On a theorem of C. Carathéodory

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The role of vector-valued functions as an instrument for treating many problems of Analysis has often been emphasized recently. The purpose of this paper is to prove a well-known theorem of Carathédory[1]) dealing with the differential equation $y'=\varphi(t,y)$, by aid of vector-valued functions. We use these functions in proving an approximation theorem, from which the theorem of Carathéodory is deduced by standard methods. The methods we apply permit us to weaken slightly the hypotheses about the function $\varphi(t,y)$.

In N° 1 we present some known results concerning vector-valued functions, then, in N° 2, we prove the approximation theorem, finally, we give the proof of the theorem of Carathéodory.

1. A function from an interval $[a,b]$ to a Banach space X will be called *vector-valued*; numerically valued functions will be termed simply *functions*. The vector-valued function $x(t)$ is called *measurable*[2]) if there exists a sequence $\{x_n(t)\}$ of continuous vector-valued functions such that $\|x_n(t)-x(t)\|$ converges to 0 for almost any t. We shall need an important criterion of measurability due to Pettis[3]). A set Γ of linear functionals on X is called *fundamental* if there exist two positive constants B and A such that $\|\gamma\|\leqslant B$ for every $\gamma\in\Gamma$ and

$$\sup_{\gamma\in\Gamma}|\gamma x|\geqslant A\|x\|$$

for every x. Then we have

Theorem of Pettis. *Let the space X be separable, and let Γ denote a fundamental set of linear functionals on X. Then the vector-valued function $x(t)$ is measurable if and only if for every $\gamma\in\Gamma$ the numerical function $\gamma x(t)$ is measurable*[4]).

[1]) See [3], p. 672.

[2]) See [2].

[3]) See [4], p. 257, or [1], p. 113.

[4]) Pettis has proved the theorem under the hypothesis that $A=B=1$. Introducing, however, in X the norm $\|x\|^*=\sup_{\gamma\in\Gamma}|\gamma x|$ we obtain another norm (equivalent to $\|x\|$), under which the constants A and B become equal to 1.

We shall use also the fact that if the function $x(t)$ is measurable, and $\|x(t)\|\leqslant s(t)$ where $s(t)$ is also measurable, then there exist continuous functions $x_n(t)$ such that $\|x_n(t)\|\leqslant s(t)$ and $\|x_n(t)-x(t)\|\to 0$ almost everywhere.

2. We shall prove now the approximation-theorem. Denote by $\boldsymbol{C}$ the space of the functions $\varphi=\varphi(u)$ defined for $\alpha\leqslant u\leqslant\beta$ and continuous there. It is a separable Banach space under the norm

$$\|\varphi\|=\max_{\alpha\leqslant u\leqslant\beta}|\varphi(u)|.$$

Every functional γ_v defined as $\gamma_v\varphi=\varphi(v)$ is linear on $\boldsymbol{C}$; moreover if S denotes a dense set in $[\alpha,\beta]$, the set Γ of functionals γ_v with $v\in S$ is fundamental in $\boldsymbol{C}$.

Every function $\varphi(t,u)$ defined for $a\leqslant t\leqslant b$, $\alpha\leqslant u\leqslant\beta$ and continuous in the variable u may be considered as a vector-valued function from $[a,b]$ to $\boldsymbol{C}$. Moreover, if for every $v\in S$ the function $\varphi(t,v)$ is measurable in t, the corresponding vector-valued function is measurable by the theorem of Pettis. We notice now that $\varphi(t,u)$ is continuous as a vector-valued function if and only if it is continuous in all the variables jointly. This, together with the remarks of N° 1, enables us to state

Theorem 1. *Let the set S be dense in $[\alpha,\beta]$ and let the function $s(t)$ be measurable. If the function $\varphi(t,u)$ defined for $a\leqslant t\leqslant b$, $\alpha\leqslant u\leqslant\beta$ is continuous for fixed t and measurable for fixed $u\in S$ and $|\varphi(t,u)|\leqslant s(t)$, then there exist continuous functions $\varphi_n(t,u)$ such that $|\varphi_n(t,u)|\leqslant s(t)$ and*

$$\lim_{n\to\infty}\max_{\alpha\leqslant u\leqslant\beta}|\varphi_n(t,u)-\varphi(t,u)|=0$$

for almost every $t\in[a,b]$.

