Il est probable qu'on ait $F_1^\lambda = F_1$ lorsque λ satisfait à la condition

$$A^\lambda < m = \min_{z \in F_1} |g_0(z)|,$$

mais je n'ai pas réussi à le démontrer.

2. Il est probable que dans l'évaluation (41) du module R le nombre N^2 puisse être remplacé par N.

3. La méthode exposée dans ce travail peut être étendue à la construction de la fonction effectuant la représentation conforme du domaine donné D sur les domaines doublement connexes plus généraux que la couronne circulaire. A cet effet, on doit remplacer la fonction frontière $\varphi(\zeta)$ par une autre convenablement choisie.

Articles cités

[1] F. Leja, *Une généralisation de l'écart et du diamètre transfini d'un ensemble*, Ann. Soc. Pol. Math. 22 (1950), p. 35-42.

[2] — *Sur une famille de fonctions analytiques*, ibidem 25 (1952), p. 1-16.

[3] — *Sur les suites de polynômes, les ensembles fermés et la fonction de Green*, ibidem 12 (1933), p. 57-71.

[4] — *Sur une suite de polynômes et la représentation conforme d'un domaine plan quelconque sur le cercle*, ibidem 14 (1935), p. 116-134.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

Haantjes and Alt curvatures in abstract metric spaces

by A. Pliś (Kraków)

Haantjes has shown that in metric spaces of the Fréchet's type the curve may have a Haantjes curvature without having an Alt curvature, and all the more without having a Menger curvature[1]). In Haantjes' example his curvature equals 2 (while the Alt upper curvature is finite). It is known, that in certain spaces (*e. g.* Euclidean, eliptical) every curve for which Haantjes curvature equals zero in every point, is geodesic. In this note we exemplify *the curve for every point p of which the Haantjes curvature is equal to zero and which in no point has an Alt curvature.* (The Alt upper curvature for that curve is equal $+\infty$.) Hence it does not possess a Menger curvature.

1. Let us consider a segment $[0, 1/2]$. The distance of two points of this segment is expressed by

$$\varrho(p_1, p_2) = |\tau_1 - \tau_2|,$$

where τ_i is a Cartesian coordinate of point p_i.

Now let us introduce for this segment a new metric

$$\varrho^*(p_1, p_2) = f(|\tau_1 - \tau_2|) \tag{1}$$

which fulfils Fréchet's axioms $\big(\varrho^*(p,p) = 0,\ \varrho^*(p,q) = \varrho^*(q,p) > 0$ for $p \neq q$, and $\varrho^*(p,q) + \varrho^*(q,r) \geqslant \varrho^*(p,r)\big)$ and by which the segment $[0, 1/2]$ becomes arc L in a certain non Euclidean space.

Owing to some properties of the function $f(u)$ the arc L has a Haantjes curvature equal to zero at every point and does not have an Alt curvature.

2. In order to define the function $f(u)$, we introduce the following notations

$$a_n = \left(\frac{1}{2}\right)^n \qquad (n = 1, 2, \ldots), \tag{2}$$

$$b_n = \frac{a_n - a_n^4}{1 - a_{n+1}^3}. \tag{3}$$

[1]) Compare [1] and [2].

We have inequalities

(4) $$a_{n+1}<b_n<a_n \qquad (n=1,2,\ldots).$$

Now let us define $f(u)$, for $0\leqslant u\leqslant 1/2$ in the following manner

(5) $$f(u)=a_n-a_n^4 \quad \text{for} \quad b_n<u\leqslant a_n \qquad (n=1,2,\ldots),$$

(6) $$f(u)=(1-a_{n+1}^3)u \quad \text{for} \quad a_{n+1}<u\leqslant b_n \qquad (n=1,2,\ldots),$$

(7) $$f(0)=0.$$

The graph of this function consists of a countable number of segments which lie either on straight lines parallel to u axis, or on straight lines going throuhg the point $(0,0)$.

It can easily be verified, that the function $f(u)$ satisfies the following conditions:

(8) *$f(u)$ is a continuous and non-decreasing function,*

(9) *$f(u)/u$ is a non-increasing function for $u>0$,*

(10) $$f(u)\leqslant u$$

as it comes from (6), (2), (9) and (7).

