITION 5. The Laplace transformation $F(s) = \int\limits_0^{\infty} e^{-st}f(t)\,dt$ will be denoted briefly by the symbol $F(s) = \mathcal{L}\{f(t)\}$.

We know that the operation $\mathcal{L}$ is reversible. It is obvious that the functions $\mathcal{L}^{-1}\{F(s)\} = f(t)$ may differ by a term $N(t)$ for which $\int\limits_0^{\infty} N(\tau)\,d\tau = 0$ for all $t > 0$ (see [2], p. 35). (It can easily be proved that the Laplace transform $\mathcal{L}\{N(t)\} = F_0(s)$ of a function $N(t)$ is identically equal to zero.) The operation $\mathcal{L}$ is completely reversible if we are dealing with continuous functions, as is here the case.

In this section we shall deal with certain functions and their estimations. Let us begin with the function

(34) $$g_\theta(t, \tau) = \mathcal{L}^{-1}\{e^{-\tau s^\theta}\} = \frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty} \exp(ts - \tau s^\theta)\,ds$$

($\mathcal{L}^{-1}$ denotes a transformation reverse to the Laplace transformation).

J. Mikusiński has proved in [6] that the function $g_\theta(t, \tau)$ is a real and continuous function for all $0 < \theta < 1$, $t > 0$ and $\tau > 0$. In addition I have proved in [10] that under the same assumption functions (34) are non-negative.

In the theory of Laplace transformation the following elementary formula is known (see e.g. [2], p. 401):

$$(35)\qquad \mathfrak{L}^{-1}\left\{\frac{1}{s^{\beta}}\right\}=\frac{t^{\beta-1}}{\Gamma(\beta)}\quad \text{for}\quad \beta>0\,.$$

Function (35) is, of course, positive for $t>0$.

Now we shall use

Tchebysheff's Theorem. *If the integrals corresponding to the functions* $F_1(s)=\int_0^\infty e^{-st}f_1(t)\,dt$ *and* $F_2(s)=\int_0^\infty e^{-st}f_2(t)\,dt$ *are absolutely convergent for* $s=s_0$, *then* $\int_0^\infty e^{-st}[f_1(t)*f_2(t)]\,dt$ *is also absolutely convergent for* $s=s_0$ *and is equal to* $F_1(s)\cdot F_2(s)$ *or* $\mathfrak{L}\{f_1(t)*f_2(t)\}=\mathfrak{L}\{f_1(t)\}\,\mathfrak{L}\{f_2(t)\}$, *which implies*

$$\mathfrak{L}^{-1}\{F_1(s)\cdot F_2(s)\}=\mathfrak{L}^{-1}\{F_1(s)\}*\mathfrak{L}^{-1}\{F_2(s)\}\,.$$

For the proof see [2], p. 161, Th. IV b.

Let us now write

$$(36)\qquad h_{\theta\beta}(t,\tau)=\mathfrak{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}\quad\text{where}\quad 0<\theta<1,\ \tau>0,\ \beta>0\,.$$

In virtue of formulae (34), (35), (36) and the theorem of Tchebysheff we have

$$(37)\qquad h_{\theta\beta}(t,\tau)=g_0(t,\tau)*\frac{t^{\beta-1}}{\Gamma(\beta)}$$

(convolution with respect to t), whence, in particular, it immediately follows from the definition of convolution (see def. 4) that function (36) is continuous and

$$(38)\qquad h_{\theta\beta}(t,\tau)=\mathfrak{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}\geqslant 0\quad\text{for}\quad 0<\theta<1,\ \beta>0,\ t>0,\ \tau>0\,.$$

Now we want to estimate function (36). For this purpose we shall take advantage of a lemma which I have proved in [9], p. 184 and which gives the formula for a transformation reverse to the Laplace transformation:

Lemma 3. *Let* $G(s)$ *be a function holomorph in an angle domain* Δ,

$$(39)\qquad |\arg(s-\lambda)|<\pi/2+\alpha\quad\textit{where}\quad \lambda>0,\ 0<\alpha<\pi/2$$

which satisfies in it the inequality

$$(40)\qquad |G(s)|\leqslant M|s|^{-\beta}\quad\textit{where}\quad M>0,\ \beta>0\,.$$

Then

$$(41)\qquad \mathfrak{L}^{-1}\{G(s)\}=\frac{1}{2\pi i}\int_C e^{st}G(s)\,ds$$

where the integration path C *is given by the equation*

$$(42)\qquad |\arg(s-\varrho)|=\frac{\pi}{2}+\alpha\quad\textit{where}\quad \varrho>\lambda$$

(see fig. 2).

We want to apply the above to the function

$$(43)\qquad G(s)=\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\quad\text{for}\quad \beta>0,\ \tau>0,\ 0<\theta<1\,.$$

We shall verify that function (43) satisfies the conditions of lemma 3 in the domain Ω provided we take an arbitrary $\lambda>0$ and choose such an $\alpha>0$ that the inequality

$$(44)\qquad \theta(\pi/2+\alpha)<\omega<\pi/2$$

takes place.

It is not difficult to verify that if $S\in\Omega$ (formula (39)) and α satisfies condition (44), then

$$(45)\qquad R(S^{\theta})\geqslant(\lambda\cos\alpha)^{\theta}\cos\omega=\sigma>0\ (^1).$$

In view of (45) we have the following estimation for function (43):

$$(46)\qquad |G(s)|\leqslant\frac{1}{|S|^{\beta}}e^{-\sigma\tau}.$$

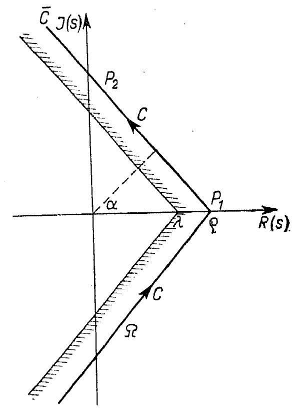

Fig. 2

Thus function (43) satisfies assumption (40) (with $M=1$).

We apply lemma 3 to function (43) and receive a formula for function (36)

$$(47)\qquad L^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}=\frac{1}{2\pi i}\int_C\frac{1}{s^{\beta}}e^{t-\tau s^{\theta}}\,ds\,,$$

where the curve C is defined by formula (42) (see fig. 2).

From formula (47) we immediately receive an estimation for function (36) in domain (39) with α satisfying condition (44):

$$(48)\qquad \left|L^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}\right|\leqslant A(t)e^{-\sigma\tau},$$

(1) By $R(z)$ we denote the real part of z.

where

(49) $$A(t) = \left| \int\limits_{\bar{C}} \frac{1}{|s|^\beta} e^{tR(s)} ds \right| < +\infty$$

and $\sigma > 0$ is given by formula (45).

Further we see that

(50) $$A(t) \leqslant B(t) + C(t),$$

where

(50a) $$B(t) = 2 \left| \int\limits_{P_1P_2} \frac{1}{|s|^\beta} e^{tR(s)} ds \right|$$

and in this integral we have $0 \leqslant R(s) \leqslant \varrho$,

(50b) $$C(t) = 2 \left| \int\limits_{\bar{C}} \frac{1}{|s|^\beta} e^{tR(s)} ds \right|$$

and in this integral we have $R(s) \leqslant 0$.

$\bar{C}$ is that part of the curve C on which the conditions $\arg(s-\varrho) = \pi/2 + a$, $R(s) < 0$ are satisfied (see fig. 2).

Remark 2. One can see from formula (45) that by choosing a suitably small and λ suitably large we can obtain an arbitrarily large σ in inequality (48). On the other hand the functions $A(t)$, $B(t)$ and $C(t)$ depend on the choice of a and also ϱ.

