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On the functional equation $f(x+y)f(x-y) = f^2(x)-f^2(y)$

by S. KUREPA (Zagreb)

The object of this note is to prove the following two theorems.

THEOREM 1. *Let $R = \{t, s, \dots\}$ be the space of all real numbers, $H = \{x, y, \dots\}$ a real Hilbert space, $x \to f(x)$ a functional defined on H with values in R and such that*

$$f(x+y)f(x-y) = f^2(x)-f^2(y) \tag{1}$$

holds for all $x, y \in H$.

If the functional $f \neq 0$ is continuous then it has one of the following forms:

(a) $f(x) = (x, x_0)$,

(b) $f(x) = A\sin(x, x_0)$,

(c) $f(x) = A\sinh(x, x_0)$

where A is a real number independent of x, the vector $x_0 \in H$ does not depend on x and (x, x_0) denotes the scalar product of x and x_0. The constant A and the vector x_0 are uniquely determined by the functional f.

THEOREM 2. *Let $M = \{x, y, \dots\}$ be the set of all real square matrices of the order n endowed with the usual topology of matrices $f(x) \not\equiv 0$ a real-valued functional which is defined on M and such that*

$$f(x+y)f(x-y) = f^2(x)-f^2(y), \qquad f(s^{-1}xs) = f(x)$$

holds true for all $x, y \in M$ and for every orthogonal matrix $s \in M$.

If the functional f is continuous on M then it has one of the following forms:

(a) $f(x) = a\cdot \mathrm{Tr}\, x$,

(b) $f(x) = A\sin a(\mathrm{Tr}\, x)$,

(c) $f(x) = A\sinh a(\mathrm{Tr}\, x)$

where real numbers a and A do not depend on x, and $\mathrm{Tr}\, x$ is the trace of the matrix x.

Proof of theorem 1. We divide the proof into two steps.

I. $H = R$. Since $f \not\equiv 0$, there are two numbers a and b such that

$$C = \int_a^b f(t)\,dt \neq 0. \tag{2}$$

If we multiply (2) by $f(s)$ and use (1) we get:

$$C\cdot f(s)=\int_a^b f(t)f(s)\,dt=\int_a^b f^2\big(\tfrac12(t+s)\big)\,dt-\int_a^b f^2\big(\tfrac12(t-s)\big)\,dt$$
$$=\int_{(a+s)/2}^{(b+s)/2} f^2(t)\,dt-\int_{(a-s)/2}^{(b-s)/2} f^2(t)\,dt\,,$$

which implies that: 1. f is an odd function, 2. f is derivable on R and 3. f' is a linear combination of

$$f^2\big(\tfrac12(a\pm s)\big)\,,\qquad f^2\big(\tfrac12(b\pm s)\big)\,.$$

Hence we conclude that f possesses derivatives of all orders.

Now we take the second derivative of (1) with respect to y and we set $y=0$. We get:

(3) $$f(x)f''(x)-\big(f'(x)\big)^2=-\tfrac12[f^2(y)]''_{y=0}\,.$$

If we take the derivative of (3) with respect to x we get

$$f(x)f'''(x)=f'(x)f''(x)\,,$$

whence

(4) $$f(x)=a\cdot x\,,\qquad f(x)=A\sin ax\qquad\text{or}\qquad f(x)=A\sinh ax$$

for all x where a and A are real constants. Thus theorem 1 holds if H is one-dimensional.

II. The general case. For a given $x\in H$, $x\neq 0$ and $t\in R$ we set:

$$f_x(t)=f(tx)\,.$$

Then:

(5) $$f_x(t+s)f_x(t-s)=f_x^2(t)-f_x^2(s)$$

for all $t,s\in R$. The continuity of f and I imply

(6) $$f_x(t)=A(x)t\qquad\text{or}\qquad f_x(t)=A(x)\sin a(x)t$$

for all $t\in R$ where $a(x)$ is a real or pure imaginary number. The rest of the proof we divide into three parts.

a) Suppose that $f_x(t)=A(x)t$ for all $x\in H$, i. e. $f(tx)=tf(x)$ for all $x\in H$ and $t\in R$.

