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REALIZATION PROBLEM FOR A CLASS OF POSITIVE CONTINUOUS-TIME SYSTEMS WITH DELAYS

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The realization problem for a class of positive, continuous-time linear SISO systems with one delay is formulated and solved. Sufficient conditions for the existence of positive realizations of a given proper transfer function are established. A procedure for the computation of positive minimal realizations is presented and illustrated by an example.

Keywords: positive realization, continuous-time system, delay, existence, computation

1. Introduction

In positive systems, inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, or water and atmospheric pollution models. A variety of models revealing the behaviour of positive linear systems can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of the state of the art in positive systems theory is given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2002). Recent developments in positive systems theory and some new results are given in (Kaczorek, 2003). The realization problem of positive linear systems without time delays has been considered in many papers and books (Benvenuti and Farina, 2004; Farina and Rinaldi, 2000; Kaczorek, 2002).

An explicit solution of equations describing discrete-time systems with time delays was given in (Busłowicz, 1982). The realization problem for positive multivariable discrete-time systems with one time delay was formulated and solved in (Kaczorek and Busłowicz, 2004). Recently, reachability, controllability and minimum energy control of positive linear discrete-time systems with time delays were considered in (Busłowicz and Kaczorek, 2004; Xie and Wang, 2003).

The main purpose of this paper is to present a method of computing positive minimal realizations for a class of positive continuous-time systems with one delay. Sufficient conditions for the solvability of the realization problem will be established and a procedure for the computation of a minimal positive realization of a proper transfer function will be presented. To the best of the author's knowledge, the realization problem for positive continuous-time linear systems with delays has not been considered yet.

2. Problem Formulation

Consider a single-input single-output continuous-time linear system with one time delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + bu(t), \tag{1a}$$

$$y(t) = cx(t) + du(t), \tag{1b}$$

where $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}$, $y = y(t) \in \mathbb{R}$ are the state vector, input and output, respectively, $A_k \in \mathbb{R}^{n\times n}$, $k = 0, 1$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^{1\times n}$, $d \in \mathbb{R}$ and $h \in \mathbb{R}_+$ is a given delay. The initial conditions for (1a) are given as

$$x_0(t) \quad \text{for} \quad t \in [-h, 0]. \tag{2}$$

Let $\mathbb{R}_+^{n\times m}$ be the set of $n \times m$ real matrices with non-negative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

Definition 1. The system (1) is called (internally) positive if for every $x_0(t) \in \mathbb{R}_+^n$, $t \in [-h, 0]$ and all inputs $u(t) \in \mathbb{R}_+$, $t \geq 0$ we have $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+$ for $t \geq 0$.

Theorem 1. *The system (1) is positive if and only if A_0 is a Metzler matrix (all off-diagonal entries are non-negative) and*

$$A_1 \in \mathbb{R}_+^{n\times n}, \quad b \in \mathbb{R}_+^n, \quad c \in \mathbb{R}_+^{1\times n}, \quad d \in \mathbb{R}_+. \tag{3}$$

Proof. The solution $x(t)$ of (1a) for $t \in [0, h]$ is given by

$$x(t) = e^{A_0 t} x_0(0) + \int_0^t e^{A_0(t-\tau)} \big[A_1 x_0(t-h-\tau) + bu(\tau)\big] d\tau. \quad (4)$$

It is well known (Kaczorek, 2002) that $e^{A_0 t} \in \mathbb{R}_+^{n\times n}$, $t \geq 0$ if and only if A_0 is a Metzler matrix. From (4) it follows that if A_0 is a Metzler matrix, $A_1 \in \mathbb{R}_+^{n\times n}$, $b \in \mathbb{R}_+^n$, then $x(t) \in \mathbb{R}_+^n$ for every $x_0(t) \in \mathbb{R}_+^n$ and all $u(t) \in \mathbb{R}_+$ for $t \in [0, h]$.

From (1b) we have that if $c \in \mathbb{R}_+^{1\times n}$, $d \in \mathbb{R}_+$, $x(t) \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+$, then $y(t) \in \mathbb{R}_+$ for $t \in [0, h]$. The deliberations can be repeated for the successive intervals $[h, 2h]$, $[2h, 3h]$, and so on.

