

CONTROLLABILITY AND OBSERVABILITY OF LINEAR DISCRETE-TIME FRACTIONAL-ORDER SYSTEMS

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In this paper we extend some basic results on the controllability and observability of linear discrete-time fractional-order systems. For both of these fundamental structural properties we establish some new concepts inherent to fractional-order systems and we develop new analytical methods for checking these properties. Numerical examples are presented to illustrate the theoretical results.

Keywords: system modeling, discrete fractional state-space systems, reachability, controllability, observability, controllability and observability Gramians.

1. Introduction

Nowadays the concept of non-integer derivative and integral is used increasingly to model the behavior of real systems in various fields of science and engineering. Originally, fractional calculus was the field of mathematical analysis aiming at the investigation of integrals and derivatives of arbitrary orders. This topic is somewhat ancient, since it started from some speculations of G.W. Leibnitz, L'Hopital (1695, 1697), and L. Euler (1730). In a letter to L'Hopital in 1695, Leibnitz raised the following question: *Can the meaning of derivatives with integer order $d^n y(t)/dx^n$ be generalized to derivatives with non-integer order, so that in the general case $n \in \mathbb{C}$?* Later, up to the middle of the 20th century, a long list of mathematicians provided important contributions to this topic, among them Laplace, Fourier, Abel, Liouville, Riemann, Grünwald, Letnikov, Hadamard, Lévy, Marchaud, Love, and Riesz.

Further steps in the development of this discipline were the organization of specialized conferences and the publication of treatises only three decades ago. The first of such conferences—it is a milestone—was organized by B. Ross in 1974.

Recent books (Oldham and Spanier, 1974; Oustaloup, 1983; Samko *et al.*, 1993; Miller and Ross, 1993; Oustaloup, 1995; Gorenflo and Mainardi, 1997; Podlubny, 1999; Kilbas *et al.*, 2006) provide a rich source of references on fractional-order calculus.

In the particular domain of control theory, several authors have been interested by this aspect since the 1960s. The first contributions (Axtell and Bise, 1990; Manabe, 1960; Oustaloup, 1983) provided generalizations of classical analysis methods for fractional-order systems (transfer function, frequency response, pole and zero analysis, etc.).

Recently, this tool has come into current use for modeling physical phenomena of real systems such as electrochemistry (Ichise *et al.*, 1971), electromagnetism and electrical machines (Lin *et al.*, 2000), thermal systems and heat conduction (Battaglia *et al.*, 2001; Cois *et al.*, 2002), transmission and acoustics (Matignon, 1994; Matignon *et al.*, 1994), viscoelastic materials (Hanyga, 2003), and robotics (Valerio and Costa, 2004). These systems exhibit hereditarily properties and long memory transients, which can be represented more accurately by fractional-order models.

Consequently, numerous contributions are regularly resorting to the fractional-order aspect in system modeling, namely, with state space representation, in parameter estimation, identification, and controller design. The CRONE approach (in French: *Commande Robuste d'Ordre Non Entier*) was elaborated to offer new solutions to control problems, such as vehicle suspension. Two noticeable conferences (the 41st IEEE CDC conference in 2002 and the IFAC FDA'04 workshop in 2004) treated the application of fractional-order calculus to automatic control theory.

The state-space representation of fractional-order systems was introduced in (Raynaud and Zergainoh, 2000; Hotzel and Fliess, 1998; Dorčák et al., 2000; Sabatier et al., 2002; Vinagre et al., 2002). It emerged that for fractional-order systems, two different interesting types can be considered: the commensurate-order and the non-commensurate-order systems. The system is of a commensurate-order if all the orders of derivatives are multiple integers of a base α , where α denotes the differentiation order. The state-space representation was exploited in the analysis of system performances. In fact, the solution of the state-space equation was derived by using the Mittag-Leffler function (Mittag-Leffler, 1904). The stability of the fractional-order system was investigated (Matignon and d'Andrèa Novel, 1996). A condition based on the principle of the argument was established to guarantee the asymptotic stability of the fractional-order system. Further, controllability and observability properties were defined, and some algebraic criteria of these two properties were derived (Matignon and d'Andrèa Novel, 1996).

A contribution to the analysis of the controllability and observability of commensurate continuous-time fractional-order systems modeled by fractional state space equations was recently made in (Bettayeb and Djennoune, 2006).

Linear discrete-time fractional-order systems modeled by a state space representation were introduced in (Dzieliński and Sierociuk, 2005; Dzieliński and Sierociuk, 2006; Dzieliński and Sierociuk, 2007). These contributions are devoted to controllability and observability analysis, the design of a Kalman filter and observers, plus adaptive feedback control for discrete fractional-order systems.

Our objective in the present paper is to contribute to the analysis of the controllability and observability of linear discrete-time fractional-order systems. To the best of our knowledge, controllability as well as some aspects of the observability of such systems have not been treated yet. Two reviewers brought to our attention the paper (Dzieliński and Sierociuk, 2007), which appeared several months after we submitted the present paper. This work is complementary to ours. We propose new concepts that are inherent to fractional-order systems, and we establish te-

stable sufficient conditions for guaranteeing the existence of these structural properties.

The remainder of this paper is organized as follows: In Section 2, we recall some fundamental definitions of fractional derivatives and fractional-order systems, modeled by continuous models. Then we expose a discrete-time model, as defined in (Dzieliński and Sierociuk, 2005), and we introduce some extra notation that reveals a new form, making it possible to take into account the past behavior of the system and to analyze the structural properties. Section 3 addresses the controllability property. The specificity of linear discrete-time fractional-order systems modeled by a state-space representation leads to interesting features that are not shown by integer-order systems.

In Section 4, similar deliberations are yielded by the study of the observability property. In Section 5, we consider the case of commensurate-order systems. Finally, in Section 6, we present some numerical results corresponding to different cases of checking the controllability and observability conditions.

2. Linear discrete-time fractional-order systems

The discrete fractional-order difference operator Δ was defined in (Dzieliński and Sierociuk, 2005) with the zero initial time as follows:

$$\Delta^\alpha x(k) = \frac{1}{h^\alpha} \sum_{j=0}^k (-1)^j \binom{\alpha}{j} x(k-j), \quad (1)$$

where the fractional order $\alpha \in \mathbb{R}^{*+}$, i.e., the set of strictly positive real numbers, $h \in \mathbb{R}^{*+}$ is a sampling period taken equal to unity in all what follows, and $k \in \mathbb{N}$ represents the discrete time. We define

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0, \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for } j > 0. \end{cases} \quad (2)$$

Let us consider now the traditional discrete-time state-space model of integer order, i.e., when α is equal to unity:

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad (3a)$$

$$y(k) = Cx(k), \quad (3b)$$

where $u(k) \in \mathbb{R}^m$ is the input vector, $y(k) \in \mathbb{R}^q$ is the output vector, and $x(k) \in \mathbb{R}^n$ is the state vector:

$$x(k) = [x_1(k) \quad x_2(k) \quad \dots \quad x_n(k)]^T.$$

Its initial value is denoted by $x_0 = x(0)$.

The first-order difference for $x(k+1)$ is defined as

$$\Delta^1 x(k+1) = x(k+1) - x(k).$$

Therefore, using Eqn. (3a) we deduce that

$$\begin{aligned} \Delta^1 x(k+1) &= Ax(k) + Bu(k) - x(k) \\ &= A_d x(k) + Bu(k), \end{aligned} \quad (4)$$

where $A_d = A - I_n$ and I_n is the identity matrix.

A generalization of this integer-order difference to a non-integer-order (or fractional-order) difference was addressed in (Dzieliński and Sierociuk, 2005; Lakshmikantham, 1988). This research was conducted to construct a linear discrete-time fractional-order state-space model, using the equations

$$\Delta^\alpha x(k+1) = A_d x(k) + Bu(k), \quad x(0) = x_0. \quad (5)$$

In this model the differentiation order α is taken the same for all the state variables $x_i(k)$, $i = 1, \dots, n$. This is referred to as a commensurate order. Besides, from Eqn. (1) we have

$$\Delta^\alpha x(k+1) = x(k+1) + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x(k-j+1). \quad (6)$$

Substituting (6) into (5) yields

$$\begin{aligned} x(k+1) &= A_d x(k) - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x(k-j+1) + Bu(k). \end{aligned} \quad (7)$$

Set $c_j = (-1)^j \binom{\alpha}{j}$. Equation (7) can be rewritten as

$$\begin{aligned} x(k+1) &= (A_d - c_1 I_n)x(k) \\ &\quad - \sum_{j=2}^{k+1} c_j x(k-j+1) + Bu(k). \end{aligned} \quad (8)$$

Let us now write

$$\begin{aligned} A_0 &= A_d - c_1 I_n, \\ A_1 &= -c_2 I_n, \\ A_2 &= -c_3 I_n, \end{aligned}$$

and further, for all $j > 0$,

$$A_j = -c_{j+1} I_n. \quad (9)$$

This leads to

$$\begin{aligned} x(k+1) &= A_0 x(k) + A_1 x(k-1) + A_2 x(k-2) \\ &\quad + \dots + A_k x(0) + Bu(k). \end{aligned} \quad (10)$$

This description can be extended to the case of non-commensurate fractional-order systems modeled in (Dzieliński and Sierociuk, 2006):

$$\Delta^\Upsilon x(k+1) = A_d x(k) + Bu(k),$$

$$x(k+1) = \Delta^\Upsilon x(k+1) + \sum_{j=1}^{k+1} A_j x(k-j+1),$$

where

$$\Delta^\Upsilon x(k+1) = \begin{bmatrix} \Delta^{\alpha_1} x_1(k+1) \\ \vdots \\ \Delta^{\alpha_n} x_n(k+1) \end{bmatrix},$$

in which $\alpha_i \in \mathbb{R}^{*+}$, $i = 1, 2, \dots$ denote any fractional orders, and for each $j = 1, 2, \dots$ we let

$$A_j = \text{diag} \left\{ -(-1)^{j+1} \binom{\alpha_i}{j+1}, i = 1, \dots, n \right\}. \quad (11)$$

Using (10) and (11), we obtain the state equation

$$x(k+1) = \sum_{j=0}^k A_j x(k-j) + Bu(k), \quad x(0) = x_0. \quad (12)$$

In this model, A_j is given by (9) in the case of a commensurate fractional-order and by (11) in the case of a non-commensurate fractional-order.

Remark 1. The model described by (12) can be classified as a discrete-time system with a time-delay in the state, whereas the models addressed in (Boukas, 2006; Debeljković *et al*, 2002; Peng Yang *et al*, 2003) assume a finite constant number of steps of time-delays. Instead, the system (12) has a varying number of steps of time-delays, equal to k , i.e., increasing with time.

Define

$$G_k = \begin{cases} I_n & \text{for } k = 0, \\ \sum_{j=0}^{k-1} A_j G_{k-1-j} & \text{for } k \geq 1. \end{cases} \quad (13)$$

In an explicit way, we have

$$\begin{aligned} G_0 &= I_n, \\ G_1 &= \sum_{j=0}^0 A_j G_{1-1-j} = A_0 G_0 = A_0, \\ G_2 &= \sum_{j=0}^1 A_j G_{2-1-j} = A_0 G_1 + A_1 G_0 = A_0^2 + A_1. \end{aligned}$$

We thus conclude that any G_k can be expressed equivalently either by a recurrent sum made of products $A_j G_{k-1-j}$ or by a recurrent sum made of products of the A_j exclusively.

Theorem 1. The solution to (12) is given by

$$x(k) = G_k x(0) + \sum_{j=0}^{k-1} G_{k-1-j} Bu(j). \quad (14)$$

Proof. In Step 1, by virtue of (12) and (13), the value of the state is

$$\begin{aligned} x(1) &= \sum_{j=0}^0 A_j x(0-j) + Bu(0) \\ &= A_0 x(0) + Bu(0) = G_1 x(0) + G_0 Bu(0). \end{aligned}$$

In Step 2 we have

$$\begin{aligned} x(2) &= \sum_{j=0}^1 A_j x(1-j) + Bu(1) \\ &= (A_0^2 + A_1)x(0) + A_0 Bu(0) + Bu(1) \\ &= G_2 x(0) + G_1 Bu(0) + G_0 Bu(1). \end{aligned}$$

The property is thus satisfied in Step 2. Assume that it is true in Step k , i.e., that (14) is satisfied. In order to complete the demonstration of this property by induction, we have to prove that it is true in Step $k + 1$.

From (12) we obtain

$$\begin{aligned} x(k+1) &= \sum_{j=0}^k A_j x(k-j) + Bu(k) \\ &= \sum_{j=0}^{k-1} A_j x(k-j) + A_k x(0) + Bu(k). \end{aligned} \quad (15)$$

In the last equation, $x(k-j)$ can be expressed similarly to $x(k)$ since the property considered is assumed to be true up to Step k . Therefore, we have (introducing an extra index l):

$$x(k-j) = G_{k-j} x(0) + \sum_{l=0}^{k-j-1} G_{k-j-1-l} Bu(l). \quad (16)$$

Thereafter, coming back to the expression of $x(k+1)$ in (15) and substituting $x(k-j)$, we obtain

$$\begin{aligned} x(k+1) &= \sum_{j=0}^{k-1} A_j \left(G_{k-j} x(0) + \sum_{l=0}^{k-j-1} G_{k-j-1-l} Bu(l) \right) \\ &\quad + A_k x(0) + Bu(k). \end{aligned} \quad (17)$$

This expression can be rewritten as follows:

$$\begin{aligned} x(k+1) &= \sum_{j=0}^{k-1} A_j G_{k-j} x(0) + A_k x(0) \\ &\quad + \sum_{j=0}^{k-1} A_j \sum_{l=0}^{k-j-1} G_{k-j-1-l} Bu(l) \\ &\quad + Bu(k). \end{aligned} \quad (18)$$

Further, we get

$$\begin{aligned} x(k+1) &= \sum_{j=0}^k A_j G_{k-j} x(0) \\ &\quad + \sum_{j=0}^{k-1} A_j \sum_{l=0}^{k-j-1} G_{k-j-1-l} Bu(l) \\ &\quad + Bu(k). \end{aligned} \quad (19)$$

The first sum becomes $G_{k+1} x(0)$, knowing that $G_0 = I_n$. Besides, in the product of the last two sums, a permutation of indexes j and l yields an equivalent summation. Hence we obtain

$$\begin{aligned} x(k+1) &= G_{k+1} x(0) + \sum_{l=0}^{k-1} \sum_{j=0}^{k-l-1} A_j G_{k-j-1-l} Bu(l) \\ &\quad + G_0 Bu(k). \end{aligned} \quad (20)$$

Next, this becomes

$$x(k+1) = G_{k+1} x(0) + \sum_{l=0}^{k-1} G_{k-l} Bu(l) + G_0 Bu(k). \quad (21)$$

Finally, we obtain that the property under study is satisfied in Step $k + 1$ and we can state that it holds in any step:

$$x(k+1) = G_{k+1} x(0) + \sum_{l=0}^k G_{k-l} Bu(l). \quad (22)$$

This completes the proof. ■

The first part of the solution of (14) represents the free system response and the last part takes the role of the convolution sum corresponding to the forced response.