By H_0 we denote the set of all $x\in H$ for which $f(x)=0$ holds. If $x\in H_0$ then $f(tx)=tf(x)=0$, i. e. $x\in H_0$ implies $tx\in H_0$ for every $t\in R$. If $x,y\in H_0$ then (1) implies $f(x+y)f(x-y)=0$, i. e. $x+y\in H_0$ or $x-y\in H_0$. Suppose that $x-y\in H_0$ and replace x,y in (1) by $x+y$, $x-y$ respectively. We get:

$$f(2x)f(2y)=f^2(x+y)-f^2(x-y)\,,$$

i. e. $f(x+y)=0$. Thus $x,y\in H_0$ implies $x+y\in H_0$ and therefore H_0 is a linear manifold. Let H_1 be the set of all vectors $x\in H$ which are orthogonal on H_0. We assert that H_1 is a one-dimensional space. Suppose that this is not so, and let e_1, e_2 be two orthonormal vectors of H_1 ($f\neq 0$). Since $e_1,e_2\in H_1$, we have $f(e_1)f(e_2)\neq 0$. Set $t_0=-f(e_1)/f(e_2)$. Then $f(e_1)+t_0f(e_2)=0$ implies $f(e_1)+f(t_0e_2)=0$, i. e. $f^2(e_1)=f^2(t_0e_2)$. Replacing x and y in (1) by e_1 and t_0e_2 respectively we find $f(e_1+t_0e_2)\cdot f(e_1-t_0e_2)=0$, whence follows $e_1+t_1e_2\in H_0$ ($t_1^2=t_0^2$). But $e_1+t_1e_2\in H_0$ leads to the conclusion that this vector is orthogonal on $e_1\in H_1$, which is impossible. In such a way H_1 is one-dimensional. The continuity of f implies that H_0 is a closed linear manifold so that every $x\in H$ can be written in the form

(7) $$x=(x,e)e+z$$

where $z\in H_0$ and e ($\|e\|=1$) is the vector which determines H_1.

Now, (7), (1) and $f(z)=0$ lead to

$$f(x)=\pm f[(x,e)e]=\pm(x,e)f(e)=\varepsilon(x)g(x)\,,$$

where $g(x)=\big(x,f(e)e\big)$ and $\varepsilon(x)=\pm1$. Obviously $g(x)$ satisfies (1). If we set $f(x)=\varepsilon(x)g(x)$ in (1) we get:

$$\varepsilon(x+y)g(x+y)\varepsilon(x-y)g(x-y)=\varepsilon^2(x)g^2(x)-\varepsilon^2(y)g^2(y)$$
$$=g^2(x)-g^2(y)=g(x+y)g(x-y)\,.$$

Thus $\varepsilon(x+y)\cdot\varepsilon(x-y)=1$. Replacing here $x+y$ by x and $x-y$ by y we get $\varepsilon(x)\cdot\varepsilon(y)=1$, whence follows $\varepsilon(x)=\varepsilon(y)$. Thus $f(x)=(x,x_0)$ with $x_0=f(e)e$ or $x_0=-f(e)e$.

b) Suppose that

(8) $$f(tx)=A(x)\sin a(x)t$$

for all $x\in H$ and $t\in R$. Setting sx instead of x in (8) we get

$$f(t\cdot sx)=A(sx)\sin a(sx)t=f(ts\cdot x)=A(x)\sin ts\,a(x)$$

which implies

(9) $$B(sx)=sB(x)\qquad\text{with}\qquad B(x)=A(x)a(x)$$

for all $x\in H$ and $s\in R$. Set (8) in (1), take the second derivative and set $t=0$. The result is

$$B(x+y)B(x-y)=B^2(x)-B^2(y)\,,$$

which together with (9) implies that the set H_0 of all $x\in H$ for which $B(x)=0$ is a linear manifold and that the set H_1 of all vectors which are orthogonal on H_0 is one-dimensional. But $B(x)=0$ implies $A(x)=0$ or $a(x)=0$, i. e. $f(x)=0$. The continuity of f implies that H_0 is a closed linear manifold, so that (7) holds also in this case. As in a) we find

$$f(x)=\pm A(e)\sin[a(e)(x,e)]$$

where $a(e)$ is a real or pure imaginary number. Thus $f(x) = A\sin(x, x_0)$ or $f(x) = A\sinh(x, x_0)$ with a suitable $x_0 \in H$ and a constant A.

c) It remains to prove that a "mixed solution" cannot occur, i. e. that either $f(tx) = A(x)t$ or $f(tx) = A(x)\sin a(x)t$ for all $x \in H$ and $t \in R$. Suppose that this is not so and let x, y be two linearly independent vectors such that

$$f_x(t) = A(x)t\,, \quad f_y(t) = C(y)\sin a(y)t \quad \text{and} \quad A(x)\cdot C(y)\cdot a(y) \neq 0\,.$$

Using I we have

$$(10) \qquad f_{x\pm y}(t) = A(x\pm y)t \quad \text{or} \quad f(x\pm y) = C(x\pm y)\sin a(x\pm y)t$$

for all $t \in R$. Furthermore (1) implies

$$(11) \qquad f_{x+y}(t)f_{x-y}(t) = f_x^2(t) - f_y^2(t) = A^2(x)t^2 - C^2(y)\sin^2 a(y)t\,.$$

Setting all possible solutions (10) in (11) we get a contradiction. For example, assuming

$$f_{x\pm y}(t) = A(x\pm y)t$$

and using the fact $C(y)a(y) \neq 0$ we find that the functions

$$t^2\,, \quad \sin^2 a(y)t$$

are linearly dependent. The other possible solutions of (10) lead to the linear dependence of the functions:

$$t^2\,, \quad t\sin a(x-y)t\,, \quad \sin^2 a(y)t \quad \text{or}$$

$$t^2\,, \quad \sin a(x+y)t\cdot\sin a(x-y)t\,, \quad \sin^2 a(y)t\,.$$

Since these functions are not linearly dependent, we find that mixed solutions cannot occur. Thus a) and b) are the only possible cases and therefore theorem 1 is proved.

Proof of theorem 2. If $x = (x_{ij})$ and $y = (y_{ij})$ are two matrices from M and if we set

$$(x, y) = \sum_{i,j=1}^{n} x_{ij}y_{ij}\,,$$

then M becomes the Hilbert (Euclidean) real space. Since the functional f satisfies all conditions of theorem 1, we find that f has one of the following forms:

$$f(x) = a(x)\,, \quad f(x) = A\sin A(x)\,, \quad f(x) = A\sinh a(x)$$

where $a(x) = (x, x_0)$ and x_0 is some constant vector-matrix from M. For every real number t and for every orthogonal matrix s we have $f(s^{-1}txs) = f(tx)$, which implies $a(s^{-1}xs) = a(x)$. Indeed, this is obvious in the first case. In other cases we have

$$\sin ta(s^{-1}xs) = \sin ta(x)\,, \quad \sinh ta(s^{-1}xs) = \sinh ta(x)\,.$$

If we take the derivative of these equations with respect to t and then set $t=0$ we get the desired result. Thus the functional $a(x)$ is orthogonally invariant. This, continuity of $a(x)$ and $a(x+y) = a(x) + a(y)$ imply $a(x) = a\operatorname{Tr}x$ with some constant a ([1], theorem 7). Q. E. D.

References

[1] S. Kurepa, *Functional equations for invariants of matrices*, Glasnik mat. fiz. i astr. 14 (1959), p. 97-113.

[2] — *Functional equation $F(x+y)F(x-y) = F^2(x) - F^2(y)$ in n-dimensional vector space*, Math. Mh., Band 64, Heft 4 (1960), p. 321-329.

[3] — *On some functional equations in Banach spaces*, Stud. Math. 19 (1960), p. 149-158.

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