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SIMILARITY TRANSFORMATION OF MATRICES TO ONE COMMON CANONICAL FORM AND ITS APPLICATIONS TO 2D LINEAR SYSTEMS

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The notion of a common canonical form for a sequence of square matrices is introduced. Necessary and sufficient conditions for the existence of a similarity transformation reducing the sequence of matrices to the common canonical form are established. It is shown that (i) using a suitable state vector linear transformation it is possible to decompose a linear 2D system into two linear 2D subsystems such that the dynamics of the second subsystem are independent of those of the first one, (ii) the reduced 2D system is positive if and only if the linear transformation matrix is monomial. Necessary and sufficient conditions are established for the existence of a gain matrix such that the matrices of the closed-loop system can be reduced to the common canonical form.

Keywords: common canonical form, similarity transformation, 2D linear system, state feedback.

1. Introduction

The notion of controlled and conditioned invariant subspaces was introduced by Basile and Marro (1969) and it has initiated the geometric approach to linear control system analysis and synthesis (Basile and Marro, 1982; Malabre *et al.*, 1997; Wonham, 1979; Kaczorek, 1992). Various canonical forms for linear 1D systems have been introduced and applied to solve the pole assignment problem, the observer design problem, the disturbance decoupling problem, etc. (Ansaklis and Michel, 1997; Kailath, 1980; Żak, 2003; Kaczorek, 1992).

The most popular models of two-dimensional (2D) linear systems are those introduced by Roesser (1975), Fornsini and Marchesini (1978) and Kurek (1985). The geometric approach to 2D linear systems was introduced by Conte and Perdon (1988), Conte *et al.* (1991), Kaczorek (1992), Karmanciolu and Lewis (1990; 1992). The problem of internally and externally stabilizing controlled and output-nulling subspaces for 2D Fornasini-Marchesini models using state-feedbacks was investigated in (Ntogramatzis, 2010).

In this paper the notion of the common canonical form for a sequence of square matrices will be introduced, necessary and sufficient conditions for the existence of a similarity transformation reducing the matrices to canonical form will be established. The common canonical form will be applied to standard and positive 2D linear systems described by the general model.

The paper is organized as follows. In Section 2, the notion of the common canonical form for a sequence of square matrices is introduced, and necessary and sufficient conditions for the existence of a similarity transformation reducing the matrices to the canonical form are established. The theory developed in Section 2 is applied to linear 2D systems in Section 3. It is shown that, using a suitable state vector linear transformation, it is possible to decompose a linear 2D system into two 2D linear subsystems such that the dynamics of the second subsystem are independent of those of the first one. It is also shown that the reduced 2D system is positive if and only if the linear transformation matrix is monomial. In Section 4, linear 2D systems with state-feedbacks are analyzed. Necessary and sufficient conditions are established for the existence of a gain matrix such that the matrices of the closed-loop system can be reduced to the common canonical form. Concluding remarks are given in Section 5.

In this paper the following notation will be used. The n-dimensional real linear space will be denoted by $\mathbb{R}^n$. The set of real $n \times m$ matrices will be denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. The set of real $n \times m$ matrices with nonnegative entries will be denoted by $\mathbb{R}_+^{n\times m}$ and the set of nonnegative integers will be denoted by $\mathbb{Z}_+$. The $n\times n$ identity matrix will be denoted by I_n.

2. Similarity transformation of matrices to one common canonical form

Consider a sequence of q real matrices of the same dimensions:

$$A_i \in \mathbb{R}^{n\times n}, \quad i = 1,2,\dots,q. \tag{1}$$

Definition 1. A linear subspace $J \in \mathbb{R}^n$ is said to be $(A_1, A_2, \dots, A_q)$-*invariant* if

$$A_i x \in J \text{ for every } x \in J \quad \text{and} \quad i = 1,2,\dots,q. \tag{2}$$

It is well known (Wonham, 1979) that every full column rank matrix $J \in \mathbb{R}^{r\times n}$ can be a basis matrix for the linear subspace $J \in \mathbb{R}^n$ if $J = \operatorname{Im} J$, where $\operatorname{Im} J$ denotes the image of J.

