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PROPER FEEDBACK COMPENSATORS FOR A STRICTLY PROPER PLANT BY POLYNOMIAL EQUATIONS[†]

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We review the polynomial matrix compensator equation $X_l D_r + Y_l N_r = D_k$ (COMP), e.g. (Callier and Desoer, 1982, Kučera, 1979; 1991), where (a) the right-coprime polynomial matrix pair (N_r, D_r) is given by the strictly proper rational plant right matrix-fraction $P = N_r D_r^{-1}$, (b) D_k is a given nonsingular stable closed-loop characteristic polynomial matrix, and (c) (X_l, Y_l) is a polynomial matrix solution pair resulting possibly in a (stabilizing) rational compensator given by the left fraction $C = X_l^{-1} Y_l$. We recall first the class of all polynomial matrix pairs (X_l, Y_l) solving (COMP) and then single out those pairs which result in a *proper* rational compensator. An important role is hereby played by the assumptions that (a) the plant denominator D_r is column-reduced, and (b) the closed-loop characteristic matrix D_k is row-column-reduced, e.g., monically diagonally degree-dominant. This allows us to get all solution pairs (X_l, Y_l) giving a proper compensator with a row-reduced denominator X_l having (sufficiently large) row degrees prescribed *a priori*. Two examples enhance the tutorial value of the paper, revealing also a novel computational method.

Keywords: linear time-invariant feedback control systems, polynomial matrix systems, row-column-reduced polynomial matrices, feedback compensator design, flexible belt device

1. Introduction and Problem Formulation

In the sequel linear time-invariant continuous-time systems will be described by their proper rational transfer matrices. $\mathbb{R}(s)$ (resp. $\mathbb{R}[s]$) denotes the field of rational functions (the ring of polynomials) with real coefficients, $\mathbb{RH}_\infty$ is the ring of rational proper-stable functions in s ('stable' means that all poles are in the open left complex half-plane), while $\underline{m} := \{1, 2, \ldots, m\}$. We are given a strictly proper plant $P(s) \in \mathbb{R}_{po}(s)^{p\times m}$. Our task is to find a proper compensator $C(s) \in \mathbb{R}_p(s)^{m\times p}$ such that the closed-loop unity feedback system $S(P, C)$ of Fig. 1 is input-output stable, see e.g., (Callier and Desoer, 1982, Sec. 4.3; Vidyasagar, 1985, Sec. 5.1), or, equivalently, the input-error transfer function of $S(P, C)$, viz.

$$H_{eu}(P,C) = \begin{bmatrix} (I_p + PC)^{-1} & -P(I_m + CP)^{-1} \\ C(I_p + PC)^{-1} & (I_m + CP)^{-1} \end{bmatrix}, \tag{1}$$

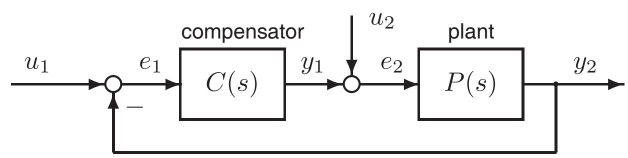

Fig. 1. Unity feedback system $S(P, C)$.

belongs to Mat($\mathbb{RH}_\infty$). For matrix fractions over $\mathbb{RH}_\infty$ leading to state-space solutions, see, e.g., (Francis, 1987, Sec. 4.4; Vidyasagar, 1985, Secs. 4.2 and 5.2). For polynomial matrix fractions, results appeared, e.g., in (Callier and Desoer, 1982; Emre, 1980; Kučera and Zagalak, 1999; Rosenbrock and Hayton, 1978; Zagalak and Kučera, 1985).

We report here the usual procedure in this context and a new extension. The extension relies on a result (Callier and Desoer, 1982, pp. 187–192), which finally gives a parametrized class of proper compensator solutions (Theorem 3 below; Callier, 2000; 2001), which is essentially similar to Theorem 2 (Kučera and Zagalak, 1999) and is contained in earlier parametrizations (Callier and Desoer, 1982, Comm. 95, pp. 193–195; Emre, 1980,

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Rem. 3.4).

The problem we handle is as follows: Represent $P(s) \in \mathbb{R}_{po}(s)^{p\times m}$ as a coprime right polynomial matrix fraction, viz.

$$P(s) = N_r(s)D_r^{-1}(s), \tag{2}$$

and represent $C(s) \in \mathbb{R}(s)^{m\times p}$ as a left polynomial matrix fraction, viz.

$$C(s) = X_l^{-1}(s)Y_l(s). \tag{3}$$

Give an $m \times m$ square closed-loop characteristic polynomial matrix $D_k(s)$ having all its zeros in the open left complex half-plane.[1] Then (i) get all solutions $(X_l, Y_l) \in$ Mat$(\mathbb{R}[s])$ of the polynomial matrix compensator equation

$$X_lD_r + Y_lN_r = D_k, \tag{COMP}$$

and (ii) pick an appropriate solution such that we get a proper compensator, viz.

$$C = X_l^{-1}Y_l \in \mathbb{R}_p(s)^{m\times p}. \tag{4}$$

The answer to Task (i) is well known, and not obvious for Task (ii).

The paper is organized as follows: Section 1 is the present introduction. The next three sections revise and slightly extend the existing theory. Section 2 handles polynomial solutions of (COMP). Section 3 discusses definitions for controlling the degrees of the entries of a polynomial matrix, which in Section 4 lead to three theorems for obtaining proper compensators by solving (COMP). To enhance the tutorial value of the paper, two example sections are added to illustrate the application of the theory. In Section 5 we succesfully design a tracking compensator for a control laboratory device, viz. a flexible belt coupled drives system (see Fig. 2). In Section 6 a comparison with other methods is drawn on a simple example chosen for straightforward numerical evidence using pencil-and-paper computations, reporting hereby a novel two-sided computational method. Finally, Section 7 finishes the paper with a conclusion.

2. Polynomial Solutions of (COMP)

For more details, see (Callier and Desoer, 1982, Thm. 6.2.39). The usual procedure starts with *coprime* right and left polynomial matrix fractions of the plant transfer matrix, i.e.,

$$\begin{aligned} P(s) &= N_r(s)D_r^{-1}(s) \\ &= D_l^{-1}(s)N_l(s) \in \mathbb{R}_{po}(s)^{p\times m}, \end{aligned} \tag{5}$$

[1] This requirement is important for obtaining feedback system input-output stability; it is not necessary for the solvability of (COMP).

for which one finds four polynomial matrices $U_r(s)$, $V_r(s)$, $U_l(s)$ and $V_l(s)$ such that with

$$U := \begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix},$$

one gets

$$U^{-1} = \begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix}.$$

The identity

$$U^{-1}U = UU^{-1} = I_{m+p} \tag{6}$$

is called the *Generalized Bézout Identity* of the plant. All solutions $(X_l, Y_l) \in$ Mat$(\mathbb{R}[s])$ of (COMP) are then given by

$$\begin{bmatrix} X_l & Y_l \end{bmatrix} = \begin{bmatrix} D_k & N_k \end{bmatrix} \begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix}, \tag{7}$$

where $N_k \in \mathbb{R}[s]^{m\times p}$ is a free polynomial matrix. Hence noting that $(X_{lp}, Y_{lp}) = (D_kV_r, D_kU_r)$ is a particular solution of (COMP), all its solutions are given by

$$X_l = X_{lp} - N_kN_l, \quad Y_l = Y_{lp} + N_kD_l. \tag{8}$$

It then follows easily that (8) is valid for *any* particular solution $(X_{lp}, Y_{lp}) \in$ Mat$(\mathbb{R}[s])$ of (COMP).

Remark 1. The solution space of (COMP) is affine in the free parameter N_k, with

$$\operatorname{rank} \begin{bmatrix} X_l & Y_l \end{bmatrix} = \operatorname{rank} \begin{bmatrix} D_k & N_k \end{bmatrix} \tag{9}$$

for $(X_{lp}, Y_{lp}) = (D_kV_r, D_kU_r)$. Moreover, as $Y_{lp} = -N_kD_l + Y_l$, it is possible to get $-N_k$ and Y_l as the quotient and remainder of the division on the right of Y_{lp} by D_l, such that one gets that $Y_lD_l^{-1}$ is strictly proper. However, this does not guarantee that X_l given in (8) is nonsingular and that $C = X_l^{-1}Y_l$ is proper. This is usually obtained by the degree control of polynomial matrices, i.e., by limiting the degrees of their entries.

3. Degree Control of Polynomial Matrices

In the sequel, $\delta_{rj}[D]$ (resp. $\delta_{cj}[D]$) denotes the j-th row (resp. column) degree of a polynomial matrix D, while $\delta_{ij}[D]$ is the degree of the (ij)-th entry. We are inspired here, see, e.g., (Callier and Desoer, 1982, Chap. 2; Kailath, 1980, Sec. 6.3; Wolovich, 1974, Sec. 2.5) by the fact that right (left) polynomial matrix fractions are invariant under multiplication on the right (or left) by a unimodular polynomial matrix. For example, if (N_r, D_r) is a right polynomial matrix fraction of $P(s) \in \mathbb{R}(s)^{p\times m}$, then so is (N_rR, D_rR), where R is any unimodular matrix. Moreover, right-coprimeness is invariant under this transformation. This allows simultaneous elementary column operations on the numerator and denominator such that the latter gets column-reduced. A similar comment is valid for left fractions, left multiplication by a unimodular matrix, and elementary row operations leading to a row-reduced left denominator.

Definition 1. An $m \times m$ polynomial matrix D is said to be *column-reduced* if it has m column degrees $k_j := \delta_{cj}[D]$ such that the limit

$$D_h := \lim_{s\to\infty} D(s)\,\mathrm{diag}\left[s^{-k_j}\right]_{j=1}^{m} \tag{10}$$

exists and is *nonsingular*.

Comment 1. An equivalent requirement is that $D_-(s) := D(s)\,\mathrm{diag}\left[s^{-k_j}\right]_{j=1}^{m}$ is a *biproper* rational matrix. One gets $D_h = D_-(\infty)$ and D_h is called the *highest column degree coefficient matrix* of $D(s)$. In (Wolovich, 1974, p. 27), "column-reduced" is termed "column-proper."

Definition 2. An $m \times m$ polynomial matrix D is said to be *row-reduced* if it has m row degrees $r_i := \delta_{ri}[D]$ such that the limit

$$D_h := \lim_{s\to\infty} \mathrm{diag}\left[s^{-r_i}\right]_{i=1}^{m} D(s) \tag{11}$$

exists and is *nonsingular*.

Comment 2. An equivalent requirement is that $D_-(s) := \mathrm{diag}\,[s^{-r_i}]_{i=1}^{m} D(s)$ is a *biproper* rational matrix. One gets $D_h = D_-(\infty)$ and D_h is called the *highest row degree coefficient matrix* of $D(s)$. In (Wolovich, 1974), "row-reduced" is termed "row-proper."

Remark 2. Column degrees and column-reducedness are appropriate tools for revealing that a right polynomial matrix fraction is proper. Indeed, it is well known, see e.g., (Callier and Desoer, 1982, p. 70; Kailath, 1980, p. 385), that with $(N(s), D(s)) \in \mathrm{Mat}(\mathbb{R}[s])$ and $D(s)$ column-reduced, $G(s) := N(s)D^{-1}(s) \in \mathbb{R}(s)^{p\times m}$ is proper (resp. strictly proper) iff for all $j \in \underline{m}$, $\delta_{cj}[N] \le \delta_{cj}[D]$ (resp. $\delta_{cj}[N] < \delta_{cj}[D]$). A similar comment can be made for left fractions, row degrees and row-reducedness. What is paramount here is the fact that the degree of an entry of a polynomial matrix is limited column-wise by its column-maximum or row-wise by its row-maximum, viz. by the corresponding column degree or row degree. This way of control is less appropriate when the matrix D_k in (COMP) is a sum having a dominant term, which is a product of a row-reduced matrix X_l and a column-reduced matrix D_r.

Theorem 1 below shows that the following is appropriate.

Definition 3. (Callier and Desoer, 1982, p. 116) An $m \times m$ polynomial matrix D is said to be *row-column-reduced* if there exist m nonnegative integers r_i, called *row powers*, and m nonnegative integers k_j, called *column powers*, such that the limit

$$D_h := \lim_{s\to\infty} \mathrm{diag}\left[s^{-r_i}\right]_{i=1}^{m} D(s)\,\mathrm{diag}\left[s^{-k_j}\right]_{j=1}^{m} \tag{12}$$

exists and is *nonsingular*.

Comment 3. An equivalent requirement is that $D_-(s) := \mathrm{diag}\,[s^{-r_i}]_{i=1}^{m} D(s)\,\mathrm{diag}\left[s^{-k_j}\right]_{j=1}^{m}$ is a *biproper* rational matrix. One gets $D_h = D_-(\infty)$ and D_h is called the *highest degree coefficient matrix* of $D(s)$. Another interpretation of the definition is that for all $(i,j) \in \underline{m} \times \underline{m}$, $\delta_{ij}[D] \le r_i + k_j$ with *nonsingular* degree-contact (nonsingular D_h). What is important here is the fact that the degrees of the entries are bilaterally controlled, i.e. with row and column powers; these powers are generically not unique, as will be observed below.

In the sequel c.r., r.r., and r.c.r. are abbreviations for column-reduced, row-reduced, and row-column-reduced. There are many r.c.r. nonsingular polynomial matrices. Rosenbrock has already considered the idea of the definition for $D_h = I$. Note here, see, e.g., (Callier and Desoer, 1982, Sec. 2.4; Kailath, 1980, Sec. 6.3) that any nonsingular polynomial matrix D can be reduced to its Smith form S by elementary row and column operations, i.e., $D = LSR$ with L and R unimodular matrices.

Fact 1. (Rosenbrock and Hayton, 1978, Lem. 1*) Denote Smith forms as* $S = \mathrm{diag}[\phi_i]_{i=1}^{m}$ *with* $\delta[\phi_1] \ge \delta[\phi_2] \ge \cdots \ge \delta[\phi_m]$*. Consider* m *nonnegative integers* r_i *such that* $r_1 \ge r_2 \ge \cdots \ge r_m$*, and* m *nonnegative integers* k_j *such that* $k_1 \ge k_2 \ge \cdots \ge k_m$*.*

Then, there exists a nonsingular r.c.r. polynomial matrix D *having the Smith form* S *and having row powers* r_i*, column powers* k_j*, and* $D_h = I$*, if and only if*

$$\sum_{i=1}^{k} \delta[\phi_i] \ge \sum_{i=1}^{k} (r_i + k_i), \quad \textit{for} \quad k \in \underline{m}, \tag{13}$$

with equality for $k = m$*.*

It follows from this fact that many nonsingular polynomial matrices can be made r.c.r. by elementary operations. The star case of row-column-reducedness is, however, probably reflected by the following case contained in the proof of Fact 1.

Definition 4. An $m \times m$ polynomial matrix D is said to be *monically diagonally degree dominant with diagonal degrees* γ_i *($i \in \underline{m}$)* if every diagonal entry of D is monic and for $(i, j) \in \underline{m} \times \underline{m}$ with $i \neq j$ and $\gamma_i := \delta_{ii}[D]$,

$$\delta_{ij}[D] < \min(\gamma_i, \gamma_j). \tag{14}$$

Comment 4. Here, without loss of generality, by symmetric permutations, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m$, giving, e.g., a degree matrix

$$\begin{bmatrix} \mathbf{5} & 3 & 2 \\ 2 & \mathbf{4} & 2 \\ 1 & 2 & \mathbf{3} \end{bmatrix}.$$

Observe that this is easily obtained: get a "diagonal degree ridge" with "top degrees on the diagonal."

Fact 2. *Let D be an $m \times m$ nonsingular polynomial matrix. Consider m nonnegative integers γ_i such that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m$. Then the following assertions are equivalent:*

(a) D is c.r. with column degrees γ_i and $D_h = I$, simultaneously D is r.r. with row degrees γ_i and $D_h = I$.

(b) D is monically diagonally degree dominant with diagonal degrees γ_i, $i \in \underline{m}$.

(c) D is r.c.r. with row powers r_i, column powers k_j and $D_h = I$, ***for all*** *pairs of m-tuples of nonnegative integers r_i and k_j such that $r_1 \geq r_2 \geq \cdots \geq r_m$, and $k_1 \geq k_2 \geq \cdots \geq k_m$, and $\gamma_i = r_i + k_i$ for all $i \in \underline{m}$.*

Proof. We show the chain of implications (a) ⇒ (b) ⇒ (c) ⇒ (a). The first implication is straightforward. For the second one, observe that here: (i) for all $i = j$, d_{ii} is monic with $\delta_{ii}[D] = \gamma_i = r_i + k_i$, (ii) for all $i > j$, $\delta_{ij}[D] < \gamma_i = r_i + k_i \leq r_i + k_j$, and (iii) for all $i < j$, $\delta_{ij}[D] < \gamma_j = r_j + k_j \leq r_i + k_j$. Hence (c) follows. The last implication follows by setting successively in (c) all r_i equal to zero, and then all k_j equal to zero. ■

Comment 5. (i) Case (a) was discovered in (Zagalak and Kučera, 1985, Lem. 3) as being essential in the proof of Fact 1, and is used in (Kučera and Zagalak, 1999, Sec. 3).