Furthermore the function $f(u)$ fulfils the inequality

(11) $$f(u)\geqslant u-u^4.$$

In fact, in the case of (5) we have $f(u)=a_n-a_n^4\geqslant u-u^4$ from the inequalities $0\leqslant u\leqslant a_n\leqslant 1/2$. In the case of (6) we have

$$f(u)=(1-a_{n+1}^3)u=u-a_{n+1}^3u\geqslant u-u^4 \quad \text{as} \quad 0<a_{n+1}<u.$$

Now we shall prove that

(12) $$f(u_1+u_2)\leqslant f(u_1)+f(u_2) \quad \text{for} \quad u_1+u_2\geqslant\frac{1}{2}.$$

For $u_1=0$ or $u_2=0$ it is a consequence of (7).

In the case of $u_1>0$ and $u_2>0$ we have

$$\frac{f(u_1+u_2)}{u_1+u_2}\leqslant\frac{f(u_1)}{u_1}$$

and

$$\frac{f(u_1+u_2)}{u_1+u_2}\leqslant\frac{f(u_2)}{u_2}$$

from (9). Let us multiply the first inequality by $(u_1+u_2)u_1$, the second by $(u_1+u_2)u_2$ and add. We obtain the inequality

$$(u_1+u_2)f(u_1+u_2)\leqslant(u_1+u_2)[f(u_1)+f(u_2)],$$

hence, since $u_1>0$ and $u_2>0$, the inequality (12).

3. The metric (1) is topologically equivalent to the Euclidean metric by the relations (10), (11), hence the interval $[0,1/2]$ is by the metric (1) really an arc.

We shall show, that the metric (1) fulfils Fréchet axioms.

In fact

I. $\varrho^*(p,p)=0$ by (1), (7),

II. $\varrho^*(p,q)=\varrho^*(q,p)\geqslant\varrho(q,p)-\varrho(q,p)^4>0$ by (1), (11),

III. $\varrho^*(p,q)+\varrho^*(q,r)=f(|\tau_p-\tau_q|)+f(|\tau_q-\tau_r|)$

$$\geqslant f(|\tau_p-\tau_q|+|\tau_q-\tau_r|)\geqslant f(|\tau_p-\tau_r|)=\varrho^*(p,r) \quad \text{by (12), (8).}$$

4. Next it will be shown that the arc L is rectifiable and its length from $p(\tau_1)$ to $p(\tau_2)$ is expressed by the formula:

(13) $$l(\tau_1,\tau_2)=|\tau_2-\tau_1|.$$

Let $\{\sigma_i\}$ be a finite subset of the interval $[\tau_1,\tau_2]$, such that

(14) $$\sigma_0=\tau_1<\sigma_1<\sigma_2<\ldots<\sigma_n<\tau_2=\sigma_{n+1}$$

and

(15) $$|\sigma_{i+1}-\sigma_i|<\varepsilon \qquad (i=0,1,2,\ldots,n),$$

where ε is an arbitrary positive number.

Denoting the length of the polygonal line with vertices $p(\sigma_i)$ by β_n we have

$$\beta_n=\sum_{i=0}^{n}\varrho(\sigma_{i+1},\sigma_i).$$

By using the inequalities (10), (11) and (14) we obtain

(16) $$(\tau_2-\tau_1)-\sum_{i=0}^{n}(\sigma_{i+1}-\sigma_i)^4\leqslant\beta_n\leqslant\tau_2-\tau_1.$$

Combining the inequalities (15) and (16), we get

$$\beta_n\geqslant\tau_2-\tau_1-\sum_{i=0}^{n}(\sigma_{i+1}-\sigma_i)^4\geqslant\tau_2-\tau_1-\sum_{i=0}^{n}(\sigma_{i+1}-\sigma_i)\varepsilon^3=(\tau_1-\tau_2)(1-\varepsilon^3),$$

hence

$$(\tau_2-\tau_1)(1-\varepsilon^3)\leqslant\beta_n\leqslant(\tau_2-\tau_1).$$

But since ε is arbitrary, the formula (13) is proved.

icm©

5. To prove that the Haantjes curvature exists in every point $p(\tau_0)$ of the arc L and identically equals zero, let us consider the following expression

$$H=\frac{l(\tau_1,\tau_2)-\varrho^*(\tau_1,\tau_2)}{l^3(\tau_1,\tau_2)}=\frac{|\tau_2-\tau_1|-\varrho^*(\tau_1,\tau_2)}{|\tau_2-\tau_1|^3}$$

as τ_1 and τ_2 approach zero.