Definition 2. The matrices (1) have a *common canonical form* if they can be expressed as

$$\bar{A}_i = \begin{bmatrix} \bar{A}_{i1} & \bar{A}_{i2} \\ 0 & \bar{A}_{i4} \end{bmatrix}, \tag{3}$$

where

$$\bar{A}_{i1} \in \mathbb{R}^{r\times r},\ \bar{A}_{i2} \in \mathbb{R}^{r\times(n-r)},$$
$$\bar{A}_{i4} \in \mathbb{R}^{(n-r)\times(n-r)}, \quad i = 1,2,\dots,q.$$

We shall show that the matrices (1) can be reduced to the common canonical form (3) by the similarity transformation

$$\bar{A}_i = TA_iT^{-1}, \quad T \in \mathbb{R}^{n\times n},\ \det(T) \neq 0. \tag{4}$$

Let $J \in \mathbb{R}^{n\times r}$ have full column rank, i.e.,

$$\operatorname{rank} J = r, \quad r = 1,2,\dots \quad (r < n). \tag{5}$$

Theorem 1. *A set of q real matrices (1) can be reduced to the common canonical form (3) by the similarity transformation (4) if and only if there exists a full column rank matrix $J \in \mathbb{R}^{n\times r}$ such that*

$$\operatorname{rank} \begin{bmatrix} J & A_iJ \end{bmatrix} = r \quad \textit{for} \quad i = 1,2,\dots,q. \tag{6}$$

Proof. By the assumption (5) there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that

$$TJ = \begin{bmatrix} I_r \\ 0 \end{bmatrix}. \tag{7}$$

Let

$$\bar{A}_i = TA_iT^{-1} = \begin{bmatrix} \bar{A}_{i1} & \bar{A}_{i2} \\ \bar{A}_{i3} & \bar{A}_{i4} \end{bmatrix}, \tag{8}$$

where

$$\bar{A}_{i1} \in \mathbb{R}^{r\times r}, \quad \bar{A}_{i2} \in \mathbb{R}^{r\times(n-r)},$$
$$\bar{A}_{i4} \in \mathbb{R}^{(n-r)\times(n-r)}, \quad i = 1,2,\dots,q.$$

We shall show that $\bar{A}_{i3} = 0$ for $i = 1,2,\dots,q$ if and only if (6) holds.

Using (8), (4) and (7), we obtain

$$\begin{aligned} \begin{bmatrix} \bar{A}_{i1} \\ \bar{A}_{i3} \end{bmatrix} &= \begin{bmatrix} \bar{A}_{i1} & \bar{A}_{i2} \\ \bar{A}_{i3} & \bar{A}_{i4} \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \\ &= TA_iT^{-1}TJ = TA_iJ = TJB \\ &= \begin{bmatrix} I_r \\ 0 \end{bmatrix} B = \begin{bmatrix} B \\ 0 \end{bmatrix} \end{aligned} \tag{9}$$

if and only if (6) holds, since $A_iJ = JB$, $i = 1,2,\dots,q$ for some $B \in \mathbb{R}^{r\times r}$. ■

From the proof we have the following procedure for computation of the matrix T and the matrices in the canonical form (3):

Procedure

Step 1. Find a full column rank matrix $J \in \mathbb{R}^{n\times r}$ satisfying the condition (6) for $i = 1,2,\dots,q$.

Step 2. Using elementary column operations (Kaczorek, 2007), find a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ satisfying (7).

Step 3. Using (8), find the canonical form of the matrices (1).

Example 1. Find a matrix T and a common canonical form for the matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 4 & 0 \\ 2 & 2 & 0 \end{bmatrix}. \tag{10}$$

Using the above procedure, we obtain the following:

Step 1. In this case we choose

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{11}$$

and the condition (6) is satisfied since

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} J & A_1J \end{bmatrix} &= \operatorname{rank} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \\ &= \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = 2 \end{aligned}$$

and

$$\operatorname{rank}\begin{bmatrix} J & A_2J \end{bmatrix} = \operatorname{rank}\left[\begin{array}{cc|cc} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] = \operatorname{rank}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = 2.$$

Step 2. It is easy to verify that the condition (7) is satisfied for the matrix

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{12}$$

Premultiplication of *J* by *T* is equivalent to the interchange of its second and third rows.