Remark 3. It can be seen from formulae (50a) and (50b) that $B(t)$ is an increasing and $C(t)$ a decreasing function of t $(t > 0)$

Remark 4. For function (50a) we have the estimation

$$B(t) \leqslant 2e^{\varrho t} \left| \int\limits_{P_1P_2} \frac{1}{|s|^\beta} ds \right| \leqslant B_0 e^{\varrho t}$$

and for function (50b)

$$C(t) < C(1) = C_0 \quad \text{for} \quad t > 1,$$

and thus ultimately

$$A(t) \leqslant B_0 e^{\varrho t} + C_0 \quad \text{for} \quad t > 1, \quad \text{where} \quad B_0 > 0,\ C_0 > 0.$$

Let us consider now a transformation $W_{\theta\beta}$, $0 < \theta < 1$, $\beta > 0$ defined by the formula

(51) $$F(t) = \theta e^{-t} \int\limits_0^\infty e^\tau f(\tau) \mathcal{L}^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau$$

where $\mathcal{L}^{-1}$ stands for a transformation on reverse to the transformation of Laplace. Formula (51) will be written briefly $F(t) = W_{\theta\beta}\{f(\tau)\}$.

Remark 5. The operator $W_{\theta\beta}$ is additive and homogeneous, i.e.

$$W_{\theta\beta}\{af(\tau) + bg(\tau)\} = aW_{\theta\beta}\{f(\tau)\} + bW_{\theta\beta}\{g(\tau)\}$$

provided the right side exists. In connection with transformation $W_{\theta\beta}$ we shall define a class of functions $L_{\theta\beta}$:

Definition 6. A complex summable function $f(\tau)$ of a real variable $\tau \geqslant 0$ is said to be a member of the class $L_{\theta\beta}$ if the following conditions are satisfied:

1° $f(\tau)$ is bounded in any finite interval $0 \leqslant \tau \leqslant T < \infty$;

2° the integral occurring on the right of formula (51) is convergent for this function.

Now we shall use the formula of Efros, which I proved in [9] and which we shall need here only in a special case (see [9], p. 187, formula 12):

(52) $$\mathcal{L}^{-1}\left\{\frac{1}{s^\beta} F^*(s^\theta)\right\} = \int\limits_0^\infty f(\tau) \mathcal{L}^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau,$$

where $0 < \theta < 1$, $\beta > 0$, $\int\limits_0^\infty e^{-\sigma\tau}|f(\tau)| d\tau < \infty$ for some $\sigma \geqslant 0$, $F^*(s) = \int\limits_0^\infty e^{-s\tau} f(\tau) d\tau$, $\mathcal{L}^{-1}$ denotes a transformation converse to the transformation of Laplace.

Theorem 5. *The transformation $W_{\theta\beta}$, $0 < \theta < 1$, $\beta > 0$ defined by formula (51) is permanent (regular) in infinity for $L_{\theta\beta}$, i.e. for every function $f(\tau) \in L_{\theta\beta}$ (def. 6) for which (def. 6) $\lim\limits_{\tau\to+\infty} f(\tau) = m$ (m finite or $+\infty$) we have as well $\lim\limits_{t\to\infty} F(t) = m$.*

Proof. It is obvious that because of the addivity and homogeneity of the operator $W_{\theta\beta}$ we may, without reducing generality, confine our considerations to the case of real functions $f(\tau)$. In the proof we shall distinguish several subsequent cases.

Case 1°: $f\{\tau\} \equiv m$. Then

(53) $$F(t) = m\theta e^{-t} \int\limits_0^\infty e^\tau \mathcal{L}^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau.$$

Now we shall apply to the integral on the right of formula (53) the formula of Efros (52), the assumptions for which are in our case plainly satisfied. Then we receive

$$\int\limits_0^\infty e^\tau \mathcal{L}^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s^\beta} \cdot \frac{1}{s^\theta - 1}\right\},$$

for in our case $F^*(s) = \mathcal{L}\{e^\tau\} = \dfrac{1}{s-1}$ (see e.g. [2], p. 401).

On the other hand, assuming that $R(s) > 0$, we have because of the properties of Laplace transformation

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}\cdot\frac{1}{s^{\theta}-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{\beta+\theta}}\cdot\frac{1}{1-s^{-\theta}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{\beta+\theta}}(1+s^{-\theta}+s^{-2\theta}+\ldots)\right\}$$
$$= \sum_{n=1}^{\infty}\frac{t^{n\theta+\beta-1}}{\Gamma(n\theta+\beta)}$$

or ultimately

$$(54)\qquad \int_0^{\infty} e^{\tau}\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}d\tau = \sum_{n=0}^{\infty}\frac{t^{n\theta+\beta-1}}{\Gamma(n\theta+\beta)}.$$

Thus returning to (53) we have

$$F(t) = m\theta e^{-t}\sum_{n=0}^{\infty}\frac{t^{n\theta+\beta-1}}{\Gamma(n\theta+\beta)}.$$

Making use of lemma 1 we receive $\lim_{t\to\infty}F(t) = m$, which proves the theorem in our case.

Case II°: $\lim_{\tau\to\infty}f(\tau) = 0$. In this case for an arbitrary $\varepsilon > 0$ there exists such a T that

$$(55)\qquad |f(\tau)| < \varepsilon/4 \quad \text{for} \quad \tau \geqslant T.$$

Now we shall estimate function (51). For this purpose we shall divide the integral $\int_0^{\infty}$ which occurs on the right of (51) into two integrals $\int_0^T + \int_T^{\infty}$ where T is the same as in inequality (55). Then we obtain an estimation

$$(56)\quad |F(t)| \leqslant M\theta e^{-t}\int_0^T e^{-\tau}\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\delta s^{\theta}}\right\}d\tau + \frac{\varepsilon}{4}\theta e^{-\tau}\int_T^{\infty} e^{\tau}\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau\varrho}\right\}d\tau,$$

where $M = \sup_{0\leqslant t\leqslant T}|f(t)|$ ($M < \infty$ for $f \in L_{\theta\beta}$ (see def. 6)).

Now we pass to further estimations. In virtue of formula (48) and remark 4 we have

$$(57)\qquad \left|\int_0^T e^{\tau}\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}d\tau\right| \leqslant e^{T}(B_0e^{\varrho t}+C_0)T.$$

Since in our case ϱ may be chosen arbitrarily (see lemma 3), we take, for instance:

$$(58)\qquad \varrho = \tfrac{1}{2}.$$

On the other hand, taking advantage of (54) and of inequality (38) we obtain

$$(59)\quad 0 \leqslant \int_T^{\infty} e^{\tau}\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}d\tau \leqslant \int_0^{\infty} e^{\tau}\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}d\tau = \sum_{n=1}^{\infty}\frac{t^{n\theta+\beta-1}}{\Gamma(n\theta+\beta)}.$$

Thus, comparing (58) and (59), we have

$$(60)\qquad |F(t)| \leqslant M\theta e^{T}T(B_0e^{-t/2}+C_0e^{-t}) + \frac{\varepsilon}{4}\theta e^{-t}\sum_{n=1}^{\infty}\frac{t^{n\theta+\beta-1}}{\Gamma(n\theta+\beta)}.$$

The first term on the right of (60) tends to zero as $t\to+\infty$, whereas the second term, in virtue of lemma 1, tends to $\varepsilon/4$. Thus we have

$$|F(t)| < \varepsilon \quad \text{for} \quad t \geqslant t_0,$$

which proves the theorem in our case.

Case III°: $\lim_{\tau\to\infty}f(\tau) = m$ (m — finite). It follows from the additivity of the transformation $W_{\theta\beta}$ that

$$(61)\qquad W_{\theta\beta}\{f(\tau)\} = W_{\theta\beta}\{m\} + W_{\theta\beta}\{f(\tau)-m\}.$$

In virtue of case I° the first expression on the right of (61) tends to m as $t\to\infty$; in virtue of case II° the second expression on the right of (61) tends to zero as $t\to+\infty$. Thus the left side of (61) tends to m as $t\to\infty$, which proves the theorem in our case.

Case IV°: $\lim_{\tau\to\infty}f(\tau) = +\infty$. In virtue of the assumption for every N there exists such a T that

$$(62)\qquad f(\tau) > 2N \quad \text{for} \quad \tau > T.$$

Proceeding in a similar way as we did in case II° we receive the following estimations:

$$F(t) \geqslant 2N\theta e^{-t}\int_T^{\infty} e^{\tau}\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}e^{-\tau s^{\theta}}\right\}d\tau - M\theta e^{-t}\int_0^T e^{\tau}\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}d\tau,$$

where $M = \sup_{0\leqslant\tau\leqslant T}|f(\tau)|$.

Further in virtue of (57) and (59) we have

$$F(t) \geqslant 2N\theta e^{-t}\sum_{n=1}^{\infty}\frac{t^{n\theta+\beta-1}}{\Gamma(n\theta+\beta)} - (M+2N)T\theta e^{T-t}(B_0e^{\varrho t}+C_0),$$

whence, assuming (58) and applying lemma 1, we have $F(t) \geqslant N$ for $t > t_1$, which proves the theorem in our case. Thus theorem 5 has been proved completely.

Remark 6. Transformation (51) $W_{\theta\beta}$ becomes for $\theta = \beta = \frac{1}{2}$ the transformation W which I used in [13], p. 140, formula 4 in the proof of the consistence of methods of the type $W_{2^k,0}$ $(k = 0, \pm 1, \pm 2, \ldots)$. Let us observe that indeed

$$W_{1/2,1/2}\{f(\tau)\} = \frac{1}{2} e^{-t} \int_0^\infty e^{\tau} f(\tau) \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}} e^{-\tau\sqrt{s}}\right\} d\tau .$$

In virtue of a known formula from the theory of Laplace transformation (see [2], p. 402, formula 18) we have

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}} e^{-\tau\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} e^{-\tau^2/4t},$$

and in this way we receive

$$W_{1/2,1/2}\{f(\tau)\} = \frac{1}{2\sqrt{\pi t}} e^{-t} \int_0^\infty e^{-\tau^2/4t+\tau} f(\tau)\, d\tau$$

In the proof of theorem 6 we shall use a certain lemma, not very difficult to prove:

LEMMA 4. *If for $n = 0, 1, 2, \ldots$*

1° the functions $f_n(u)$ are defined and continuous for $u \geqslant 0$,

2° the integrals $\int_0^\infty |f_n(u)| du$ are convergent,

3° the series $\sum_{n=0}^\infty f_n(u)$ converges uniformly in every finite interval,

4° $\sum_{n=0}^\infty |f_n(u)| < +\infty$

then the integral $\int_0^\infty \sum_{n=0}^\infty f_n(u)\, du$ is convergent and equal to $\sum_{n=0}^\infty \int_0^\infty f_n(u)\, du$.

THEOREM 6. *If a sequence $x = \{\xi_n\}$ is limitable by the method $B_{a\gamma}$ to the number ξ and under the following assumptions*

(63) $$0 < \theta < 1, \quad \delta > \theta(\gamma+1) - 1$$

its transform $B_{\theta a,\delta}$ exists, then the sequence x is limitable by the method $B_{\theta a,\delta}$ to the number ξ.

Proof. In the same way as in the proof of theorem 3 we may assume without loss of generality that $\gamma \geqslant 0$.

Let us write

(64) $$\beta = \delta - \theta(\gamma+1) + 1 > 0$$

and take into account the transformation $W_{\theta\beta}$ where θ is a number satisfying the hypothesis, and β is given by formula (64). Let us put in formula (51) function (2) instead of $f(\tau)$. Then we receive

$$W_{\theta\beta}\{b_{a\gamma n}(\tau)\} = a\theta e^{-t} \int_0^\infty \frac{\tau^{an+\gamma}}{\Gamma(an+\gamma+1)} \mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}} e^{-\tau s^{\theta}}\right\} d\tau .$$

Applying formula (52) (formula of Efros) to the second number we obtain

$$W_{\theta\beta}\{b_{a\gamma n}(\tau)\} = a\theta e^{-t} \mathcal{L}^{-1}\left\{\frac{1}{s^{\theta an+\theta\gamma+\theta+\beta}}\right\}$$

or

$$W_{\theta\beta}\{b_{a\gamma n}(\tau)\} = a\theta e^{-t} \frac{t^{\theta an+\theta\gamma+\theta+\beta-1}}{\Gamma(\theta+an+\theta\gamma+\theta+\beta)} = a\theta e^{-t} \frac{t^{\theta an+\delta}}{\Gamma(\theta an+\delta+1)} .$$

Putting for β value (64) we receive ultimately

(65) $$W_{\theta\beta}\{b_{a\gamma n}(\tau)\} = b_{\theta a,\delta,n}(t)$$

or

(66) $$\int_0^\infty \frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} \mathcal{L}^{-1}\left\{\frac{1}{s^{\delta-\theta(\gamma+1)+1}} e^{-\tau s^{\theta}}\right\} = \frac{t^{\theta an+\delta}}{\Gamma(\theta an+\delta+1)} .$$

Now we shall prove that

(67) $$W_{\theta\beta}\{B_{a\gamma}(\tau, x)\} = B_{\theta a,\delta}(t, x)$$

or

(68) $$\int_0^\infty \sum_{n=1}^\infty \frac{\tau^{an+\gamma}}{\Gamma(an+\gamma+1)} \xi_n \mathcal{L}^{-1}\left\{\frac{1}{s^{\delta-\theta(\gamma+1)+1}} e^{-\tau s^{\theta}}\right\} d\tau = \sum_{n=0}^\infty \frac{t^{\theta an+\delta}}{\Gamma(\theta an+\delta+1)} \xi_n .$$

In order to prove the formula (68) it is enough to prove that on the left of (68) the order of the signs $\int_0^\infty$ and $\sum_{n=0}^\infty$ may be changed; then the theorem follows from formula (66).

For this purpose we apply lemma 4 to the function

$$f_n(u) = \frac{u^{an+\gamma}}{\Gamma(an+\gamma+1)} \xi_n \mathcal{L}^{-1}\left\{\frac{1}{s^{\delta-\theta(\gamma+1)+1}} e^{-us^{\theta}}\right\}.$$

Assumption 1° of lemma 4 is obviously satisfied.

Assumptions 2° and 3° are satisfied because the sequence $x = \{\xi_n\}$ in virtue of the assumption of our theorem is limitable by the method $B_{a\gamma}$.

Finally let us notice that, since function (36) is non-negative, in view of (66) in our case we have

$$\int_0^\infty |f_n(u)|\, du = \frac{t^{\theta an+\delta}}{\Gamma(\theta an+\delta+1)} |\xi_n| .$$

Assumption 4° of lemma 4 is satisfied, since for the sequence $x = \{\xi_n\}$ the transform $B_{\theta a,\delta}(t, x)$ exists, and consequently the series on the right of (68) is absolutely convergent. Thus the correctness of formula (68) and consequently that of formula (67) has been proved.

In view of the fact that the sequence x is limitable by the method $B_{a\gamma}$ to the number ξ, the following relation must be true

$$\lim_{\tau\to\infty} B_{a\gamma}(\tau, x) = \xi .$$

In virtue of equality (67) and theorem 5 we must have also

$$\lim_{t\to\infty} B_{\theta a,\delta}(t, x) = \xi . \tag{69}$$

Equality (69) proves our theorem, which ends the proof of theorem 6.

Remark 7. Theorem 6 is more general than my theorem given in [14] or the theorem of Borwein given in [1], for in this theorem δ may be smaller than $\theta\gamma$.

In the proof of theorem 7 we shall use the following lemma.

LEMMA 5. *If $f(u)$ is a complex function of a real variable defined and continuous for $u \geqslant 0$ and $F^*(s) = \int_0^\infty e^{-su} f(u)\,du$, where the integral on the right is absolutely convergent for $R(s) > 0$, then $\lim_{s\to 0+} sF^*(s) = m$.*

THEOREM 7. *If a sequence $x = \{\xi_n\}$ is limitable by the method $B_{a\gamma}$ to the number ξ and its transform with respect to the method of Abel exists, then the sequence x is limitable by the method of Abel to the number ξ.*

Proof. In virtue of the assumption

$$B_{a\gamma}(t, x) = ae^{-t} \sum_{n=0}^{\infty} \frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)} \xi_n \tag{70}$$

exists,

$$\lim_{t\to\infty} B_{a\gamma}(t, x) = \xi \tag{71}$$

and the transform with respect to the Abel method

$$A(t, x) = (1-t) \sum_{n=0}^{\infty} t^n \xi_n , \tag{72}$$

exists.