3. We pass now to the theorem of Carathéodory. Let the function $\varphi(t,u)$ satisfy the hypotheses of theorem 1 and let $s(t)$ be integrable. Consider the differential equation

$$y'=\varphi(t,y);$$

we seek a function $y(t)$ absolutely continuous such that $y'(t)=\varphi(t,y(t))$ almost every where and $y(\tau)=\eta$ where $\tau\in(a,b)$, $\eta\in(\alpha,\beta)$. The equivalent form of the equation is

$$y(t)=\eta+\int_\tau^t\varphi(\vartheta,y(\vartheta))\,d\vartheta.$$

Let p,q be chosen so that

$$\int_p^\tau s(t)\,dt<\min(\beta-\eta,\eta-\alpha),\qquad \int_\tau^q s(t)\,dt<\min(\beta-\eta,\eta-\alpha).$$

By theorem 1 there exist continuous functions $\varphi_n(t,u)$ such that $|\varphi_n(t,u)|\leqslant s(t)$ and

$$\lim_{n\to\infty}\max_{a\leqslant u\leqslant\beta}|\varphi_n(t,u)-\varphi(t,u)|=0 \tag{1}$$

for $t\epsilon[a,\beta]-A$ where $|A|=0$.

Consider now the equation

$$y'=\varphi_n(t,y), \qquad y(\tau)=\eta;$$

by a well-known existence-theorem of Peano there exists a solution of this equation $y_n(t)$; these functions are defined in (p,q) and

$$y_n(t)=\eta+\int_\tau^t\varphi_n(\vartheta,y_n(\vartheta))\,d\vartheta. \tag{2}$$

Since

$$|y_n(t_1)-y_n(t_2)|\leqslant\Big|\int_{t_1}^{t_2}\varphi_n(\vartheta,y_n(\vartheta))\,d\vartheta\Big|\leqslant\Big|\int_{t_1}^{t_2}s(\vartheta)\,d\vartheta\Big|,$$

the functions $y_n(t)$ are equicontinuous; it is obvious that these functions are uniformly bounded. Hence, by Arzelà's theorem, there exists a sequence $\{y_{n_k}(t)\}$ converging uniformly to a function $y(t)$ in every interval I interior to (p,q). By (1)

$$\lim_{k\to\infty}\big|\varphi(t,y_{n_k}(t))-\varphi_n(t,y_{n_k}(t))\big|=0$$

for every $t\epsilon(p,q)-A$ and by continuity of φ in u

$$\lim_{n\to\infty}\big|\varphi(t,y(t))-\varphi(t,y_n(t))\big|=0$$

for every $t\epsilon[a,\beta]$. Hence, by Lebesgue's theorem on integration of sequences, we may pass to the limit in (2), whence

$$y(t)=\eta+\int_\tau^t\varphi(\vartheta,y(\vartheta))\,d\vartheta.$$

Thus we have proved

THEOREM 2. *Let the function $\varphi(t,u)$ defined for $a\leqslant t\leqslant b$, $a\leqslant u\leqslant\beta$ be continuous for fixed t, and measurable for fixed $u\epsilon S$, suppose further that $|\varphi(t,u)|\leqslant s(t)$ where $s(t)$ is integrable; then there exists a function $y(t)$ satisfying the differential equation*

$$y'=\varphi(t,y)$$

almost everywhere and passing through the point (τ,η); the function $y(t)$ is defined for $p<t<q$ where p,q are chosen so that

$$\int_p^\tau s(t)\,dt<\min(\beta-\eta,\eta-a), \qquad \int_\tau^q s(t)\,dt<\min(\beta-\eta,\eta-a).$$

The theorem of Carathéodory in the original version, *i. e.* for systems of equations

$$y_i'=\varphi_i(t,y_1,\ldots,y_n),$$

where φ_i are continuous for fixed t in all the variables $y_1,y_2,\ldots,y_n$ jointly, and measurable for fixed $y_1,\ldots,y_n$, may be proved in the same manner. We have merely to replace the space C in N° 2 by the space of continuous functions of n variables.

References

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[3] C. Carathéodory, *Reelle Funktionen*, Leipzig-Berlin 1927.

[4] B. J. Pettis, *Differentiation in Banach spaces*, Duke Math. Journal 5 (1939), p. 254-269.

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