Step 3. Using (8) and (12), we obtain

$$\begin{aligned} \bar{A}_1 &= TA_1T^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 2 \end{array}\right], \end{aligned}$$

$$\bar{A}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{A}_{12} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \bar{A}_{14} = [2],$$

$$\begin{aligned} \bar{A}_2 &= TA_2T^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 \\ 0 & 4 & 0 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \left[\begin{array}{cc|c} 0 & 3 & 1 \\ 2 & 0 & 2 \\ \hline 0 & 0 & 4 \end{array}\right], \end{aligned}$$

$$\bar{A}_{21} = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}, \quad \bar{A}_{22} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \bar{A}_{24} = [4].$$

3. Linear 2D systems

3.1. Standard 2D systems. Consider the general model of 2D linear systems,

$$x_{i+1,j+1} = A_0x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1}, \tag{13}$$

where $i,j \in \mathbb{Z}_+ = \{0,1,\dots\}$, $x_{i,j} \in \mathbb{R}^n$ is the state vector and $A_k \in \mathbb{R}^{n\times n}$, $k = 0,1,2\dots$.

The boundary conditions for (13) have the form

$$x_{i,0} \in \mathbb{R}^n, \;\; i \in \mathbb{Z}_+ \quad \text{and} \quad x_{0,j} \in \mathbb{R}^n, \;\; j \in \mathbb{Z}_+. \tag{14}$$

Let $\bar{x}_{ij} \in \mathbb{R}^n$ be a new state vector related to x_{ij} by the equality

$$\bar{x}_{ij} = \begin{bmatrix} \bar{x}_{ij}^{(1)} \\ \bar{x}_{ij}^{(2)} \end{bmatrix} = Tx_{ij}, \tag{15}$$

where $T \in \mathbb{R}^{n\times n}$ is a nonsingular matrix satisfying the relations (7) and (8).

Substitution of (15) into (13) yields

$$\begin{aligned} \begin{bmatrix} \bar{x}_{i+1,j+1}^{(1)} \\ \bar{x}_{i+1,j+1}^{(2)} \end{bmatrix} &= \begin{bmatrix} \bar{A}_{01} & \bar{A}_{02} \\ 0 & \bar{A}_{04} \end{bmatrix}\begin{bmatrix} \bar{x}_{ij}^{(1)} \\ \bar{x}_{ij}^{(2)} \end{bmatrix} \\ &+ \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{14} \end{bmatrix}\begin{bmatrix} \bar{x}_{i+1,j}^{(1)} \\ \bar{x}_{i+1,j}^{(2)} \end{bmatrix} \\ &+ \begin{bmatrix} \bar{A}_{21} & \bar{A}_{22} \\ 0 & \bar{A}_{24} \end{bmatrix}\begin{bmatrix} \bar{x}_{i,j+1}^{(1)} \\ \bar{x}_{i,j+1}^{(2)} \end{bmatrix}. \end{aligned} \tag{16}$$

Let $J \in \mathbb{R}^{n\times r}$ be a basis matrix for an *r*-dimensional $(A_1, A_2, \dots, A_q)$-invariant subspace:

$$J = \operatorname{Im} J. \tag{17}$$

Theorem 2. *If the boundary conditions (14) satisfy*

$$x_{i0} \in J, \;\; i \in \mathbb{Z}_+ \quad \textit{and} \quad x_{0j} \in J, \;\; j \in \mathbb{Z}_+, \tag{18}$$

then

$$\bar{x}_{ij}^{(2)} = 0 \quad \textit{for all} \quad i,j \in \mathbb{Z}_+ \tag{19a}$$

and

$$\bar{x}_{ij}^{(1)} \in J \quad \textit{for all} \quad i,j \in \mathbb{Z}_+. \tag{19b}$$