Let p stand for an integral non-negative number such that

$$ap + \gamma \geqslant 0 \tag{73}$$

and let us write

$$\delta = ap + \gamma \geqslant 0 . \tag{74}$$

In virtue of remark 1 the limitability of a sequence $x = \{\xi_0, \xi_1, \xi_2, \ldots\}$ by the method $B_{a\gamma}$ to the number ξ is equivalent to the limitability of the sequence $x_p = \{\xi_p, \xi_{p+1}, \xi_{p+2}, \ldots\}$ by the method $B_{a\delta}$ to the number ξ, i.e. the transform

$$B_{a\delta}(t, x_p) = ae^{-t} \sum_{n=0}^{\infty} \frac{t^{an+\delta}}{\Gamma(an+\delta+1)} \xi_{p+n} \tag{75}$$

exists and

$$\lim_{t\to\infty} B_{a\delta}(t, x_p) = \xi . \tag{76}$$

By means of the formulae for the Laplace transformation, making use of assumption (72) and applying lemma 4, we can easily show that

$$\mathcal{L}\{B_{a\gamma}(t, x_p)\} = \frac{a}{(s+1)} \sum_{n=0}^{\infty} \frac{1}{(s+1)^{an}} \xi_{p+n} .$$

Multiplying by s both members of this relation and substituting in the second member

$$\frac{1}{(s+1)^a} = t \tag{77}$$

we receive

$$s\mathcal{L}\{B_{a\delta}(t, x_p)\} = \frac{a(1-t^{1/a})}{1-t} A(t, x_p) , \tag{78}$$

where $A(t, x)$ stands for function (72), and x_p is given by def. 1.

In virtue of (76) and lemma 5 the left side of (78) tends to ξ as $s\to 0+$. Thus in virtue of (77) the right side of (78) also tends to ξ as $t\to 1-$.

The first factor on the right of (78) tends to 1 as $t\to 1-$, and ultimately we receive

$$\lim_{t\to 1-} A(t, x_p) = \xi ,$$

i.e. the sequence x_p is limitable by the method of Abel to the number ξ. But it is known that the method of Abel is capable of displacement to the left and right, and therefore the sequence $x = \{\xi_n\}$ is also limitable by the method of Abel to the number ξ. Thus theorem 7 has been proved.

Remark 8. Since the transform of a bounded sequence with respect to Abel's method always exists, it follows from theorem 7 that if a bounded sequence is limitable by the method $B_{a\gamma}$ to the number ξ, then it is also limitable by the method of Abel, and consequently also by the method C_1, to the number ξ.

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Sur une fonction extrémale liée à l'écart arithmétique d'un ensemble

par B. Szafirski (Kraków)

Introduction. Dans le présent travail je me propose de résoudre un problème de M. F. Leja, relatif à l'existence et aux applications de certaines fonctions extrémales liées aux ensembles fermés et bornés de l'espace euclidien R^m à m dimensions.

Si $E\subset R^m$ est un ensemble fermé et borné et si $q^{(n)}=\{q_1,\dots,q_n\}$ est le $n^{\text{ème}}$ système extrémal de points de E, c'est-à-dire un système de points tels que

$$\sum_{1\leqslant i<k\leqslant n}|q_i-q_k|=\sup_{x^{(n)}\subset E}\sum_{1\leqslant i<k\leqslant n}|x_i-x_k|\,,$$

où $x^{(n)}=\{x_1,\dots,x_n\}$ désignent un système quelconque de n points de l'ensemble E, alors la suite de fonctions

$$\Phi_n(x)=\frac{1}{n}\sum_{i=1}^{n}|x-q_i|\,,\qquad n=1,2,\dots,$$

converge en tout point de l'espace R^m. Dans la première partie je donne la démonstration de ce théorème, ainsi que certaines propriétés de la fonction extrémale ainsi obtenues.

La seconde partie est consacrée à une généralisation naturelle des notions et des théorèmes de la première partie.

Première partie

1. Soit E un ensemble borné et fermé de points de l'espace euclidien R^m à m dimensions, et $|p-q|$ la distance des points p et q. Désignons par $p^{(n)}$ un système de n points quelconques $p_1,\dots,p_n$ de E, par $A(p^{(n)})$ les sommes

$$A(p^{(n)})=\sum_{1\leqslant i<k\leqslant n}|p_i-p_k| \tag{1}$$