(ii) In Statement (c) it should be stressed that "**for all** pairs" is essential, as it implies that D must be r.c.r. for many possibilities of row and column powers, giving flexibility in applications, e.g., if $(\gamma_1, \gamma_2) = (5, 3)$, then D has to be r.c.r. with $D_h = I$ for twelve pairs of two-tuples of row and column powers starting with $(0, 0)$ $(5, 3)$, $(1, 0)$ $(4, 3)$, $(1, 1)$ $(4, 2)$, ... and ending with ..., $(4, 2)$ $(1, 1)$, $(4, 3)$ $(1, 0)$, $(5, 3)$ $(0, 0)$. Moreover, if in (c) "**for all** pairs" is replaced by "**for one** pair", then (c) is **not** equivalent to (a), as can be readily seen from the following example:

$$D(s) = \begin{bmatrix} s^6 & s^3 & s^3 \\ 0 & s^4 & s^2 \\ s^3 & s^2 & s^2 \end{bmatrix}. \tag{15}$$

Here D is r.c.r. with $D_h = I$ and row powers $(3, 2, 1)$ and column powers $(3, 2, 1)$, but neither c.r., nor r.r.

4. Proper Compensators

Proper compensators are obtained from (COMP) by the appropriate degree control of the polynomial matrix data and solutions. An important observation about the solutions is: if $C := X_l^{-1} Y_l$ exists and is proper, then, without loss of generality, X_l is r.r.

Theorem 1. (Callier and Desoer, 1982, p. 187) *Consider $P(s) = N_r(s) D_r^{-1}(s) \in \mathbb{R}_{po}(s)^{p \times m}$, where the pair $(N_r, D_r) \in Mat(\mathbb{R}[s])$ is right-coprime and D_r is c.r. with column degrees k_j ($j \in \underline{m}$) and highest column degree coefficient matrix D_{rh}. Let $D_k(s) \in \mathbb{R}[s]^{m \times m}$ be nonsingular and consider (COMP), i.e. $X_l D_r + Y_l N_r = D_k$.*

Then (COMP) has a polynomial matrix solution (X_l, Y_l) such that

(i) X_l is r.r. with row degrees r_i ($i \in \underline{m}$) and highest row degree coefficient matrix X_{lh}, and

(ii) $C(s) := X_l^{-1} Y_l \in \mathbb{R}_p(s)^{m \times p}$,

if and only if

(a) D_k is r.c.r. with row powers r_i ($i \in \underline{m}$) and column powers k_j ($j \in \underline{m}$), and

(b) for the given D_k, (X_l, Y_l) is a polynomial matrix solution of (COMP) such that the row degrees of Y_l satisfy $\delta_{ri}[Y_l] \leq r_i$ for all $i \in \underline{m}$.

Moreover, under these conditions, there holds $D_{kh} = X_{lh} D_{rh}$.

Comment 6. Theorem 1 is closely related to (Rosenbrock and Hayton, 1978, Cor. 1 of Thm. 5) and (Emre, 1980, Thm. 3.1). Extending the analysis in (Callier and Desoer, 1982, p. 183), it can be shown that, under the assumptions and conditions of Theorem 1, the feedback system

$S(P,C)$ of Fig. 1 has a polynomial matrix system description, e.g., (Callier and Desoer, 1982),

$$S(P,C): \begin{cases} D_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\xi(t) \\ = \begin{bmatrix} Y_l\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & X_l\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} D_r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ N_r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}\xi(t) \\ \qquad + \begin{bmatrix} 0 & -I_m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}. \end{cases} \tag{16}$$

The system $S(P,C)$ is well formed (Callier and Desoer, 1982, Sec. 3.3) and has its zero-input dynamics fixed *a priori* by the data D_r, N_r, and D_k; its zero-state dynamics are ultimately fixed by a solution X_l, Y_l of (COMP). Moreover, when all degree control parameters are positive, it is possible to get (A,B,C,D), a Fuhrmann-inspired state-space realization (Fuhrmann, 1976, Sec. VI), where the plant and the characteristic matrix D_k fix A and C, and the compensator fixes B and D. The order n of the system $S(P,C)$ equals the number of independent initial conditions $\xi_j(0_-)^{(k)}$, $j \in \underline{m}$, $k = \{0,1,\dots\}$ needed to determine a solution $\xi(t)$ of the homogeneous equation $D_k(\frac{\mathrm{d}}{\mathrm{d}t})\xi(t) = 0$ obtained in (16) by putting $u_1 = 0$, $u_2 = 0$, $t \geq 0$.

In view of Theorem 1 and earlier comments, the important existence question is: Assuming that Condition (a) holds, when does (b) hold? The answer is: by choosing the row powers r_i of the closed-loop characteristic denominator D_k sufficiently large. Indeed, one has the following result, whose proof is given for the reader's convenience.

Theorem 2 (Particular proper compensator). (Callier and Desoer, 1982, p. 190) *Let $P(s) \in \mathbb{R}_{po}(s)^{p\times m}$ have coprime right and left polynomial matrix fractions reading*

$$P(s) = N_r(s)D_r(s)^{-1} = D_l(s)^{-1}N_l(s), \tag{17}$$

with D_r c.r. with column degrees k_j ($j \in \underline{m}$) and D_l r.r. with row degrees μ_i ($i \in \underline{p}$). Let $\mu := \max_{i\in\underline{p}} \mu_i$, i.e., μ is the greatest observability index of the plant, cf. (Kailath, 1980, p. 431). Let

(a) $D_k \in \mathbb{R}[s]^{m\times m}$ be r.c.r. with row powers r_i and column powers k_j, and

(b) $r_i \geq \mu - 1$ for all $i \in \underline{m}$.

Then for the given data D_k, N_r and D_r, (COMP) has a polynomial matrix solution (X_l, Y_l) such that $\delta_{\mathrm{r}i}[Y_l] \leq r_i$ for all $i \in \underline{m}$. Hence (by Theorem 1) X_l is r.r. with row degrees r_i and $C(s) = X_l(s)^{-1}Y_l(s) \in \mathbb{R}_p(s)^{p\times m}$.

Proof. All polynomial matrix solutions of (COMP) are given by the pairs (X_l, Y_l) given by (8), where $N_k \in \mathrm{Mat}(\mathbb{R}[s])$ is a free parameter. In $Y_{lp} = -N_kD_l + Y_l$, choose $-N_k$ and Y_l to be the quotient and remainder of the division of Y_{lp} on the right by D_l. Then $Y_lD_l^{-1}$ is strictly proper such that $\delta_{\mathrm{c}j}[Y_l] < \delta_{\mathrm{c}j}[D_l]$ for all $j \in \underline{p}$. Hence for all $i \in \underline{m}$ $\delta_{\mathrm{r}i}[Y_l] \leq \max_{i\in\underline{m}} \delta_{\mathrm{r}i}[Y_l] = \max_{j\in\underline{p}} \delta_{\mathrm{c}j}[Y_l] < \max_{j\in\underline{p}} \delta_{\mathrm{c}j}[D_l] = \max_{i\in\underline{p}} \delta_{\mathrm{r}i}[D_l] := \mu$. Thus for all $i \in \underline{m}$ one gets $\delta_{\mathrm{r}i}[Y_l] \leq \mu - 1 \leq r_i$. ■

Comment 7. The idea of the proof of Theorem 2 is used in the proof of (Rosenbrock and Hayton, 1978, Thm. 6). However, one does not need to resort to the division of Y_{lp} on the right by D_l to obtain an existence result; see, e.g., (Emre, 1980, Thm. 4.1; Kraffer and Zagalak, 2002, Lem. 4.2; Antoniou and Vardulakis, 2005) for another method.

The following result constitutes, together with Theorem 2, a generalization of Theorem 2 in (Kučera and Zagalak, 1999) (modulo dualization and the fact that row-column-reducedness of D_k is used instead of the more restricted assumption that D_k is simultaneously row- and column-reduced, see Fact 2). It forms a parametrized class of solutions to (COMP) leading to a proper compensator. Its simple proof here is based on Theorem 1.

Theorem 3 (Parametrization of proper compensators). *Let the assumptions of Theorem 1 hold with $P(s)$ having coprime right and left polynomial matrix fractions as in (17), and let D_k satisfy Condition (a) of Theorem 1 with row powers r_i ($i \in \underline{m}$). Moreover, assume that a particular solution (X_{lo}, Y_{lo}) of (COMP) exists such that X_{lo} is r.r. with row degrees r_i ($i \in \underline{m}$), and $C_o := X_{lo}^{-1}Y_{lo} \in \mathbb{R}_p(s)^{m\times p}$.*

Then all *polynomial matrix solutions (X_l, Y_l) of (COMP) such that X_l is r.r. with row degrees r_i ($i \in \underline{m}$), and $C := X_l^{-1}Y_l \in \mathbb{R}_p(s)^{m\times p}$ are given by*

$$X_l = X_{lo} - N_kN_l, \quad Y_l = Y_{lo} + N_kD_l, \tag{18}$$

where $N_k \in \mathbb{R}[s]^{m\times p}$ has the property that for all $i \in \underline{m}$ and for all $j \in \underline{p}$,

$$\delta_{ij}[N_k] \leq r_i - \mu_j = r_i - \delta_{\mathrm{r}j}[D_l]. \tag{19}$$

Proof. Define $\Delta_\mu(s) := \mathrm{diag}\,[s^{\mu_i}]_{i=1}^p$ and $\Delta_r(s) := \mathrm{diag}\,[s^{r_i}]_{i=1}^m$. Note that by Theorem 1, $\Delta_r^{-1}Y_{lo}$ is proper. Observe also that condition (b) of Theorem 1 is equivalent to requiring that $\Delta_r^{-1}Y_l$ be proper. Finally, as D_l

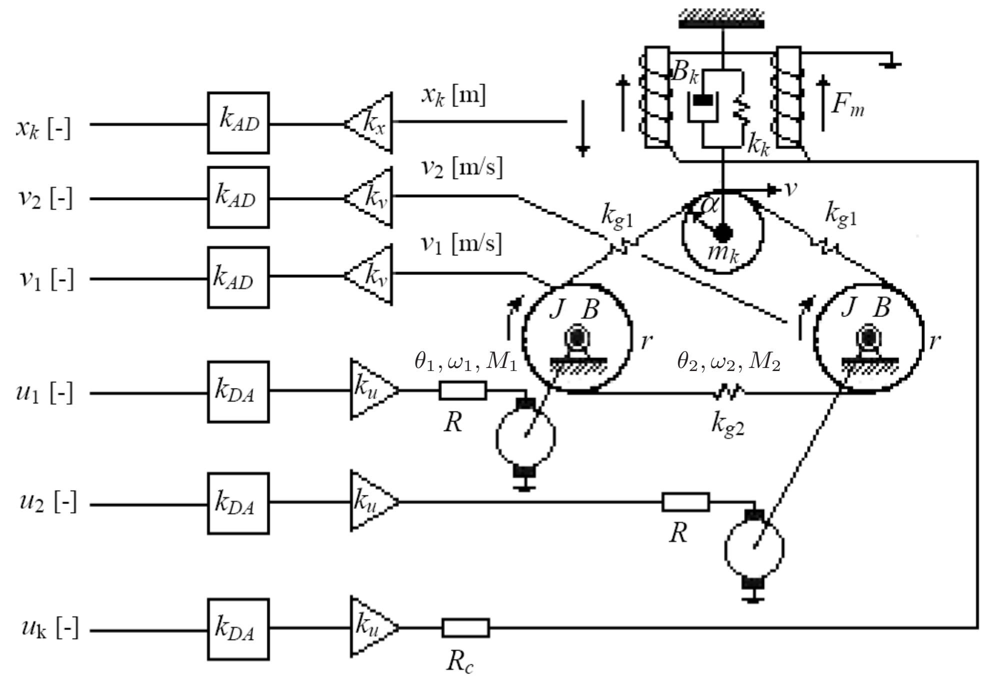

Fig. 2. Laboratory flexible belt coupled drive system.

is r.r. with row-degrees μ_i, one gets by Comm. 2 that $D_l = \Delta_\mu D_{l-}$, where D_{l-} is biproper. Hence upon substituting this in the second equation of (18) premultiplied by Δ_r^{-1}, one gets $\Delta_r^{-1} Y_l = \Delta_r^{-1} Y_{lo} + \Delta_r^{-1} N_k \Delta_\mu D_{l-}$, where $\Delta_r^{-1} Y_{lo}$ is proper and D_{l-} is biproper. Thus $\Delta_r^{-1} Y_l$ is proper iff $\Delta_r^{-1} N_k \Delta_\mu$ is proper. The result follows from this and Theorem 1. ■

Comment 8. The parametrization in (18) is only relatively new as it is contained in earlier affine parametrizations generated by Emre (1980, Rem. 3.4) and Callier and Desoer (1982, Comm. 95, pp. 193–195), i.e., solving algebraic linear systems with fewer equations than unknowns: the solution contains free parameters which can be adapted to those of Theorem 3. This is also the case for (Antoniou and Vardulakis, 2005, Thm. 8) and in a similar fashion for (Kraffer and Zagalak, 2002), which handles essentially the case $r_i = \mu - 1$ for all $i \in \underline{m}$.

5. Control Laboratory Tracking Device Example

Our first example concerns a control laboratory device, viz. the flexible belt coupled drives system of Fig. 2. The system has three actuators, two motors and a tensioner, equipped with pulleys, which are centered at the vertices of an isosceles triangle of varying height. A continuous flexible belt is moved by these pulleys. The lower two pulleys (driven pulleys) have fixed centers and are driven by two identical voltage controlled electric DC servomotors, with common drive voltage $u_d = u_1 = u_2$. The upper pulley (jockey pulley) has a center that can be vertically displaced by a tensioner bar, controlled by a voltage driven electromagnet and containing a spring-dashpot system whose spring may be simplified to a linear spring. The actuators operate simultaneously to control the speed of the belt and its tension. The vertical displacement of the center of the jockey pulley and thus of the tensioner is a measure of the tension in the belt. It is assumed that the belt speeds at the driven pulleys are the circumferential speeds of these pulleys due to the servomotors, and that the belt speed is the average of the belt speeds at the driven pulleys. The outputs of interest are the belt speed and the vertical position of the tensioner. Assuming that the belt angle α is constant, a linear initial model (ProTyS, Inc., 2003; Hagadoorn and Readman, 2004) is given by

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) &= \mathcal{C}x(t), \end{aligned} \tag{20}$$

a minimal state-space realization specified by

$$\mathcal{A} = \begin{bmatrix} 0 & 0.0310 & -0.0310 & 0 & 0 \\ -44286 & -3.0786 & 0 & 34197 & 0 \\ 44286 & 0 & -3.0786 & -34197 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \\ 514.80 & 0 & 0 & -1128.4 & -66.667 \end{bmatrix}, \tag{21}$$

$$\mathcal{B} = \begin{bmatrix} 0 & 0 \\ 2076.4 & 0 \\ 2076.4 & 0 \\ 0 & 0 \\ 0 & -51.000 \end{bmatrix}, \tag{22}$$

$$\mathcal{C} = \begin{bmatrix} 0 & 0.0008060 & 0.0008060 & 0 & 0 \\ 0 & 0 & 0 & -66.000 & 0 \end{bmatrix}. \tag{23}$$

In the sequel we shall use the notation of Fig. 2 as much as possible. The states contained in x are successively: horizontal belt length change due to the electric drives $x_h = r(\theta_1 - \theta_2)$ [m], left motor angular velocity $\omega_1 = \mathrm{d}\theta_1/\mathrm{d}t$ [rad/s], right motor angular velocity $\omega_2 = \mathrm{d}\theta_2/\mathrm{d}t$ [rad/s], tensioner vertical position x_k [m] and tensioner vertical speed v_k [m/s]. The inputs in u are successively: common drive voltage u_d and electromagnet voltage u_k. The outputs in y are: belt speed $r(\omega_1 + \omega_2)/2$ and tensioner vertical position x_k. Both u and y are measured in computer units [–].

The dynamic responses of the currents of the drive circuits of the motors and electromagnet are much faster than those of the mechanical variables, whence the inductances of the former circuits are neglected. This gives for the motor torques $M_i = \beta u_d - \beta_e \omega_i \ \ (i = 1, 2)$, where β is the drive voltage constant and β_e is a constant due to the back electromotive force (Messner and Tilbury, 1999). Moreover, for the vertical force applied by the electromagnet $F_m = \beta_k u_k$, where β_k is the electromagnet voltage constant. An important parameter is the oblique belt length change, which, assuming that the belt position at the jockey pulley is $r(\theta_1 + \theta_2)/2$, gives for the left case $r[(\theta_1+\theta_2)/2-\theta_1]+(\cos\alpha)x_k = -(1/2)x_h+(\cos\alpha)x_k$, and the same result for the right case. The most important state differential equations are the second, third and last one (for the notation see above and Fig. 2). They concern: (i) the balances of the torques applied to the left and right driven pulleys, namely,

$$J\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = -(B+\beta_e)\omega_1 - \Big(\frac{1}{2}k_{g1} + k_{g2}\Big) r x_h + k_{g1} r(\cos\alpha)x_k + \beta u_d,$$

$$J\frac{\mathrm{d}\omega_2}{\mathrm{d}t} = -(B+\beta_e)\omega_2 + \Big(\frac{1}{2}k_{g1} + k_{g2}\Big) r x_h - k_{g1} r(\cos\alpha)x_k + \beta u_d,$$

and (ii) the balance of the vertical forces applied to the tensioner, namely,

$$m_k\frac{\mathrm{d}v_k}{\mathrm{d}t} = -B_k v_k - (k_k + 2k_{g1}\cos^2\alpha)x_k + k_{g1}(\cos\alpha)x_h - \beta_k u_k,$$

where k_{g1} and k_{g2} are belt stiffness constants.

Our task is to design a proper compensator such that the closed-loop unity feedback system can track reference signals. These concern here the outputs, viz., belt speed and tensioner vertical position of the model given above represented by a strictly proper plant $\mathcal{P}(s) \in \mathbb{R}_{po}(s)^{2\times 2}$, i.e.,

$$\mathcal{P}(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}. \tag{24}$$

Design specifications include the capability of responding to setpoint values that may change from time to time (step changes). In absolute value physical capabilities and limitations on actuators require the drive input and tensioner voltages to stay below 0.8 [–] and 0.4 [–], respectively, for the belt speed and tensioner position below 0.25 [–] and 0.35 [–], respectively.

A conventional solution to the task relies on a state space approach, viz., linear quadratic optimal control (Athans and Falb, 1966), involving a state-variable feedback control law and an asymptotic estimator for providing an estimate of the state-variables not present in the output, and where, moreover, preliminary use of integral control is needed because our plant has no integral action of its own.

Our solution to the task relies on polynomial equations and is inspired by the conventional approach, but achieved by the design of a compensator, which is directly obtained; the estimator and integral control occur implicitly and are not done separately. As in state space design, one uses constant, positive weights for tuning. An important advantage of the present technique is that the arbitrariness is reduced to fewer degrees of freedom than those required to select the weights in a higher-order state space system.

Near physical symmetry with respect to the vertical axis through the tensioner bar in Fig. 2 reveals that a simplified model can be obtained from the initial model (20)–(24). First, we replace the state vector $(x_h, \omega_1, \omega_2, x_k, v_k)^{\mathrm{T}}$ by $(x_h, (\omega_1 + \omega_2)/2, (\omega_1 - \omega_2)/2, x_k, v_k)^{\mathrm{T}}$. During tests it is then observed that the states $x_h = r(\theta_1 - \theta_2)$ and $(\omega_1 - \omega_2)/2$ remain small within the range of design specifications, whence they may be set to zero and removed. Thus we get a final simplified model in terms of a minimal state-space realization (A, B, C) specified by

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \left[\begin{array}{ccc|cc} -3.0786 & 0 & 0 & 2076.4 & 0 \\ 0 & 0 & 1.000 & 0 & 0 \\ 0 & -1128.4 & -66.667 & 0 & -51.000 \\ \hline 0.0008060 & 0 & 0 & 0 & 0 \\ 0 & -66.000 & 0 & 0 & 0 \end{array}\right]. \tag{25}$$

The states are here successively: average driven pulley angular velocity $(\omega_1 + \omega_2)/2$ [rad/s], tensioner vertical position x_k [m], and tensioner vertical speed v_k [m/s]. The inputs are: common drive input voltage u_d, and electromagnet voltage u_k, and the outputs are: average belt speed $r(\omega_1 + \omega_2)/2$ and tensioner vertical position x_k. Both inputs and outputs are in computer units [–].

The simplified model can be represented by the coprime right polynomial matrix fraction

$$P(s) = N_r(s)D_r(s)^{-1}, \tag{26}$$

where the polynomial matrices N_r and D_r are given by

$$\begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix} = \left[\begin{array}{cc} 1.6736 & 0 \\ 0 & 3366.0 \\ \hline s+3.0786 & 0 \\ 0 & s^2+66.667s+1128.4 \end{array}\right]. \tag{27}$$

For future reference, record that the column degrees of D_r are $(k_1, k_2) = (1, 2)$, the right and left fractions of P coincide (the transfer function is diagonal), and its observability index equals two, the largest row degree in the left denominator (Kailath, 1980).

We choose now an appropriate matrix D_k for the plant (26)–(27) to set up an appropriate compensator equation (COMP).

Guided by LQ-optimal control methods, we compute first the polynomial matrix

$$W(s) = \begin{bmatrix} 0.70711s+2.7459 & 0 \\ 0 & 2.2361s^2+172.49s+4206.7 \end{bmatrix}, \tag{28}$$

whose zeros are exclusively in the open left half-plane. The matrix W is obtained from spectral factorization given by

$$W^{\mathrm{T}}(-s)W(s) = \begin{bmatrix} N_r^{\mathrm{T}}(-s) & D_r^{\mathrm{T}}(-s) \end{bmatrix} S \begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix}, \tag{29}$$

where S is a positive-definite matrix, a design parameter weighting the importance of inputs and outputs relative to the partial states of $N_r D_r^{-1}$. The weight S is selected as

$$S = \mathrm{diag}(1, 1, 0.5, 5). \tag{30}$$

The spectral factor W is r.c.r. with row and column powers given by $(r_1, r_2) = (0, 0)$ and $(k_1, k_2) = (1, 2)$. Hence, as was to be expected, by Theorem 1 with $D_k = W$ only a constant compensator can be obtained from the equation (COMP), and we have to find a matrix D_k that is r.c.r. with identical column powers and strictly larger row powers for obtaining a dynamical compensator that (i) contains a full rank integrator, capable of tracking input reference signals, and (ii) provides for "good observer dynamics."

Our final choice is the polynomial matrix

$$D_k(s) = \begin{bmatrix} d_{11}^k(s) & 0 \\ 0 & d_{22}^k(s) \end{bmatrix}, \tag{31}$$

where $d_{11}^k(s) = s^2 + 7.8832s + 15.533$ and $d_{22}^k(s) = s^4+95.138s^3+3325.8s^2+38183s+105350$. It is obtained by scaling the diagonal elements to monic polynomials of the diagonal matrix QW, where Q is a "good augmented dynamics" scaling factor selected as

$$Q(s) = \begin{bmatrix} s+4 & 0 \\ 0 & (s+4)(s+14) \end{bmatrix}, \tag{32}$$

and W is the spectral factor in (28).

The matrix D_k is trivially monically diagonally dominant with diagonal degrees $(2, 4)$, and r.c.r. with row and column powers $(r_1, r_2) = (1, 2)$ and $(k_1, k_2) = (1, 2)$.

For the data (D_r, N_r, D_k) in (27) and (31), by Theorems 1 and 2, proper compensators (X_l, Y_l) with $\delta_{ri}[X_l] = r_i$ $(i = 1, 2)$ for $(r_1, r_2) = (1, 2)$ exist. A particular proper compensator is given by

$$C(s) = \begin{bmatrix} s+4.8047 & 0 \\ 0 & s^2+28.471s+299.30 \end{bmatrix}^{-1} \times \begin{bmatrix} 0.44302 & 0 \\ 0 & -4.1287s-69.037 \end{bmatrix}. \tag{33}$$

This compensator, obtained by the method in (Kraffer and Zagalak, 2002, Lem. 4.2), is not unique. Non-uniqueness is a generic property, not valid given our data should the row powers of D_k equal the lower bound $\mu - 1 = 1$ given by Theorem 2. As one of the row powers D_k, given above, exceeds the lower bound, extra degrees of freedom are generated that are available to accommodate the open-loop system consisting of the compensator C and the plant P such that $[I + PC]^{-1}(0) = 0$, i.e., as a system which is able to track steps under unity feedback.

For the data (D_r, N_r, D_k) in (27) and (31), the affine set of all proper compensators, with $\delta_{ri}[X_l] = r_i$ $(i = 1, 2)$, may be centered on the compensator given by (33). Since here $(r_1, r_2) = (1, 2)$ and $(\mu_1, \mu_2) = (1, 2)$, a parametrization of this set is given by Theorem 3 by

$$X_l(s) = \begin{bmatrix} s+4.8047 & 0 \\ 0 & s^2+28.471s+299.30 \end{bmatrix} - N_k(s)\begin{bmatrix} 1.6736 & 0 \\ 0 & 3366.0 \end{bmatrix}, \quad (34)$$

$$Y_l(s) = \begin{bmatrix} 0.44302 & 0 \\ 0 & -4.1287s-69.037 \end{bmatrix} + N_k(s)\begin{bmatrix} s+3.0786 & 0 \\ 0 & s^2+66.667s+1128.4 \end{bmatrix}, \quad (35)$$

where the free parameter N_k has the form

$$N_k(s) = \begin{bmatrix} n_1 & 0 \\ m_{21}s+n_{21} & n_2 \end{bmatrix}, \quad (36)$$

all coefficients being free.

In order to obtain a simple, i.e., diagonal compensator, we set $m_{21} = n_{21} = 0$ and satisfy the tracking objective by finding a diagonal constant value of N_k such that $[I + PC]^{-1}(0) = 0$, here $X_l(0) = 0$. As a consequence, the value of N_k is unique,

$$N_k = \begin{bmatrix} 2.8709 & 0 \\ 0 & 0.08892 \end{bmatrix}, \quad (37)$$

and specifies the compensator given by

$$X_l(s) = \begin{bmatrix} s & 0 \\ 0 & s^2+28.471s \end{bmatrix}, \quad (38)$$

$$Y_l(s) = \begin{bmatrix} 2.8709s+9.2811 & 0 \\ 0 & 0.08892s^2+1.7993s+31.299 \end{bmatrix}. \quad (39)$$

As was to be expected by the internal model principle, the compensator contains a full rank integrator, and, except for its pole at zero, is stable. Moreover, it is minimum phase. This is considered good for practical applications.

Having designed our tracking compensator using the simplified model in (26)–(27), we return to the initial model (20)–(24) to verify the corresponding closed-loop design performance.

The results are shown in Fig. 3. Verification using the actual physical system is shown in Fig. 4. In both figures the upper three curves concern tensioner variables, and the lower three concern belt speed variables. The nonzero displacement of the tensioner at the start of the run in Fig. 4 is due to (unmodeled) static friction. The results are satisfactory.

The steps of the method to design a tracking compensator were as follows: (a) select a convenient representation of the plant-sensor-actuator, in the form of a coprime right polynomial matrix fraction, (b) develop the right hand-side matrix for (COMP) that corresponds to satisfactory dynamic response, based on a suitable balance of weighted spectral factorization and additionally selected poles, and (c) customize the solution of (COMP) such that the open-loop system satisfies the internal model principle, that is, endow the system with the ability to generate, and hence also to track, a given class of signals.

6. Example for Comparison with Other Methods

For educational purposes and for straightforward numerical evidence, we introduce a simple example, which is amenable to hand calculations. Consider the following unstable plant with a strictly proper transfer matrix:

$$P(s) = \begin{bmatrix} \dfrac{s+1}{s(s-2)} & 0 \\ \dfrac{1}{s(s-1)} & \dfrac{1}{s-1} \end{bmatrix}, \quad (40)$$

with coprime right and left polynomial matrix fractions as in (17) respectively given by

$$D_r(s) = \begin{bmatrix} s(s-2) & 0 \\ 1 & s-1 \end{bmatrix}, \quad N_r(s) = \begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix}, \quad (41)$$

and

$$D_l(s) = \begin{bmatrix} 0 & s(s-1) \\ 2-s & s-1 \end{bmatrix}, \quad N_l(s) = \begin{bmatrix} 1 & s \\ -1 & 1 \end{bmatrix}. \quad (42)$$

D_r is c.r. with $D_h = I$ and column degrees $k_1 = 2$ and $k_2 = 1$. D_l is r.r. with row degrees $\mu_1 = 2$ and $\mu_2 = 1$. Therefore, $\mu := 2$. Hence by Theorems 1 and 2 an appropriate choice of the row and column powers of a r.c.r. characteristic matrix D_k of (COMP) is $k_1 = 2$, $k_2 = 1$, and $r_1 = 1$, $r_2 = 1$. Having Fact 2 and Comm. 6 in mind, we choose then

$$D_k(s) = \begin{bmatrix} (s+4)(s^2+4s+8) & 0 \\ 0 & (s+6)(s+8) \end{bmatrix}. \quad (43)$$

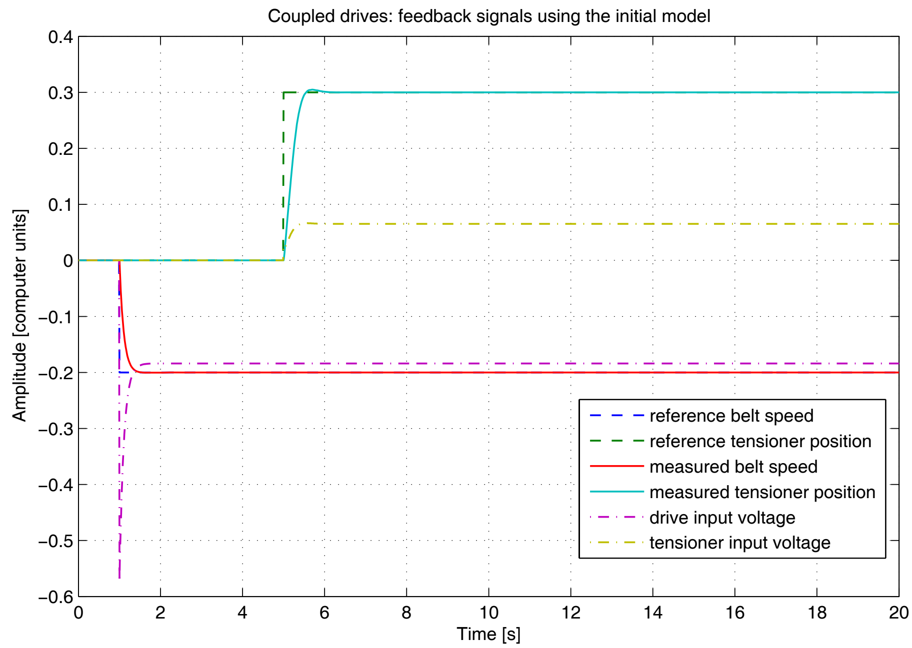

Fig. 3. Initial model verification.

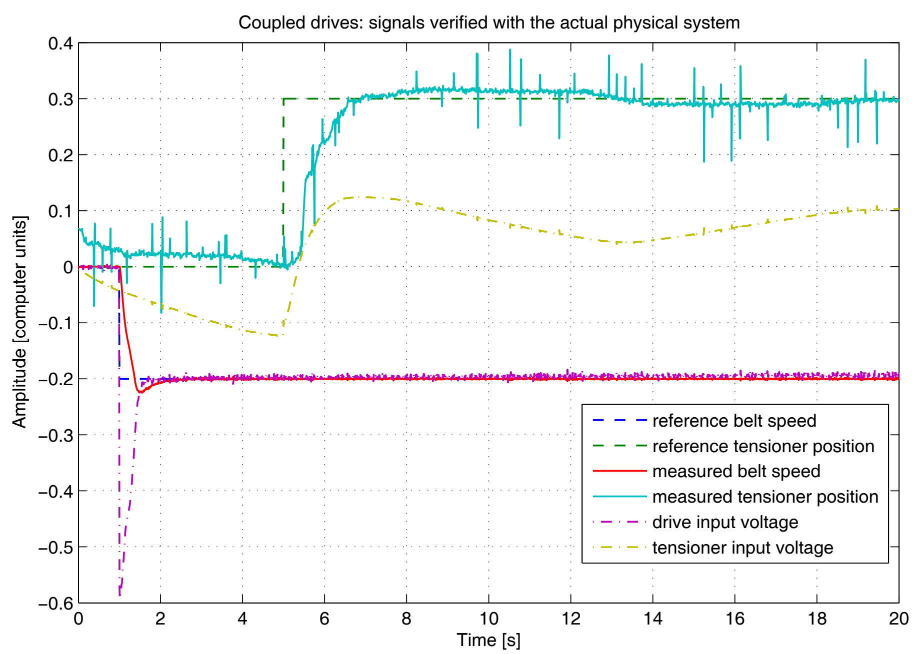

Fig. 4. Physical system verification.

It is then possible by Section 2 to consider the polynomial matrix solutions of (COMP) as in (8) with

$$3X_{lp}(s) = \begin{bmatrix} 0 & s^3+8s^2+24s+32 \\ 0 & -s^2-14s-48 \end{bmatrix}, \quad (44)$$

and

$$3Y_{lp}(s) = \begin{bmatrix} s^3+8s^2+24s+32 & -s^4-7s^3-16s^2-8s+32 \\ -s^2-14s-48 & s^3+16s^2+76s+96 \end{bmatrix}. \quad (45)$$

A proper compensator generating solution is obtained as in the proof of Theorem 2 by dividing Y_{lp} on the right by D_l, giving

$$3Y_{lo}(s) = \begin{bmatrix} 120 & 12s-12 \\ -80 & 77s+112 \end{bmatrix}, \quad 3N_{ko}(s) = \begin{bmatrix} s^2+7s+14 & s^2+10s+44 \\ -s-16 & -s-16 \end{bmatrix}, \quad (46)$$

and

$$3X_{lo}(s) = \begin{bmatrix} 3s+30 & -12 \\ 0 & 3s-32 \end{bmatrix}. \quad (47)$$

Applying Theorem 3 one finds that the degree limitations on the entries of N_k in (18) result in

$$N_k(s) = \begin{bmatrix} 0 & n_{12} \\ 0 & n_{22} \end{bmatrix}, \quad (48)$$

where n_{12} and n_{22} are two real free parameters. Hence for the given plant and D_k all the solutions of Theorem 3 generating a proper compensator are given by

$$X_l(s) = \begin{bmatrix} s+(10+n_{12}) & -(4+n_{12}) \\ n_{22} & s-\left(\frac{32}{3}+n_{22}\right) \end{bmatrix}, \quad (49)$$

and

$$Y_l(s) = \begin{bmatrix} -n_{12}s+(40+2n_{12}) & (4+n_{12})s-(4+n_{12}) \\ -n_{22}s+\left(2n_{22}-\frac{80}{3}\right) & \left(\frac{77}{3}+n_{22}\right)s+\left(\frac{112}{3}-n_{22}\right) \end{bmatrix}. \quad (50)$$

Observe that for $n_{12} = -4$ and $3n_{22} = -32$, the second column of X_l is zero at zero, whence the value of $(I_p + PC)^{-1}$ is zero at zero, and the feedback system $S(P, C)$ of Fig. 1 will asymptotically track steps and reject disturbances of the same form at the output of the plant.

We compare now with the parametrization generated by Callier and Desoer (1982, Comm. 95, pp. 193–195). For the data given here, Theorem 1 asks to search the solutions (X_l, Y_l) of (COMP) of the form

$$X_l(s) = \begin{bmatrix} s+x_0^{11} & x_0^{12} \\ x_0^{21} & s+x_0^{22} \end{bmatrix}, \quad Y_l(s) = \begin{bmatrix} y_1^{11}s+y_0^{11} & y_1^{12}s+y_0^{12} \\ y_1^{21}s+y_0^{21} & y_1^{22}s+y_0^{22} \end{bmatrix}. \quad (51)$$

Observe now that our D_r has three characteristic values $s_1 = 0$, $s_2 = 1$, and $s_3 = 2$, with corresponding characteristic vectors $l_1^{\mathrm{T}} = [\ 1 \quad 1\ ]^{\mathrm{T}}$, $l_2^{\mathrm{T}} = [\ 0 \quad 1\ ]^{\mathrm{T}}$, and $l_3^{\mathrm{T}} = [\ 1 \quad -1\ ]^{\mathrm{T}}$. Moreover, $D_r(-1)$ is nonsingular. Now, as part of a solution of (COMP), Y_l is such that D_r must divide $D_k - Y_l N_r$ on the right, which is here equivalent to the interpolation conditions

$$D_k(s_i)l_i = Y_l(s_i)N_r(s_i)l_i \ \text{ for } \ i \in \underline{3}. \quad (52)$$

Hence the eight unknown coefficients of Y_l must satisfy six linear equations. Solving the latter allows us to parametrize six coefficients as an affine function of two free coefficients, viz. y_1^{12} and y_1^{22}. Finally, the matrix identity

$$X_l = (D_k - Y_l N_r)\, D_r^{-1} \quad (53)$$

at $s = -1$ gives four scalar linear equations, permitting to parametrize the four unknown coefficients of X_l as affine functions of y_1^{12} and y_1^{22}. There result successively

$$Y_l(s) = \begin{bmatrix} (4-y_1^{12})s+(32+2y_1^{12}) & y_1^{12}s-y_1^{12} \\ \left(\frac{77}{3}-y_1^{22})s-(78-2y_1^{22}\right) & y_1^{22}s+(63-y_1^{22}) \end{bmatrix} \quad (54)$$

and

$$X_l(s) = \begin{bmatrix} s+(6+y_1^{12}) & -y_1^{12} \\ \left(-\frac{77}{3}+y_1^{22}\right) & s+(15-y_1^{22}) \end{bmatrix}. \quad (55)$$

This parametrization agrees with (49)–(50) for $y_1^{12} = 4 + n_{12}$ and $3y_1^{22} = 77+3n_{22}$, and needs for the case at hand fewer numerical computations. For more on interpolation methods, see (Antsaklis and Gao, 1993).

Similar results follow by the Fuhrmann-realization inspired method of (Emre, 1980, Rem. 3.4). The idea is to relate (COMP) to the rational matrix equation

$$X_l + Y_l N_r D_r^{-1} = D_k D_r^{-1}, \quad (56)$$

which can be solved over polynomial matrices. The parametrization of Y_l is obtained by equating the strictly proper parts of $Y_l N_r D_r^{-1}$ and $D_k D_r^{-1}$, as described by

$$\begin{bmatrix} Y_0 & Y_1 & \cdots & Y_{\mu-1} \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\mu-1} \end{bmatrix} = \bar{C}, \qquad (57)$$

where (A, B, C) is the realization of $N_r D_r^{-1}$ while $(A, B, \bar{C})$ is the realization of the strictly proper part of $D_k D_r^{-1}$ and

$$Y_l(s) = Y_0 + Y_1 s + \cdots + Y_{\mu-1} s^{\mu-1}. \qquad (58)$$

Recall that $\mu := 2$. Hence the eight unknown coefficients of the (2×2)-matrices Y_0 and Y_1 must satisfy six linear equations. Solving the latter allows us to parametrize six coefficients as an affine function of two free coefficients, viz. y_1^{12} and y_1^{22} of Y_1. We successively get

$$\begin{bmatrix} A & B \\ C & 0 \\ \bar{C} & 0 \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 44 & 32 & 0 & 0 & 0 \\ -1 & -15 & 63 & 0 & 0 \end{array}\right], \qquad (59)$$

and

$$Y_l(s) = \begin{bmatrix} (4 - y_1^{12})s + (32 + 2y_1^{12}) & y_1^{12}s - y_1^{12} \\ \left(\frac{77}{3} - y_1^{22}\right)s - \left(78 - 2y_1^{22}\right) & y_1^{22}s + (63 - y_1^{22}) \end{bmatrix}. \qquad (60)$$

This parametrization is identical to (54). Note here that to every Y_l there corresponds a unique X_l, i.e., (55). For more on realization methods, see (Emre, 1980).

Now, the parametrization (54)–(55) can also be obtained by Kraffer and Zagalak (2002, Algorithm, Sec. 5.3), and Antoniou and Vardulakis (2005, Algorithm, p. 21). However, more interesting is the following bilateral degree control method, or "two-sided method" for short, which (up to our knowledge) is new.

Choose, according to the row and column powers of D_k, the left and right polynomial basis matrices $S_l(s)$ and $S_r(s)$ given by

$$S_l(s) := \begin{bmatrix} 1 & s & 0 & 0 \\ 0 & 0 & 1 & s \end{bmatrix}, \quad S_r(s) := \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix}, \qquad (61)$$

and define

$$\Omega(s) := \begin{bmatrix} X_l(s) & Y_l(s) \end{bmatrix}, \qquad (62a)$$

$$F(s) := \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}. \qquad (62b)$$

Here $F(\cdot)$ is c.r. with column-degrees $k_1 = 2$ and $k_2 = 1$, and $\Omega(\cdot)$ is r.r. with row-degrees $r_1 = r_2 = 1$, and by (62), Eqn. (COMP) becomes

$$\Omega(s) F(s) = D_k(s). \qquad (63)$$

Now $\Omega(\cdot)$ and $F(\cdot)$ have a unique matrix representation with

$$X_l(s) := \begin{bmatrix} x_1^{11}s + x_0^{11} & x_1^{12}s + x_0^{12} \\ x_1^{21}s + x_0^{21} & x_1^{22}s + x_0^{22} \end{bmatrix},$$

$$Y_l(s) := \begin{bmatrix} y_1^{11}s + y_0^{11} & y_1^{12}s + y_0^{12} \\ y_1^{21}s + y_0^{21} & y_1^{22}s + y_0^{22} \end{bmatrix}, \qquad (64)$$

$$\Omega(s) = S_l(s)\Omega, \qquad (65)$$

where

$$\Omega = (\omega_{ij}) = \begin{bmatrix} x_0^{11} & x_0^{12} & y_0^{11} & y_0^{12} \\ x_1^{11} & x_1^{12} & y_1^{11} & y_1^{12} \\ x_0^{21} & x_0^{22} & y_0^{21} & y_0^{22} \\ x_1^{21} & x_1^{22} & y_1^{21} & y_1^{22} \end{bmatrix}.$$

Moreover,

$$F(s) = F S_r(s), \qquad (66)$$

where

$$F = \begin{bmatrix} 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix representation of $D_k(\cdot)$ given by

$$D_k(s) = S_l(s) D S_r(s), \qquad (67)$$

where

$$D = \begin{bmatrix} 32 & 24 - n_1 & 8 - n_2 & 0 & -n_3 \\ n_1 & n_2 & 1 & n_3 & 0 \\ 0 & -n_4 & -n_5 & 48 & 14 - n_6 \\ n_4 & n_5 & 0 & n_6 & 1 \end{bmatrix},$$

is, however, nonunique with 6 real free parameters n_i, $i \in \underline{6}$, because the (2×2)-zero polynomial matrix has the two-sided representation

$$0 = S_l(s) N S_r(s), \tag{68}$$

where

$$N = \begin{bmatrix} 0 & -n_1 & -n_2 & 0 & -n_3 \\ n_1 & n_2 & 0 & n_3 & 0 \\ 0 & -n_4 & -n_5 & 0 & -n_6 \\ n_4 & n_5 & 0 & n_6 & 0 \end{bmatrix}.$$

Finally, by (63), and (65)–(67), Eqn. (COMP) reduces to the system of linear equations

$$\Omega F = D, \tag{69}$$

which for given F and D must be solved for Ω, where, by Theorem A of Appendix, F has a full row rank, i.e., $\operatorname{rank} F = 4$.

Now the data of the problem meet the assumptions of Theorem 2, whence (for some values of the parameters) System (69) must have a solution. To find the latter, one resorts to elementary column operations on (69), whereby F is reduced to its column echelon form, revealing zero column(s). That is, there exists a nonsingular matrix T such that $FT = [\ F_1 \quad 0\ ]$, where F_1 is lower triangular nonsingular. Performing the column operations on the compound matrix $[\ F^{\mathrm{T}} \quad D^{\mathrm{T}}\ ]^{\mathrm{T}}$, we get

$$\begin{bmatrix} F \\ D \end{bmatrix} T = \begin{bmatrix} F_1 & 0 \\ D_1 & D_2 \end{bmatrix}. \tag{70}$$

A necessary and sufficient condition for the solvability of Equation (69) is $D_2 = 0$, giving a linear system of equations in the parameters n_i, revealing hereby the final free parameters.

For the problem at hand we find

$$\begin{bmatrix} F_1 & 0 \\ D_1 & D_2 \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 8-n_2 & -n_3 & 32+n_3 & -n_3 & 8-n_1-2n_2-n_3 \\ 1 & 0 & n_1 & n_3 & 2-n_1+n_2+n_3 \\ -n_5 & 14-n_6 & -14+n_6 & 62-n_6 & 76-n_4-2n_5-2n_6 \\ 0 & 1 & n_4-1 & 1+n_6 & 2-n_4+n_5+n_6 \end{array}\right], \tag{71}$$

where upon setting $D_2 = 0$,

$$n_1 = 4, \quad n_2 = 2 - n_3, \quad n_4 = \frac{80}{3}, \quad n_5 = \frac{74}{3} - n_6, \tag{72}$$

that is, the six parameters n_i have been reduced to two, namely, n_3 and n_6.

The substitution of the relations (72) in D_1 and omitting the zero column(s), followed by subtracting the fourth column from the third column (which reduces F_1 to the identity matrix), gives the bottom block

$$\Omega = \begin{bmatrix} 6+n_3 & -n_3 & 32+2n_3 & -n_3 \\ 1 & 0 & 4-n_3 & n_3 \\ -\frac{74}{3}+n_6 & 14-n_6 & -76+2n_6 & 62-n_6 \\ 0 & 1 & \frac{74}{3}-n_6 & 1+n_6 \end{bmatrix}, \tag{73}$$

where Ω is the parametrized solution of (69). Finally by (62a), (64), and (65) the solution of (COMP) is found to be the (54)–(55) modulo $n_3 = y_1^{21}$ and $n_6 = y_1^{22} - 1$.

Comment 9. The two-sided method works well by using a linear system of equations whose dimension is smaller than those of (Antoniou and Vardulakis, 2005; Kraffer and Zagalak, 2002), at the small cost of introducing initially more parameters (due to the nonunique representation of the zero polynomial matrix), which afterwards are reduced to the final free parameters. The two-sided method gives also with fewer calculations the solutions of the examples in (Antoniou and Vardulakis, 2005; Kraffer and Zagalak, 2002). Moreover, it works for the general case that D_k is row-column-reduced: no special case needed as in the former papers.

7. Conclusion

Theorem 3 confirms that, in a polynomial matrix context with data- and parameter-degree control, it is possible to characterize all proper feedback compensators $C = X_l^{-1} Y_l$ whose denominator X_l is row-reduced with sufficiently large prescribed row degrees r_i.

This result can be shown to be consistent with a similar result using matrix fractions over $\mathbb{R}\mathcal{H}_\infty$ in (Vidyasagar, 1985, Sec. 5.2). Hence it should be useful for appropriate plant feedback stabilization in an optimization or tracking context, see also (Callier and Desoer, 1982, Sec. 7.3; Kučera and Zagalak, 1999). It is obvious that the results can be dualized for the case that the plant is given as a left polynomial matrix fraction.

The theoretical part was complemented by two examples to illustrate the theory for (i) the design of a tracking compensator for a control laboratory device (flexible belt coupled drives system), and (ii) comparison with other methods, one of which is new. These examples show that a systematic procedure for a good choice of the right-hand side characteristic matrix D_k of Equation (COMP) by feedback control considerations is in order, and thus a task for future research.

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Appendix

Theorem A. *Let the assumptions of Theorem 1 hold and let* $P(\cdot)$ *have a full generic row rank* p. *Consider the right coprime fraction* $P(\cdot) = N_r(\cdot)D_r(\cdot)^{-1}$ *with* $D_r(\cdot)$ *c.r. Let* $k_j,\ j \in \underline{m}$ *be the column degrees of* D_r. *Consider*

$$F(s) := \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}$$

and let

$$S_r(s) := \text{block diag}\left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_j} \end{bmatrix} \right\}_{j\in\underline{m}} .$$

Let F *be the* $(m+p) \times (\sum_{j=1}^m k_j + m)$ *real matrix defined by* $F(s) = FS_r(s)$.

Then F has full row rank, i.e., $\operatorname{rank} F = m + p$.

Proof. According to Antoniou and Vardulakis (2005, Cor. 4), Bitmead *et al.* (1978, Thm. 1), Kailath (1980, footnote, p. 413),

$$\operatorname{rank} F = m + p - \sum_{i:\mu_i<1} (1 - \mu_i), \tag{74}$$

where μ_i are the row-degrees of D_l in the left coprime fraction $P(\cdot) = D_l(\cdot)^{-1} N_l(\cdot)$ with $D_l(\cdot)$ r.r. Denote by "grank" the generic rank, i.e., over $\mathbb{R}(s)$, of a rational matrix. Then, because $D_l(\cdot)$ is nonsingular and $D_l(\cdot)P(\cdot) = N_l(\cdot)$, $\operatorname{grank} P(\cdot) = \operatorname{grank} N_l(\cdot) = p$, i.e., $N_l(\cdot)$ has a full generic row rank. This implies

$$\mu_i > 0, \quad \forall i \in \underline{p}. \tag{75}$$

Indeed, suppose $\exists\ i \in \underline{p}$ such that $\mu_i = 0$. Then with $P(\cdot)$ strictly proper, $\delta_{ri}[N_l] < \delta_{ri}[D_l] = \mu_i = 0$, whence the i-th row of $N_l(\cdot)$ is zero. This contradicts the full generic row rank of $N_l(\cdot)$, whence (75) holds. Finally, (74) and (75) imply $\operatorname{rank} F = m + p$. ■

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