Proof. From (7) it follows that, if boundary conditions satisfy the condition (18), then

$$\bar{x}_{i0}^{(2)} = 0 \quad \text{for} \quad i \in \mathbb{Z}_+, \tag{20a}$$

$$\bar{x}_{0j}^{(2)} = 0 \quad \text{for} \quad j \in \mathbb{Z}_+. \tag{20b}$$

From (16) we have

$$\begin{aligned} \bar{x}_{i+1,j+1}^{(2)} &= \bar{A}_{04}\bar{x}_{ij}^{(2)} + \bar{A}_{14}\bar{x}_{i+1,j}^{(2)} \\ &\quad + \bar{A}_{24}\bar{x}_{i,j+1}^{(2)}, \quad i,j \in \mathbb{Z}_+, \end{aligned} \tag{21}$$

and, taking into account (20b), we obtain (17a).

From (16) we also have

$$\begin{aligned} \bar{x}_{i+1,j+1}^{(1)} &= \bar{A}_{01}\bar{x}_{ij}^{(1)} + \bar{A}_{02}\bar{x}_{ij}^{(2)} \\ &\quad + \bar{A}_{11}\bar{x}_{i+1,j}^{(1)} + \bar{A}_{12}\bar{x}_{i+1,j}^{(2)} \\ &\quad + \bar{A}_{21}\bar{x}_{i,j+1}^{(1)} + \bar{A}_{22}\bar{x}_{i,j+1}^{(2)}, \quad i,j \in \mathbb{Z}_+. \end{aligned} \tag{22}$$

After substitution of (17a) into (22) we obtain

$$\bar{x}_{i+1,j+1}^{(1)} = \bar{A}_{01}\bar{x}_{ij}^{(1)} + \bar{A}_{11}\bar{x}_{i+1,j}^{(1)} + \bar{A}_{21}\bar{x}_{i,j+1}^{(1)}, \quad i,j \in \mathbb{Z}_+,$$

and this implies (17b). ∎

3.2. Positive 2D system.

Definition 3. (Kaczorek, 2001) The 2D linear system (13) is called *positive* if $x_{i,j} \in \mathbb{R}_+^n$ for $i, j \in \mathbb{Z}_+$ for all boundary conditions,

$$x_{i,0} \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+, \qquad x_{0,j} \in \mathbb{R}_+^n, \quad j \in \mathbb{Z}_+. \tag{23}$$

Theorem 3. (Kaczorek, 2001) *The 2D linear system (13) is positive if and only if*

$$A_k \in \mathbb{R}_+^{n\times n}, \quad k = 0, 1, 2. \tag{24}$$

A matrix $A \in \mathbb{R}^{n\times n}$ is called *monomial* if in each of its rows and in each of its columns only one entry is positive and the remaining entries are zero.

Theorem 4. *Let the linear 2D system (13) be positive. The reduced 2D system (16) is positive if and only if the transformation matrix $T \in \mathbb{R}_+^{n\times n}$ is monomial.*

Proof. It is well known (Kaczorek, 2001) that the inverse matrix $A^{-1} \in \mathbb{R}_+^{n\times n}$ if and only if A is a monomial matrix. From (15) we have $\bar{x}_{i0} = Tx_{i0} \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ and $\bar{x}_{0j} = Tx_{0j} \in \mathbb{R}_+^n$, $j \in \mathbb{Z}_+$ for any boundary conditions (23) if and only if the matrix T is a monomial one.

Similarly, from (8) we have $\bar{A}_i \in \mathbb{R}_+^{n\times n}$ for $i = 0, 1, 2$ if and only if T is a monomial matrix. ∎

Example 2. Consider the 2D system (13) with the matrices (10) and

$$A_0 = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}. \tag{25}$$

The 2D system (13) with (10) and (25) is positive since by Theorem 3 its matrices have nonnegative entries.

For the matrix (11) and (25) we obtain

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} J & A_0J \end{bmatrix} &= \operatorname{rank}\left[\begin{array}{cc|cc} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \\ &= \operatorname{rank}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = 2, \end{aligned}$$

and the transformation matrix T is monomial and has the form (12).

Using (8) and (25) we obtain

$$\begin{aligned} \bar{A}_0 &= TA_0T^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 2 \\ \hline 0 & 0 & 1 \end{array}\right], \end{aligned}$$

$$\bar{A}_{01} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{02} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \bar{A}_{04} = [1].$$

The reduced 2D system has the form

$$\begin{aligned} \begin{bmatrix} \bar{x}^{(1)}_{i+1,j+1} \\ \bar{x}^{(2)}_{i+1,j+1} \end{bmatrix} &= \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 2 \\ \hline 0 & 0 & 1 \end{array}\right]\begin{bmatrix} \bar{x}^{(1)}_{ij} \\ \bar{x}^{(2)}_{ij} \end{bmatrix} \\ &+ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 2 \end{array}\right]\begin{bmatrix} \bar{x}^{(1)}_{i+1,j} \\ \bar{x}^{(2)}_{i+1,j} \end{bmatrix} \\ &+ \left[\begin{array}{cc|c} 0 & 3 & 1 \\ 2 & 0 & 2 \\ \hline 0 & 0 & 4 \end{array}\right]\begin{bmatrix} \bar{x}^{(1)}_{i,j+1} \\ \bar{x}^{(2)}_{i,j+1} \end{bmatrix}, \end{aligned}$$

and it is a positive 2D system.

These deliberations can be extended to linear 2D systems with delays as well as 1D discrete-time and continuous-time linear systems with delays.

4. Linear 2D systems with state-feedbacks

Consider the linear 2D system

$$x_{i+1,j+1} = A_0x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1} + Bu_{ij}, \qquad i, j \in \mathbb{Z}_+, \tag{26}$$

subject to the boundary conditions (14), where $x_{i,j} \in \mathbb{R}^n$, $u_{i,j} \in \mathbb{R}^m$ are respectively the state and input vectors, $A_k \in \mathbb{R}^{n\times n}$, $k = 0, 1, 2$, $B \in \mathbb{R}^{n\times m}$.

Let us assume that for a given matrix $J \in \mathbb{R}^{n\times r}$ the condition

$$\operatorname{rank}\begin{bmatrix} J & A_iJ \end{bmatrix} = \operatorname{rank} J = r \tag{27}$$

is not satisfied for $i = 0$ but it is satisfied for $i = 1, 2$.

We are looking for a gain matrix $K \in \mathbb{R}^{n\times n}$ of the state feedback

$$u_{ij} = Kx_{ij} \tag{28}$$

such that the closed-loop system

$$x_{i+1,j+1} = (A_0+B_0K)x_{ij}+A_1x_{i+1,j}+A_2x_{i,j+1} \tag{29}$$

satisfies the condition

$$\operatorname{rank}\begin{bmatrix} J & (A_0 + BK)J \end{bmatrix} = r. \tag{30}$$

Theorem 5. *Let the matrix $\hat{A}_0$ satisfy the condition*

$$\operatorname{rank}\begin{bmatrix} J & \hat{A}_0J \end{bmatrix} = \operatorname{rank} J, \quad \hat{A}_0 = A_0 + BK. \tag{31}$$

There exists a gain matrix K such that (30) is met if and only if

$$\operatorname{rank} B = \operatorname{rank}\begin{bmatrix} B & \hat{A}_0 - A_0 \end{bmatrix}. \tag{32}$$

Proof. If the condition (31) is satisfied, then by the Kronecker-Capelly theorem, the equation

$$BK = \hat{A}_0 - A_0, \tag{33}$$

has a solution K if and only if (32) holds. ■

Example 3. Consider the 2D system (26) with the matrices (10) and

$$A_0 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}. \tag{34}$$

In this case, for the matrix (11) the condition (6) is satisfied for $i = 1, 2$, but it is not satisfied for $i = 0$ since

$$\operatorname{rank} \begin{bmatrix} J & A_0 J \end{bmatrix} = \operatorname{rank} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] = 3. \tag{35}$$

Let

$$\hat{A}_0 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix}. \tag{36}$$

In this case the condition (32) is satisfied since

$$\begin{aligned} \operatorname{rank} B = \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} B & \hat{A}_0 - A_0 \end{bmatrix} \\ &= \operatorname{rank} \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & -1 \\ 0 & 2 & -1 & 0 & -2 \end{array}\right] = 2 \end{aligned} \tag{37}$$

and the solution K of the equation

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} K = - \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \tag{38}$$

is

$$K = - \begin{bmatrix} 2 & 0 & 1 \\ 0.5 & 0 & 1 \end{bmatrix}.$$

The matrix (36) satisfies the condition (31) since

$$\operatorname{rank} \begin{bmatrix} J & \hat{A}_0 J \end{bmatrix} = \operatorname{rank} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] = 2.$$

Using (8) and (36), we obtain

$$\begin{aligned} \tilde{A}_0 &= T \hat{A}_0 T^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 3 \\ \hline 0 & 0 & 1 \end{array}\right], \end{aligned}$$

$$\tilde{A}_{01} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_{02} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \tilde{A}_{04} = [1]. \tag{39}$$

Hence, the reduced closed-loop system has the form

$$\begin{aligned} \begin{bmatrix} \bar{x}^{(1)}_{i+1,j+1} \\ \bar{x}^{(2)}_{i+1,j+1} \end{bmatrix} &= \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 3 \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \bar{x}^{(1)}_{ij} \\ \bar{x}^{(2)}_{ij} \end{bmatrix} \\ &+ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 2 \end{array}\right] \begin{bmatrix} \bar{x}^{(1)}_{i+1,j} \\ \bar{x}^{(2)}_{i+1,j} \end{bmatrix} \\ &+ \left[\begin{array}{cc|c} 0 & 3 & 1 \\ 2 & 0 & 2 \\ \hline 0 & 0 & 4 \end{array}\right] \begin{bmatrix} \bar{x}^{(1)}_{i,j+1} \\ \bar{x}^{(2)}_{i,j+1} \end{bmatrix}. \end{aligned} \tag{40}$$

In a similar way, these deliberations can be easily extended to linear 2D systems described by the general model

$$\begin{aligned} x_{i+1,j+1} = {} & A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ & + B_0 u_{ij} + B_1 u_{i+1,j} \\ & + B_2 u_{i,j+1}, \qquad i, j \in \mathbb{Z}_+ \end{aligned} \tag{41}$$

with state-feedback, where $x_{ij} \in \mathbb{R}^n$ and $u_{ij} \in \mathbb{R}^m$ are respectively the state and input vectors and $A_k \in \mathbb{R}^{n \times n}$ for $k = 0, 1, 2$.

5. Concluding remarks

The notion of the common canonical form (3) for the sequence of q real square matrices (1) was introduced. Necessary and sufficient conditions for the existence of the similarity transformation (4) reducing the matrices to the canonical form (3) were established (Theorem 1). A procedure for computation of the matrix T of the similarity transformation and of the common canonical form of the matrices (1) was proposed. Using the procedure and a linear state vector transformation, it was shown that a linear 2D system can be decomposed into two linear 2D subsystems. The dynamics of the second subsystem are independent of those of the first one. If the boundary conditions satisfy the assumption (18), then the state vector of the second subsystem is zero (Theorem 2). It was shown that the reduced 2D system (16) is positive if and only if the transformation matrix T is monomial (Theorem 4). Necessary and sufficient conditions were established for the existence of a gain matrix K such that the matrices of the closed-loop 2D linear system can be reduced to the common canonical form. The common canonical form can also be applied to the stability analysis and stabilization of linear 1D and 2D systems. The discussion can be extended to linear 1D and 2D systems with delays.

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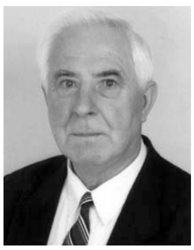

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