The corresponding transition matrix is defined as

$$\Phi(k, 0) = G_k, \quad \Phi(0, 0) = G_0 = I_n. \quad (23)$$

Remark 2. $\Phi(k, 0)$ exhibits the particularity of being time-varying in the sense that it is composed of a number of terms A_j which grows with k . This is due to the fractional-order feature of the model, which takes into account all the past values of the states.

Theorem 2. *The state transition matrix $\Phi(k, 0)$ has the following properties:*

1. $\Phi(k, 0)$ is a solution of the homogeneous state equation

$$\Phi(k+1, 0) = \sum_{j=0}^k A_j \Phi(k-j, 0), \quad \Phi(0, 0) = I_n.$$

2. The semi-group property is not satisfied:

$$\Phi(k_2, 0) \neq \Phi(k_2, k_1)\Phi(k_1, 0), \quad \forall k_2 > k_1 > 0.$$

Proof. (Part 1) From (23) we deduce

$$\Phi(k, 0) = \sum_{j=0}^{k-1} A_j \Phi(k-1-j, 0).$$

Then we directly have

$$\Phi(k+1, 0) = \sum_{j=0}^k A_j \Phi(k-j, 0).$$

(Part 2) Since, by definition, we have

$$\Phi(k_1, 0) = \sum_{j=0}^{k_1-1} A_j \Phi(k_1-1-j, 0),$$

$$\Phi(k_2, 0) = \sum_{j=0}^{k_2-1} A_j \Phi(k_2-1-j, 0),$$

$$\Phi(k_2, k_1) = \sum_{j=k_1}^{k_2-1} A_j \Phi(k_2-1-j, 0),$$

it is can be easily checked that

$$\Phi(k_2, k_1)\Phi(k_1, 0) \neq \Phi(k_2, 0).$$

■

3. Reachability and controllability

In this section we discuss a fundamental question for dynamic systems modeled by (12) in the case of a non-commensurate fractional order. This question is to determine whether it is possible to transfer the state of the system from a given initial value to any other state. We attempt below to extend two concepts of state reachability (or controllability-from-the-origin) and controllability (or controllability-to-the-origin) to the present case. We are interested in completely state reachable and controllable systems.

Definition 1. The linear discrete-time fractional-order system modeled by (12) is *reachable* if it is possible to find a control sequence such that an arbitrary state can be reached from the origin in a finite time.

Definition 2. The linear discrete-time fractional-order system modeled by (12) is *controllable* if it is possible to find a control sequence such that the origin can be reached from any initial state in a finite time.

Definition 3. For the linear discrete-time fractional-order system modeled by (12) we define the following:

1. The controllability matrix:

$$C_k = [G_0 B \quad G_1 B \quad G_2 B \quad \cdots \quad G_{k-1} B]. \quad (24)$$

2. The reachability Gramian:

$$W_r(0, k) = \sum_{j=0}^{k-1} G_j B B^T G_j^T, \quad k \geq 1. \quad (25)$$

It is easy to show that $W_r(0, k) = C_k C_k^T$.

3. The controllability Gramian, provided that A_0 is non-singular:

$$W_c(0, k) = G_k^{-1} W_r(0, k) G_k^{-T}, \quad k \geq 1. \quad (26)$$

Note that $G_1 = A_0$, and the existence of $W_r(0, 1)$ imposes A_0 to be nonsingular. However, this is not that restrictive a condition because a discrete model is often obtained by sampling a continuous one. Thus, in the remainder of this paper we assume that A_0 is non-singular.

Theorem 3. The linear discrete-time fractional-order system modeled by (12) is reachable if and only if there exists a finite time K such that $\text{rank}(C_K) = n$ or, equivalently, $\text{rank}(W_r(0, K)) = n$. Furthermore, the input sequence

$$U_K = [u^T(K-1) \quad u^T(K-2) \quad \cdots \quad u^T(0)]^T$$

that transfers $x_0 = 0$ at $k = 0$ to $x_f \neq 0$ at $k = K$ is given by

$$U_K = C_K^T W_r^{-1}(0, K) x_f. \quad (27)$$

Proof. (Sufficiency) Let x_f be the final state to be reached. From (14) we have

$$x_f(k) = G_k x_0 + \sum_{j=0}^{k-1} G_{k-1-j} B u(j).$$

With $x_0 = 0$, this gives

$$x_f(k) = C_k U_k, \quad (28)$$

where $U_k = [u^T(k-1) \quad u^T(k-2) \quad \cdots \quad u^T(0)]^T$. Equation (28) has a unique solution U_k at Step $k = K$ if $\text{rank}(C_K) = n$. Besides, we have $W_r(0, K) = C_K C_K^T$. Hence, if $\text{rank}(C_K) = n$, then $\text{rank}(W_r(0, K)) = n$. It follows that $W_r(0, K)$ is a positive definite non-singular matrix. At Step $k = K$ we have

$$x_f(K) = C_K U_K. \quad (29)$$

Substituting (27) into (29), we get

$$x_f(K) = C_K C_K^T W_r^{-1}(0, K) x_f = x_f.$$

We conclude that the system (12) is reachable.

(Necessity) This part is by contradiction. Assume that the system (19) is reachable but $\text{rank}(\mathcal{C}_k) = n$ for any $k > 0$, which implies that the rows of \mathcal{C}_k are linearly dependent for any $k > 0$. It results that there exists a non-zero constant $1 \times n$ row vector v such that

$$v\mathcal{C}_k = 0.$$

From (28) we have

$$vx_f(k) = v\mathcal{C}_k\mathcal{U}_k = 0,$$

which implies that $x_f(k) = 0$ for any $k > 0$, i.e., the system is not reachable. This is a contradiction, which completes the proof. ■

Remark 3. In the case of an integer order, it is well known that the rank of \mathcal{C}_k cannot increase for any $k \geq n$. This results from the Cayley-Hamilton theorem. On the contrary, in the case of the linear discrete-time non-commensurate fractional-order system (12), the rank of \mathcal{C}_k can increase for values of $k \geq n$. In other words, it is possible to reach the final state x_f in a number of steps greater than n . This is due to the nature of the elements G_k which build up the controllability matrix \mathcal{C}_k and which exhibit the particularity of being time-varying, in the sense that they are composed of a number of terms A_j that grows with k , as has already mentioned in Remark 2. The full rank of (\mathcal{C}_k) can be reached at some Step $k = K$ equal to or greater than n .

Theorem 4. *The linear discrete-time fractional-order system modeled by (12) is controllable if and only if there exists a finite time K such that $\text{rank}(W_c(0, K)) = n$. Furthermore, an input sequence $\mathcal{U}_K = [u^T(K-1) \ u^T(K-2) \ \dots \ u^T(0)]^T$ that transfers $x_0 \neq 0$ at $k = 0$ to $x_f = 0$ at $k = K$ is given by*

$$\mathcal{U}_K = -\mathcal{C}_K^T G_K^{-T} W_c^{-1}(0, K)x_0. \quad (30)$$

Proof. (Sufficiency) Let $x_f = 0$ be the final state to be reached at some finite time K from an initial state $x_0 \neq 0$. From (14) we have

$$x_f = G_K x_0 + \mathcal{C}_K \mathcal{U}_K = 0,$$

which gives

$$x_0 = -G_K^{-1} \mathcal{C}_K \mathcal{U}_K. \quad (31)$$

If we get $\text{rank}(W_c(0, K)) = n$ for some K , then $W_c^{-1}(0, K)$ exists. Substituting (30) into (31) yields

$$\begin{aligned} x_0 &= G_K^{-1} \mathcal{C}_K \mathcal{C}_K^T G_K^{-T} W_c^{-1}(0, K)x_0 \\ &= W_c(0, K) W_c^{-1}(0, K)x_0 = x_0. \end{aligned}$$

(Necessity) The proof is by contradiction. Assume that (12) is controllable but $\text{rank}(W_c(0, k)) < n$ for any $k > 0$. Since G_k is full rank for $k \geq 0$, we have $\text{rank}(W_c(0, k)) = \text{rank}(W_r(0, k)) = \text{rank}(\mathcal{C}_k)$. It follows that there exists a non-zero constant $1 \times n$ row vector w such that

$$w\mathcal{C}_k = 0.$$

Since $x_f = 0$, from (14) we have

$$wx_f = wG_k x_0 + w\mathcal{C}_k \mathcal{U}_k = 0.$$

This implies that $wG_k x_0 = 0$, i.e., $x_0 = 0$. This is a contradiction, which completes the proof. ■

4. Observability

In this section we aim at extending the concept of observability to the system of equations (12) and (3b), in the case of a non-commensurate fractional order. We are interested in completely state observable systems.

Definition 4. The linear discrete-time fractional-order system modeled by (12) and (3b) is *observable at time $k = 0$* if and only if there exists some $K > 0$ such that the state x_0 at time $k = 0$ can be uniquely determined from the knowledge of $u_k, y_k, k \in [0, K]$.

Definition 5. For the linear discrete-time fractional-order system modeled by (12) and (3b) we define the following:

1. The observability matrix:

$$\mathcal{O}_k = \begin{bmatrix} CG_0 \\ CG_1 \\ CG_2 \\ \vdots \\ CG_{k-1} \end{bmatrix}. \quad (32)$$

2. The observability Gramian:

$$W_o(0, k) = \sum_{j=0}^{k-1} G_j^T C^T C G_j. \quad (33)$$

It is easy to show that $W_o(0, k) = \mathcal{O}_k^T \mathcal{O}_k$.

Theorem 5. *The linear discrete-time fractional-order system modeled by (12) and (3b) is observable if and only if there exists a finite time K such that $\text{rank}(\mathcal{O}_K) = n$ or, equivalently, $\text{rank}(W_o(0, K)) = n$. Furthermore, the initial state x_0 at $k = 0$ is given by*

$$x_0 = W_o^{-1}(0, K) \mathcal{O}_K^T [\tilde{\mathcal{Y}}_K - \mathcal{M}_K \tilde{\mathcal{U}}_K] \quad (34)$$

with

$$\begin{aligned} \tilde{U}_K &= [u^T(0) \quad u^T(1) \quad \dots \quad u^T(K-1)]^T, \\ \tilde{Y}_K &= [y^T(0) \quad y^T(1) \quad \dots \quad y^T(K-1)]^T, \end{aligned}$$

and

$$\mathcal{M}_K = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ CG_0B & 0 & \dots & 0 & 0 \\ CG_1B & CG_0B & \dots & 0 & 0 \\ CG_2B & CG_1B & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CG_{K-2}B & CG_{K-3}B & \dots & CG_0B & 0 \end{bmatrix}. \tag{35}$$

Proof. (Sufficiency) From the output equation (3b) and Eqn. (14) we have

$$\begin{aligned} y(0) &= Cx(0) = CG_0x(0), \\ y(1) &= Cx(1) = CG_1x(0) + CG_0Bu(0), \\ y(2) &= Cx(2) = CG_2x(0) + CG_1Bu(0) + CG_0Bu(1), \end{aligned}$$

and, at last,

$$\begin{aligned} y(k-1) &= Cx(k-1) \\ &= CG_{k-1}x(0) + CG_{k-2}Bu(0) \\ &\quad + CG_{k-3}Bu(1) + \dots \\ &\quad + CG_0Bu(k-2). \end{aligned} \tag{36}$$

The above relations can be written in the following condensed form:

$$\tilde{Y}_k = \mathcal{O}_k x(0) + \mathcal{M}_k \tilde{U}_k,$$

where

$$\begin{aligned} \tilde{U}_k &= [u^T(0) \quad u^T(1) \quad \dots \quad u^T(k-1)]^T, \\ \tilde{Y}_k &= [y^T(0) \quad y^T(1) \quad \dots \quad y^T(k-1)]^T. \end{aligned}$$

At time $k = K$, we can write

$$\tilde{Y}_K = \mathcal{O}_K x(0) + \mathcal{M}_K \tilde{U}_K.$$

It follows that

$$\mathcal{O}_K x(0) = \tilde{Y}_K - \mathcal{M}_K \tilde{U}_K.$$

Then

$$\mathcal{O}_K^T \mathcal{O}_K x(0) = \mathcal{O}_K^T (\tilde{Y}_K - \mathcal{M}_K \tilde{U}_K),$$

which becomes

$$W_o(0, K)x(0) = \mathcal{O}_K^T (\tilde{Y}_K - \mathcal{M}_K \tilde{U}_K).$$

If $\text{rank}(\mathcal{O}_K) = n$ or, equivalently, if $\text{rank}(W_o(0, K)) = n$, then $W_o(0, K)$ is positive definite. Consequently, we obtain

$$x(0) = W_o^{-1}(0, K) \mathcal{O}_K^T (\tilde{Y}_K - \mathcal{M}_K \tilde{U}_K).$$

(Necessity) The proof is by contradiction. Assume that the system of equations (12) and (3b) is observable but $\text{rank}(\mathcal{O}_k) < n$ for any $k > 0$. Then the columns of \mathcal{O}_k are linearly dependent for any $k > 0$, i.e., there exists a non-zero constant column $n \times 1$ vector z such that

$$\mathcal{O}_k z = 0.$$

Let us choose $x(0) = z$. From the relation

$$\tilde{Y}_k = \mathcal{O}_k x(0) + \mathcal{M}_k \tilde{U}_k,$$

we deduce

$$\mathcal{O}_k z = \tilde{Y}_k - \mathcal{M}_k \tilde{U}_k = 0.$$

Hence the initial state $x(0) = z$ is not detected. This is in contradiction with the assumption that the system of equations (12) and (3b) is observable. This completes the proof. ■

Remark 4. From the Cayley-Hamilton theorem, it is well known that for integer-order systems the rank of the observability matrix \mathcal{O}_k cannot increase at Step $k \geq n$. Here, too, it is remarkable that this is not true in the case of the discrete-time non-commensurate fractional-order system of (12) and (3b). Indeed, $\text{rank}(\mathcal{O}_k)$ can increase for values $k \geq n$. We can state that the observability of this type of systems can possibly be obtained in a number of steps greater than n . This is due to the same reasons as those exposed above in Remark 3 for controllability. In (Dzieliński and Sierociuk, 2006), the observability condition for the discrete-time fractional-order system as modeled in (12), with a non-commensurate order, is that the rank of \mathcal{O}_k should be equal to n at most at Step $k = n$. Our result shows that the full rank of (\mathcal{O}_k) can be reached at some step $k = K$ greater than n . This can be considered as an extension of the previous result in (Dzieliński and Sierociuk, 2006).

Remark 5. The property of reconstructibility (Antsaklis and Michel, 1997) can also be studied in this case. Note that if A_0 is non-singular, then observability and reconstructibility are equivalent.

5. Commensurate fractional-order case

In this section we address the particular case of commensurate fractional-order systems. The terms A_j are expressed by (9). It is clear then that the matrices G_k defined by (13) are polynomials in A_0 , i.e.,

$$G_k = A_0^k + \beta_{1k} A_0^{k-1} + \beta_{2k} A_0^{k-2} + \dots + \beta_{kk} I_n,$$

where the real coefficients β_{j_k} are calculated from the coefficients c_j . In particular, we have

$$G_n = A_0^n + \beta_{1_n} A_0^{n-1} + \beta_{2_n} A_0^{n-2} + \dots + \beta_{n_n} I_n.$$

From the Cayley-Hamilton theorem, A_0^n is a linear combination of $A_0^{n-1}, A_0^{n-2}, \dots, I_n$. We deduce that G_{k+n} , for all $k \geq 0$ are linearly dependent on $G_{n-1}, G_{n-2}, \dots, I_n$. This implies the following results:

Corollary 1. *The linear discrete-time fractional-order system modeled by (12) and (3b) in the commensurate case is reachable if and only if $\text{rank}(\mathcal{C}_n) = n$ or, equivalently, $\text{rank}(W_r(0, n)) = n$. On the other hand, this system is controllable if and only if $\text{rank}(W_c(0, n)) = n$.*

Corollary 2. *The linear discrete-time fractional-order system modeled by (12) and (3b) in the commensurate case is observable if and only if $\text{rank}(\mathcal{O}_n) = n$ or, equivalently, $\text{rank}(W_o(0, n)) = n$.*

Remark 6. We therefore observe that the controllability and observability criteria for the commensurate fractional-order case are similar to those of the integer-order case, in the sense that if a state cannot be reached in n steps, then it is not reachable at all, and that if an initial state cannot be deduced from n steps of input-output data, then it is not observable at all. The result given in (Dzieliński and Sierociuk, 2006) which states that a necessary and sufficient condition for the discrete-time fractional-order system as modeled in (12) and (3b) to be observable is that the rank of \mathcal{O}_k should be equal to n at most at Step $k = n$ is true only in the case of commensurate fractional-order systems.

6. Numerical examples

6.1. Reachability. Consider the following discrete-time non-commensurate fractional-order of dimension $n = 4$, with

$$\alpha_1 = 0.2, \quad \alpha_2 = 0.3, \quad \alpha_3 = 0.6, \quad \alpha_4 = 0.7,$$

$$A_d = \begin{bmatrix} -0.7 & -1 & 4 & -0.5 \\ 1 & -1.6 & 1.5 & 0.8 \\ 2 & -3 & -0.1 & 2.5 \\ -0.8 & 0.7 & 1.8 & -0.4 \end{bmatrix},$$

$$B = \begin{bmatrix} 10 & 10 & 10 & 10 \end{bmatrix}^T.$$

We determined $\text{rank}(\mathcal{C}_k)$ over a set of $N = 20$ samples. We found $\text{rank}(\mathcal{C}_k) = 4$ at $K = 5$ and

$$\mathcal{C}_K = \begin{bmatrix} 10.00 & 20.00 & 40.80 & 84.90 & 173.31 \\ 10.00 & 20.00 & 41.05 & 84.77 & 175.66 \\ 10.00 & 20.00 & 41.20 & 84.63 & 177.03 \\ 10.00 & 20.00 & 41.05 & 85.12 & 174.78 \end{bmatrix}.$$

We chose the final state

$$x_f = \begin{bmatrix} 1 & -0.5 & 3 & 0.3 \end{bmatrix}^T.$$

The input sequence that permitted to transfer the state from the origin to x_f according to (27) is

$$\mathcal{U}_K = \begin{bmatrix} 30.31 & 60.61 & 210.91 & -64.38 & -26.85 \end{bmatrix}^T.$$

Table 1 gives the values of the state variables at each step. We see that the final state has been reached within a number of steps of the input data sequence greater than the system dimension. This comes up to be a particu-

Table 1. Values of the state variables in the transfer steps.

k	$x_1(k)$	$x_2(k)$	$x_3(k)$	$x_4(k)$
0	0	0	0	0
1	-268.49	-268.49	-268.49	-268.49
2	-1180.76	-1180.76	-1180.76	-1180.76
3	-273.93	-280.65	-284.67	-280.65
4	-81.96	-94.43	-100.46	-103.96
5	1.00	-0.50	3.00	0.30

larity of discrete non-commensurate fractional-order systems. This is not satisfied in the case of discrete commensurate fractional-order systems for which the full rank, n , if it can be reached, cannot be reached beyond the number of steps $K = n$. The states show some values of large magnitude. Nevertheless, the objective is reached.

6.2. Observability. We considered the system with $\alpha_1 = 0.2, \alpha_2 = 0.3, \alpha_3 = 0.6, \alpha_4 = 0.7$,

$$A_d = \begin{bmatrix} -0.4 & -1 & 4 & -0.5 \\ 1 & 5 & 1.5 & 0.8 \\ 2 & -3 & -5.9 & 2.5 \\ -0.8 & 0.7 & 1.8 & -1.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

We determined $\text{rank}(\mathcal{O}_k)$ over a set of $N = 20$ samples. We found $\text{rank}(\mathcal{O}_k) = 4$ at $K = 5$ and

$$\mathcal{O}_K = \begin{bmatrix} 1.00 & 1.00 & 1.00 & 1.00 \\ 2.00 & 2.00 & 2.00 & 2.00 \\ 4.08 & 4.10 & 4.12 & 4.10 \\ 8.45 & 8.459 & 8.33 & 8.51 \\ 17.06 & 17.95 & 18.34 & 17.09 \end{bmatrix}.$$

We chose the following input sequence over 5 steps:

$$\tilde{U}_K = \begin{bmatrix} 1 & -0.2 & 5 & 10 & -0.6 \end{bmatrix}^T.$$

In this example, we see that the second row of \mathcal{O}_K is the doubled first row. The output sequence must be chosen so as to take into account this dependence. Let us denote by \tilde{Y}_K^* the zero-input response of the system

$$\begin{aligned} \tilde{Y}_K^* &= \tilde{Y}_K - \mathcal{M}_k * \tilde{U}_K \\ &= [y^*(0) \quad y^*(1) \quad y^*(2) \quad y^*(3) \quad y^*(4)]^T. \end{aligned}$$

The output sequence \tilde{Y}_K must be then chosen so as to get $y^*(1) = 2y^*(0)$.

A candidate output sequence is, e.g.,

$$\tilde{Y}_K = \begin{bmatrix} 1 & 6 & -2 & 7 & 3 \end{bmatrix}^T.$$

According to (34), the initial state

$$x_0 = \begin{bmatrix} 1.22 & -3.27 & 1.63 & 0.41 \end{bmatrix}^T$$

is detected. The corresponding determinant of the observability gramian is

$$\det[W_o(0, K)] = 4.97 \times 10^{-5}.$$

The singular value decomposition of $W_o(0, K)$ gives

$$\Sigma = \text{diag}(1613.86, 0.38, 9.80 \times 10^{-4}, 8.34 \times 10^{-5}).$$

We observe that, except the first one, the singular values are quite small: the corresponding states are weakly observable. The plot of the simulated output, starting from the detected initial state x_0 , is illustrated in Fig. 1. The simulated output sequence is identical to the chosen initial output sequence. It is possible to consider other examples with stronger observability. For this purpose, let us consider the same example in which the output matrix C is successively changed into

$$C = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 10 & 10 & 10 & 10 \end{bmatrix}.$$

The determinant of the observability Gramian takes the values $\det[W_o(0, K)] = 19.422$ and $\det[W_o(0, K)] = 4972$, respectively. This shows that the state variables may become strongly observable.

7. Conclusion

In this paper we investigated the structural properties of the controllability and observability of linear discrete-time

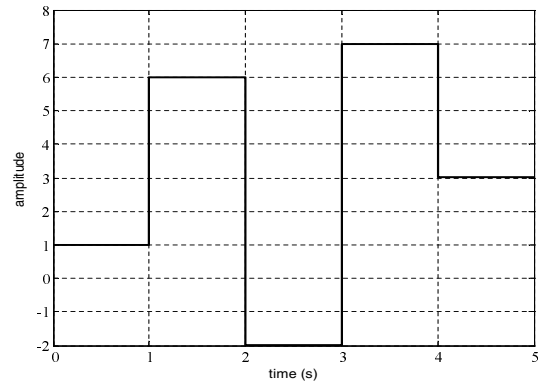


Fig. 1. Computed values of the output sequence.

fractional-order systems. We established a new formulation of the state-space equation and showed that this new formulation makes it possible to analyze effectively these properties. In addition, it reveals new controllability and observability conditions in both cases of non-commensurate and commensurate fractional-orders. We verified the theoretical results stated in this paper with suitable numerical examples.

Acknowledgments

The authors would like to thank the anonymous reviewers for their numerous constructive remarks that have improved this paper.

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Received: 17 June 2007

Revised: 21 October 2007