The necessity can be shown in much the same way as for positive continuous-time linear systems without delays (Kaczorek, 2002). ■

The transfer function of the system (1) is given by

$$T(s) = c[I_n s - A_0 - A_1 e^{-hs}]^{-1} b + d. \quad (5)$$

Definition 2. Matrices

$$A_0 \in M, \quad A_1 \in \mathbb{R}_+^{n\times n}, \quad b \in \mathbb{R}_+^n, \quad c \in \mathbb{R}_+^{1\times n}, \quad d \in \mathbb{R}_+, \quad (6)$$

where M stands for the set of Metzler matrices, are called a *positive realization* of a given proper transfer function $T(s)$ if they satisfy the equality (5). The realization (5) is called *minimal* if the dimension n of A_0 and A_1 is minimal among all realizations of $T(s)$.

The positive realization problem can be formulated as follows: Given a proper transfer function $T(s)$, find a positive minimal realization (6) of $T(s)$. Necessary conditions and sufficient conditions for the solvability of this problem will be established and a procedure for the computation of a positive minimal realization will be proposed.

3. Problem Solution

The transfer function (5) can be rewritten in the form

$$T(s) = \frac{c \operatorname{Adj} H(s) b}{\det H(s)} + d = \frac{n(s)}{d(s)} + d, \quad (7)$$

where $\operatorname{Adj} H(s)$ denotes the adjoint matrix for $H(s)$,

$$H(s) = [I_n s - A_0 - A_1 w], \quad w = e^{-hs},$$
$$n(s) = c \operatorname{Adj} H(s) b = b_{n-1}s^{n-1} + \cdots + b_1 s + b_0, \quad (8)$$
$$d(s) = \det H(s) = s^n + d_{n-1}s^{n-1} + \cdots + d_1 s + d_0,$$

and the coefficients $b_k = b_k(w)$ and $d_k = d_k(w)$, $k = 0, 1, \ldots, n-1$ are polynomials in $w = e^{-hs}$ with real coefficients.

From (7) we have

$$d = \lim_{s\to\infty} T(s) \quad (9)$$

since $\lim_{s\to\infty} H^{-1}(s) = 0$. The strictly proper part of $T(s)$ is given by

$$T_{sp}(s) = T(s) - d = \frac{n(s)}{d(s)}. \quad (10)$$

Therefore, the positive realization problem has been reduced to finding matrices

$$A_0 \in M, \quad A_1 \in \mathbb{R}_+^{n\times n}, \quad b \in \mathbb{R}_+^n, \quad c \in \mathbb{R}_+^{1\times n} \quad (11)$$

for the given strictly proper transfer function (10).

Let us assume that the given proper transfer function $T(s)$ has the form (7) with the denominator $d(s)$ with coefficients d_k, $k = 0, 1, \ldots, n-1$, which are first-degree polynomials in w, i.e.,

$$d(s) = s^n - (a_{2n-1}w + a_{2n-2})s^{n-1} - (a_{2n-3}w + a_{2n-4})s^{n-2} - \cdots - (a_3 w + a_2)s - (a_1 w + a_0), \quad (12)$$

and with the numerator $n(s)$ of the form (8) with coefficients

$$b_k = q_{k,n-1}w^{n-1} + q_{k,n-2}w^{n-2} + \cdots + q_{k1}w + q_{k0}, \quad k = 0, 1, \ldots, n-1. \quad (13)$$

Lemma 1. *The coefficient a_0 of (12) is equal to zero, i.e., $a_0 = 0$, if and only if*

$$\det A_0 = 0. \quad (14)$$

If $d(s)$ has the form (12) and $n \geq 2$, then

$$\det A_1 = 0. \quad (15)$$

Proof. Note that the substitution of $s = w = 0$ into $d(s)$ (defined by (8)) yields $\det[-A_0] = -a_0$. From (8) we have

$$\det[-A_0 - A_1 w] = w^n \det[-A_1] + \cdots + \det[-A_0]. \quad (16)$$

If $d(s)$ has the form (12) and $n \geq 2$, then from (16) it follows that (15) holds. ■

Lemma 2. *If the pair* (A_0, A_1) *has one of the following forms:*

$$A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-6} & 0 & \cdots & 1 & 0 & 0 \\ a_{2n-4} & 0 & \cdots & 0 & 1 & a_{2n-2} \end{bmatrix}, \tag{17a}$$

$$A_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & \cdots & 0 & 0 \\ a_3 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-5} & 0 & \cdots & 0 & 0 \\ a_{2n-3} & 0 & \cdots & 0 & a_{2n-1} \end{bmatrix}, \tag{17b}$$

$$\bar{A}_0 = A_0^T, \qquad \bar{A}_1 = A_1^T, \tag{17c}$$

$$\hat{A}_0 = PA_0P, \quad \hat{A}_1 = PA_1P, \tag{17d}$$

$$P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{17e}$$

$$\tilde{A}_0 = \hat{A}_0^T, \quad \tilde{A}_1 = \hat{A}_1^T, \tag{17f}$$

$$A_0' = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-6} & 0 & \cdots & 1 & 0 & a_{2n-4} \\ 0 & 0 & \cdots & 0 & 1 & a_{2n-2} \end{bmatrix}, \tag{17g}$$

$$A_1' = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & \cdots & 0 & 0 \\ a_3 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-5} & 0 & \cdots & 0 & a_{2n-3} \\ 0 & 0 & \cdots & 0 & a_{2n-1} \end{bmatrix}, \tag{17h}$$

$$\bar{A}_0' = (A_0')^T, \quad \bar{A}_1' = (A_1')^T, \tag{17i}$$

$$\hat{A}_0' = PA_0'P, \quad \hat{A}_1' = PA_1'P, \tag{17j}$$

$$\tilde{A}_0' = (\hat{A}_0')^T, \quad \tilde{A}_1' = (\hat{A}_1')^T, \tag{17k}$$

then

$$\begin{aligned} &\det[I_n s - A_0 - A_1 w] = \det[I_n s - \bar{A}_0 - \bar{A}_1 w] \\ &= \det[I_n s - \hat{A}_0 - \hat{A}_1 w] = \det[I_n s - \tilde{A}_0 - \tilde{A}_1 w] \\ &= \det[I_n s - A_0' - A_1' w] = \det[I_n s - \bar{A}_0' - \bar{A}_1' w] \\ &= \det[I_n s - \hat{A}_0' - \hat{A}_1' w] = \det[I_n s - \tilde{A}_0' - \tilde{A}_1' w] \\ &= s^n - (a_{2n-1}w - a_{2n-2})s^{n-1} - (a_{2n-3}w - a_{2n-4})s^{n-2} \\ &\quad - \cdots - (a_3 w + a_2) - (a_1 w + a_0). \end{aligned} \tag{18}$$

Proof. The expansion of the determinant with respect to the first row for (17a) and (17b) yields

$$\det\,[I_n s - A_0 - A_1 w]$$

$$= \begin{vmatrix} s & 0 & \cdots & 0 & 0 & -1 \\ -a_0 - a_1 w & s & \cdots & 0 & 0 & 0 \\ -a_2 - a_3 w & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{2n-6} - a_{2n-5} w & 0 & \cdots & -1 & s & 0 \\ -a_{2n-4} - a_{2n-3} w & 0 & \cdots & 0 & -1 & s - a_{2n-2} - a_{2n-1} w \end{vmatrix}$$

$$= s^{n-1} - (s - a_{2n-2} - a_{2n-1} w) + (-1)^{n+2}$$

$$\times \begin{vmatrix} -a_0 - a_1 w & s & \cdots & 0 & 0 \\ -a_2 - a_3 w & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{2n-6} - a_{2n-5} w & 0 & \cdots & -1 & s \\ -a_{2n-4} - a_{2n-3} w & 0 & \cdots & 0 & -1 \end{vmatrix}$$

$$\begin{aligned} &= \cdots = s_1^n - (a_{2n-1}w + a_{2n-2})s^{n-1} - (a_{2n-3}w + a_{2n-4})s^{n-2} \\ &\quad - \cdots - (a_3 w + a_2) - (a_1 w + a_0). \end{aligned}$$

Taking into account that $\det X^T = \det X$ in (17c), we obtain

$$\begin{aligned} \det[I_n s - \bar{A}_0 - \bar{A}_1 w] &= \det[I_n s - A_0 - A_1 w]^T \\ &= \det[I_n s - A_0 - A_1 w]. \end{aligned}$$

Note that $P^{-1} = P$ and

$$\begin{aligned} \det[I_n s - \hat{A}_0 - \hat{A}_1 w] &= \det\left\{P^{-1}[I_n s - A_0 - A_1 w]P\right\} \\ &= \det[I_n s - A_0 - A_1 w]. \end{aligned}$$

The proof for the pair (17f) is similar to that for the pair (17c). The proof for the pairs (17g)–(17k) proceeds in much the same way. ■

Remark 1. The matrix A_0 is a Metzler matrix and the matrix A_1 has non-negative entries if and only if the coefficients a_k, $k = 0, 1, \ldots, 2n-3, 2n-1$ of the polynomial (12) are non-negative and a_{2n-2} is arbitrary.

Remark 2. The dimension $n \times n$ of the matrices (17) is the smallest possible for (10).

If the pair (A_0, A_1) has the form (17a) and (17b), then the adjoint matrix $\mathrm{Adj}\, H(s)$ has the form

$$\mathrm{Adj}\, H(s) = \begin{bmatrix} h_{11}(s) & h_{12}(s) & \cdots & h_{1n}(s) \\ h_{21}(s) & h_{22}(s) & \cdots & h_{2n}(s) \\ \cdots & \cdots & \cdots & \cdots \\ h_{n1}(s) & h_{n2}(s) & \cdots & h_{nn}(s) \end{bmatrix}, \tag{19}$$

where

$$\begin{aligned}
h_{11}(s) &= s^{n-1} - s^{n-2}(a_{2n-1}w + a_{2n-2}),\\
h_{12}(s) &= 1, \quad h_{13}(s) = s, \ldots, h_{1n}(s) = s^{n-2},\\
h_{21}(s) &= s^{n-3}\big[s-(a_{2n-1}w+a_{2n-2})\big](a_1w+a_0),\\
h_{22}(s) &= s^{n-1} - s^{n-2}(a_{2n-1}w + a_{2n-2})\\
&\quad - s^{n-3}(a_{2n-3}w + a_{2n-4})\\
&\quad - \cdots - (a_3w + a_2),\\
h_{23}(s) &= a_1w + a_0, \ldots,\\
h_{2,n-1}(s) &= s^{n-4}(a_1w + a_0),\\
h_{2n}(s) &= s^{n-3}(a_1w + a_0),\\
h_{31}(s) &= s^{n-4}\big[s - (a_{2n-1}w + a_{2n-2})\big]\\
&\quad \times \big[s(a_3w + a_2) + a_1w + a_0\big],\\
h_{32}(s) &= s^{n-2} - s^{n-3}(a_{2n-1}w + a_{2n-2})\\
&\quad - s^{n-4}(a_{2n-3}w + a_{2n-4})\\
&\quad - \cdots - (a_5w + a_4),\\
h_{33}(s) &= s^{n-1} - s^{n-2}(a_{2n-1}w + a_{2n-2})\\
&\quad - \cdots - s(a_5w + a_4), \ldots,\\
h_{3,n-1}(s) &= s^{n-5}\big[s(a_3w + a_2) + a_1w + a_0\big],\\
h_{3n}(s) &= s^{n-4}\big[s(a_3w + a_2) + a_1w + a_0\big]\\
&\ \vdots\\
h_{n-1,1}(s) &= \big[s - (a_{2n-1}w + a_{2n-2})\big]\\
&\quad \times \big[s^{n-3}(a_{2n-5}w + a_{2n-6})\\
&\quad + \cdots + s(a_3w + a_2) + a_1w + a_0\big],\\
h_{n-1,2}(s) &= s^2 - s(a_{2n-1}w + a_{2n-2})\\
&\quad - (a_{2n-3}w + a_{2n-4}),\\
h_{n-1,3}(s) &= s^3 - s^2(a_{2n-1}w + a_{2n-2})\\
&\quad - s(a_{2n-3}w + a_{2n-4}),\\
&\ \vdots\\
h_{n-1,n-1}(s) &= s^{n-1} - s^{n-2}(a_{2n-1}w + a_{2n-2})\\
&\quad - s^{n-3}(a_{2n-3}w + a_{2n-4}),\\
h_{n-1,n}(s) &= s^{n-3}(a_{2n-5}w + a_{2n-6})\\
&\quad + \cdots + s(a_3w + a_2) + a_1w + a_0,\\
h_{n,1}(s) &= s^{n-2}(a_{2n-3}w + a_{2n-4})\\
&\quad + \cdots + s(a_3w + a_2) + a_1w + a_0,\\
h_{n2}(s) &= s, \quad h_{n3}(s) = s^2, \ldots,\\
h_{n,n-1}(s) &= s^{n-2}, \quad h_{nn} = s^{n-1}.
\end{aligned}$$

The substitution of (19) into (7) yields

$$\begin{aligned}
\frac{c\,\mathrm{Adj}\, H(s)b}{\det H(s)} &= \frac{1}{d(s)}\begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}\\
&\quad \times \begin{bmatrix} h_{11}(s) & h_{12}(s) & \cdots & h_{1n}(s) \\ h_{21}(s) & h_{22}(s) & \cdots & h_{2n}(s) \\ \cdots & \cdots & \cdots & \cdots \\ h_{n1}(s) & h_{n2}(s) & \cdots & h_{nn}(s) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\\
&= \frac{n(s)}{d(s)}
\end{aligned} \tag{20}$$

From the comparison of the coefficients at the same powers of s and w of the numerators of (20), we obtain

$$Gx = q, \tag{21}$$

where

$$\begin{cases} G = \begin{bmatrix} g_{11} & \cdots & g_{1n^2} \\ g_{m1} & \cdots & g_{mn^2} \end{bmatrix},\\ x = \begin{bmatrix} x_1 & x_2 & \ldots & x_{n^2} \end{bmatrix}^T\\ \quad = \begin{bmatrix} b_1c_1 & b_1c_2 & \ldots & b_1c_n, b_2c_1, \ldots, b_nc_n \end{bmatrix}^T,\\ q = \begin{bmatrix} q_1 & q_2 & \ldots & q_m \end{bmatrix}^T. \end{cases} \tag{22}$$

The entries g_{ij} of G depend on the entries of the matrices A_0 and A_1, and the entries q_i of q depend on the coefficients q_{kl} of the polynomial (13).

For example, if

$$A_0 = \begin{bmatrix} 0 & 1 \\ a_0 & a_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ a_1 & a_3 \end{bmatrix},$$

then

$$\mathrm{Adj}\, H(s) = \mathrm{Adj}[Is - A_0 - A_1w] = \begin{bmatrix} s - (a_3w + a_2) & 1 \\ a_1w + a_0 & s \end{bmatrix}.$$

The comparison of the coefficients at the same powers of s and w in the equality

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s-(a_3w+a_2) & 1 \\ a_1w+a_0 & s \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = n(s) = q_1 s + q_2 w + q_3$$

yields

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -a_3 & a_1 & 0 & 0 \\ -a_2 & a_0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1c_1 \\ b_1c_2 \\ b_2c_1 \\ b_2c_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

By Lemma A of Appendix, if

$$\operatorname{rank} G = \operatorname{rank} \begin{bmatrix} G, & q \end{bmatrix}, \tag{23}$$

then (21) has a non-negative solution $x \in \mathbb{R}_+^{n^2}$ if

$$\sum_{i=1}^{r} \frac{u_i^T G^T q u_i}{s_i} \geq 0 \text{ for all } s_i > 0, \quad i = 1, \ldots, r, \tag{24}$$

where $r = \operatorname{rank} G^T G$, s_i is an eigenvalue of $G^T G$ and u_i is the corresponding eigenvector, i.e.,

$$G^T G u_i = s_i u_i, \quad i = 1, \ldots, n^2, \tag{25}$$

$\|u_i\| = 1$. From the structure of the vector x defined by (22) it follows that

$$x_i x_{k+n} = x_k x_{i+n} \text{ for } i \neq k \text{ and } i,k = 1, \ldots, n. \tag{26}$$

Knowing the solution x of (21), we may find $b \in \mathbb{R}_+^n$ and $c \in \mathbb{R}_+^{1\times n}$ if the conditions (26) are satisfied. Therefore, the following result has been proved:

Theorem 2. *There exists a positive minimal realization (6) of $T(s)$ if the following conditions are satisfied:*

(i) $T(\infty) = \lim_{s\to\infty} T(s) \in \mathbb{R}_+$,

(ii) the coefficients a_k, $k = 0, 1, \ldots, 2n-3, 2n-1$ of the polynomial (12) are non-negative, i.e.,

$$a_k \geq 0 \text{ for } k = 0, 1, \ldots, 2n-3, 2n-1, \tag{27}$$

(a_{2n-2} can be arbitrary),

(iii) the conditions (24) and (26) are satisfied.

If the conditions of Theorem 2 are satisfied, then a positive minimal realization (6) of $T(s)$ can be found using the following procedure:

Procedure

Step 1. Using (9) and (10), find d and the strictly proper transfer function $T_{sp}(s)$.

Step 2. Knowing the coefficients a_k, $k = 0, 1, \ldots, 2n-1$ of the polynomial (12), find the matrices (17a) and (17b) (or (17c)–(17k)).

Step 3. Comparing the coefficients at the same powers of s and w of (20), find the entries of G and q.

Step 4. Find the solution $x \in \mathbb{R}_+^{n^2}$ of (21).

Step 5. Knowing x, find b and c.

Example 1. Given the transfer function

$$T(s) = \frac{2s^3 - 2ws^2 - (2w+1)s - 2w}{s^3 - (w+1)s^2 - (w+2)s - (2w+1)}, \tag{28}$$

find its positive minimal realization (6).

Solution. Using the above procedure, we obtain the following:

Step 1. From (9) and (10) we get

$$d = \lim_{s\to\infty} T(s) = 2 \tag{29}$$

and

$$T_{sp}(s) = T(s) - d = \frac{2s^2 + 3s + 2(w+1)}{s^3 - (w+1)s^2 - (w+2)s - (2w+1)}. \tag{30}$$

Step 2. Taking into account the fact that $d(z) = s^3 - (w+1)s^2 - (w+2)s - (2w+1)$ ($a_0 = a_3 = a_4 = a_5 = 1$, $a_1 = a_2 = 2$), and using (17a) and (17b), we obtain

$$A_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{31}$$

Step 3. In this case, we get (32a) and (32b).

The comparison of the coefficients at the same powers of s and w of (32) yields (21) with

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 2 \\ 2 \end{bmatrix}. \tag{33}$$

For (33), the conditions (23) and (24) are satisfied.

$$\mathrm{Adj}[Is - A_0 - A_1 w] = \begin{bmatrix} s^2-(w+1)s & 1 & s \\ (2w+1)s-(w+1)(2w+1) & s^2-(w+1)s-(w+2) & w+1 \\ (w+2)s+(2w+1) & s & s^2 \end{bmatrix}, \tag{32a}$$

$$\begin{aligned} c\,\mathrm{Adj}[Is - A_0 - A_1 w]b &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} s^2-(w+1)s & 1 & s \\ (2w+1)s-(w+1)(2w+1) & s^2-(w+1)s-(w+2) & 2w+1 \\ (w+2)s+(2w+1) & s & s^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= 2s^2 + 3s + 2(w+1). \end{aligned} \tag{32b}$$

Step 4. Equation (21) with (33) has the solution

$$x = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}^T. \tag{34}$$

It is easy to check that the solution (34) satisfies the conditions (26).

Step 5. The matrices b and c have the form

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}. \tag{35}$$

The desired minimal positive realization is given by (29), (31) and (35). ♦

4. Concluding Remarks

The realization problem for a class of positive single-input single-output continuous-time systems with one delay has been formulated and solved. Special forms (17) of the pairs of matrices were introduced. Sufficient conditions for the existence of a positive minimal realization (6) of a proper transfer function $T(s)$ were established. A procedure for the computation of a minimal positive realization of a proper transfer function was presented and illustrated by an example. The deliberations can be extended to multi-input multi-output continuous-time linear systems with many time delays. An extension to singular linear systems with time delays is also possible.

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Appendix

Consider the matrix equation

$$Ax = b, \tag{A1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$.

It is assumed that (A1) has a solution, i.e.,

$$\mathrm{rank}\begin{bmatrix} A, & b \end{bmatrix} = \mathrm{rank}\, A. \tag{A2}$$

Lemma A. *Let the assumption (A2) be satisfied. The equation (A1) has a non-negative solution* $x \in \mathbb{R}^n_+$ *if*

$$\sum_{i=1}^{r} \frac{u_i^T A^T b u_i}{s_i} \geq 0 \ \textit{for all}\ s_i > 0,\ i = 1, \dots, r, \tag{A3}$$

where $r = \mathrm{rank}\, A^T A$, s_i *is an eigenvalue of* $A^T A$ *and* u_i *is the associated eigenvector, i.e.,*

$$A^T A u_i = s_i u_i, \quad i = 1, \dots, n, \tag{A4}$$

and $\|u_i\| = 1$.

Proof. Premultiplying (A1) by A^T, we obtain

$$A^T\, Ax = A^T b. \tag{A5}$$

The premultiplication of (A5) by u_i^T yields

$$u_i^T\, A^T\, Ax = u_i^T A^T b, \quad i = 1, \dots, n, \tag{A6}$$

and using (A4) we obtain

$$s_i u_i^T x = u_i^T A^T b, \quad i = 1, \dots, n. \tag{A7}$$

Taking into account the fact that $s_i = 0$, $i = r+1, \dots, n$, from (A7) we obtain

$$x = \sum_{i=1}^{r} u_i^T x u_i = \sum_{i=1}^{r} \frac{u_i^T A^T b u_i}{s_i} \tag{A8}$$

for all $s_i > 0$, $i = 1, \dots, r$. Therefore, Eqn. (A1) has a non-negative solution $x \in \mathbb{R}_+^n$ if the condition (A3) is satisfied. ■