By virtue of definition (1) and the inequalities (10), (11) we have

$$0\leqslant H\leqslant\frac{|\tau_2-\tau_1|-(|\tau_2-\tau_1|-|\tau_2-\tau_1|^4)}{|\tau_2-\tau_1|^3}=|\tau_2-\tau_1|$$

which implies

(17) $$\lim_{\substack{\tau_2\to 0\\ \tau_1\to 0}} H=0.$$

Hence the Haantjes curvature $K_H=\sqrt{24\lim H}=0$.

6. We shall show now that the Alt curvature does not exist in any point of the arc L.

Let $\bar{\tau}$ be an arbitrary number of the interval $[0,1/2]$. Let us define two sequences tending to $\bar{\tau}$ in the following manner:

(18) $$\tau_n^{(1)}=\bar{\tau}+a_{n+1}\quad\text{in the case of } \bar{\tau}\leqslant\frac{1}{4},$$

(19) $$\tau_n^{(2)}=\bar{\tau}+b_{n+1}\quad\text{in the case of } \bar{\tau}\leqslant\frac{1}{4},$$

(18′) $$\tau_n^{(1)}=\bar{\tau}-a_{n+1}\quad\text{in the case of } \bar{\tau}\geqslant\frac{1}{4},$$

(19′) $$\tau_n^{(2)}=\bar{\tau}-b_{n+1}\quad\text{in the case of } \bar{\tau}\geqslant\frac{1}{4}.$$

It is evident that

(20) $$\tau_n^{(1)}\in\left[0,\frac{1}{2}\right],\qquad \tau_n^{(2)}\in\left[0,\frac{1}{2}\right]$$

by (2), (4) and (18), (19) or (18′), (19′),

(21) $$\tau_n^{(1)}\neq\tau_n^{(2)},\qquad \tau_n^{(1)}\neq\bar{\tau},\qquad \tau_n^{(2)}\neq\bar{\tau}$$

by (2), (4) and (18), (19) or (18′), (19′),

(22) $$\tau_n^{(1)}\xrightarrow[n\to\infty]{}\bar{\tau},\qquad \tau_n^{(2)}\xrightarrow[n\to\infty]{}\bar{\tau}.$$

Let us now consider a triangle Δ_n with vertices $p(\bar{\tau})$, $p(\tau_n^{(1)}),p(\tau_n^{(2)})$ and denote the lengths of its sides by a_n, β_n, γ_n.

We have

(23) $$a_n=f(a_{n+1})\leqslant a_{n+1}$$

by definition (1) and inequality (10),

(24) $$\beta_n=f(b_{n+1})=f(a_{n+1})=a_n$$

by the relations (1)-(5),

(25) $$\gamma_n=f(a_{n+1}-b_{n+1})\leqslant f(a_{n+1})=a_n$$

by the relations (1)-(8).

If we denote by r_n the radius of the circle circumscribed about the triangle Δ_n (see definition of Alt curvature), we have by (24)

(26) $$r_n=\frac{a_n^2}{\sqrt{4a_n^2-\gamma_n^2}}\leqslant\frac{a_n^2}{\sqrt{3a_n^2}}<a_n\leqslant a_{n+1}$$

by virtue of (25), (26).

The inequality (26) and notations (2) imply that $r_n\to 0$, hence by (20), (21), (22) the Alt upper curvature at the point $p(\bar{\tau})$ equals $+\infty$ which was to be proved.

References

[1] J. Haantjes, *Distance geometry. Curvature in abstract metric spaces*, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam 50 (1947), p. 496-508.

[2] — *Characteristic local property of geodesics in certain metric spaces*, ibid. 54 (1951), p. 66-73.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES