OBSERVER DESIGN FOR SYSTEMS WITH UNKNOWN INPUTS

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Design procedures are proposed for two different classes of observers for systems with unknown inputs. In the first approach, the state of the observed system is decomposed into known and unknown components. The unknown component is a projection, not necessarily orthogonal, of the whole state along the subspace in which the available state component resides. Then, a dynamical system to estimate the unknown component is constructed. Combining the output of the dynamical system, which estimates the unknown state component, with the available state information results in an observer that estimates the whole state. It is shown that some previously proposed observer architectures can be obtained using the projection operator approach presented in this paper. The second approach combines sliding modes and the second method of Lyapunov resulting in a nonlinear observer. The nonlinear component of the sliding mode observer forces the observation error into the sliding mode along a manifold in the observation error space. Design algorithms are given for both types of observers.

Keywords: state observation, unknown input observer (UIO), uncertain systems, projection operators, second method of Lyapunov

1. Introduction

Observers use the plant input and output signals to generate an estimate of the plant's state, which is then employed to close the control loop. Observers are utilized to augment or replace sensors in a control system. The observer was first proposed and developed by Luenberger in the early sixties of the last century (Luenberger, 1966; 1971; 1979). Since the early developments, observers for plants with both known and unknown inputs have been developed resulting in the so-called unknown input observer (UIO) architectures, such as, for example, those in (Bhattacharyya, 1978; Chen and Patton, 1999; Chen *et al.*, 1996; Corless and Tu, 1998; Darouach *et al.*, 1994; Hostetter and Meditch, 1973; Hou and Müller, 1992; Hou *et al.*, 1999; Hui and Żak, 1993; 2005; Kudva *et al.*, 1980; Kurek, 1983; Krzemiński and Kaczorek, 2004; Sundareswaran *et al.*, 1977; Wang *et al.*, 1975; Yang and Wilde, 1988). More recently, observer architectures utilizing the concept of sliding modes were proposed for uncertain systems, see, for example, (Edwards and Spurgeon, 1998; Ha *et al.*, 2003; Hui and Żak, 1990; Koshkouei and Zinober, 2004; Utkin *et al.*, 1999; Walcott and Żak, 1987; 1988; Walcott *et al.*, 1987; Żak and Walcott, 1990; Żak and Hui, 1993; Żak, 2003; Żak *et al.*, 1993). Other methods of observer design for linear systems developed up to 1983 are reported by O'Reilly in (1983).

Observers for systems with unknown inputs play an essential role in robust model-based fault detection (Chen and Patton, 1999; Edwards *et al.*, 2000; Edwards and Spurgeon, 1998; Jiang *et al.*, 2004; Saif and Xiong, 2003). The basic idea behind the use of observers for fault detection is to form residuals from the difference between the actual system outputs and the estimated outputs using an observer. Once a fault occurs, the residuals are expected to react by becoming greater than a prespecified threshold. When the system under consideration is subject to unknown disturbances or unknown inputs, their effect has to be decoupled from the residuals to avoid false alarms.

In this paper, we present design procedures for full- and reduced-order observers for systems with unknown inputs. The unknown input can be a combination of unmeasurable or unmeasured disturbances, unknown control action, or unmodeled system dynamics. The first design method uses a projection operator approach to the state estimation where the state of the system, whose state is to be estimated, is decomposed into known and unknown components. The unknown component is, in general, a skew projection, that is, not necessarily orthogonal, of the whole state along the subspace in which the available state component resides. We then construct a dynamical system to estimate the unknown component. Finally, we combine the output of the dynamical system, which estimates the unknown state component, with the available state information to obtain the observer that estimates the whole state. In the second design method, we employ a sliding mode approach combined with the second method of Lyapunov. We include design algorithms and illustrate the results with numerical examples.

2. Modeling of Systems with Unknown Inputs

The class of dynamical systems that we consider is modeled by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \tag{1}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}, \tag{2}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, the input matrix $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ and the output matrix $\boldsymbol{C} \in \mathbb{R}^{p\times n}$. We assume that the model parameters $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ are known. We further assume that some or all of the inputs are unknown, and that the first m_1 components of $\boldsymbol{u}$ are known and the remaining $m_2 = m - m_1$ inputs are unknown. We partition the input matrix $\boldsymbol{B}$ corresponding to the known and unknown inputs as

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \end{bmatrix},$$

where $\boldsymbol{B}_1 \in \mathbb{R}^{n\times m_1}$ and $\boldsymbol{B}_2 \in \mathbb{R}^{n\times m_2}$. Let

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}.$$

Then, the system model (1) can be represented as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2. \tag{3}$$

The vector function $\boldsymbol{u}_2$ may also model lumped uncertainties or nonlinearities in the plant. We assume that the pair $(\boldsymbol{A}, \boldsymbol{C})$ is detectable.

3. A Projection Operator Approach to State Observation of Systems with Unknown Inputs

In our discussion in this section, we assume that the matrix $\boldsymbol{B}_2$ has full column rank. We begin our presentation by noticing that because the system output $\boldsymbol{y}$ is known it would seem reasonable to decompose the state $\boldsymbol{x}$ as

$$\begin{aligned}\boldsymbol{x} &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} + \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}\\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} + \boldsymbol{M}\boldsymbol{y},\end{aligned} \tag{4}$$

where $\boldsymbol{M}$ is an $n \times p$ real matrix, and the unknown part of the decomposition is $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}$. Let $\boldsymbol{q} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}$, then $\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y}$, and we have

$$\begin{aligned}\dot{\boldsymbol{q}} &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\dot{\boldsymbol{x}}\\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2)\\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{u}_2\\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1)\\ &\quad + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{u}_2.\end{aligned}$$

If $\boldsymbol{M}$ is chosen so that $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$, then the dynamics of $\boldsymbol{q}$ depend only on the known quantities $\boldsymbol{u}_1$ and $\boldsymbol{y}$:

$$\dot{\boldsymbol{q}} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1). \tag{5}$$

Note that if we start the above dynamical system with the initial condition $\boldsymbol{q}(0) = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}(0)$, then $\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{C}\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y}$ for all $t \geq 0$. But since $\boldsymbol{x}(0)$ is assumed to be unknown,

$$\tilde{\boldsymbol{x}} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y} \tag{6}$$

is only an approximation of $\boldsymbol{x}$. To improve the convergence rate or to ensure the convergence, we add an extra term to the right-hand side of (5) to obtain

$$\begin{aligned}\dot{\boldsymbol{q}} &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\Big(\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1\\ &\qquad + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{y})\Big)\\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\Big(\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1\\ &\qquad + \boldsymbol{L}\boldsymbol{C}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{y})\Big).\end{aligned} \tag{7}$$

Let $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$. We will show that

$$\dot{\boldsymbol{e}} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}$$

and $\boldsymbol{e}(t) \to \boldsymbol{0}$ as $t \to \infty$ under mild conditions.

Because

$$\text{rank}\ (\boldsymbol{M}\boldsymbol{C}\boldsymbol{B}_2) \leq \text{rank}\ (\boldsymbol{C}\boldsymbol{B}_2) \leq \text{rank}\ (\boldsymbol{B}_2),$$

the equality $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ makes it necessary that

$$\text{rank}\ (\boldsymbol{C}\boldsymbol{B}_2) = \text{rank}(\boldsymbol{B}_2), \tag{8}$$

which we assume throughout the paper. This rank condition also implies that there must be at least as many independent outputs as unknown inputs for the method to work.

We will show that, in order to arrive at a reduced-order observer using the above presented approach, it is critical that the term $L(y - Cq - CMy)$ be premultiplied by $(I - MC)$, or equivalently, L have $(I - MC)$ as a left factor. Indeed, let $\tilde{P} = I - MC$. Then, if $\tilde{P}$ is a projection, that is, $\tilde{P}^2 = \tilde{P}$, then the subspace $\mathcal{V} = \tilde{P}\mathbb{R}^n$ is invariant under $\tilde{P}$. It follows that $\dot{q}$ in (7) lies in $\mathcal{V}$. If the initial condition $q(0)$ is also in $\mathcal{V}$, then the trajectories of the system will reside in $\mathcal{V}$ for $t \geq 0$. If the term $L(y - Cq - CMy)$ is not premultiplied by $(I - MC)$ or L does not have $(I - MC)$ as a left factor, then the trajectory will not stay in $\mathcal{V}$ and, in general, it would not be possible to transform the full-order observer into a reduced-order one.

The condition $q(0) \in \mathcal{V}$ alone is not sufficient to guarantee that the observation error e tends to $\mathbf{0}$. The reason is that we do not know $x(0)$ and it is not obvious how to choose $q(0)$ so that the $e(t)$ converges to $\mathbf{0}$. Unless $q(0)$ is chosen appropriately, the observation error $e(t)$ stays in a hyperplane not containing $\mathbf{0}$ and thus $e(t)$ cannot converge to $\mathbf{0}$.

Another difficulty that must be overcome is the fact that, for the error dynamics matrix $(I - MC)(A - LC)$ to be asymptotically stable, it is not sufficient for $(A - LC)$ to be asymptotically stable. It is possible for a product of a projection matrix and an asymptotically stable matrix to be unstable as the following simple example shows:

Example 1. Let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & -3 \\ 3 & -2 \end{bmatrix}.$$

It is easy to check that A is asymptotically stable while PA is unstable. Furthermore, the system $\dot{x} = Ax$ restricted to the range of P is governed by $\dot{z} = z$, which is also unstable. ⧫

We now analyze the convergence properties of the proposed full-order observer and then use the results of our analysis to propose a new type of a reduced-order observer for uncertain systems. Consider the dynamical system model given by (6) and (7). We will now show that $\tilde{x} \to x$ as $t \to \infty$. To this end let

$$e(t) = x(t) - \tilde{x}(t)$$

denote the estimation error. Then, using $(I - MC)B_2 = O$ and $y = Cx$, we have

$$\begin{aligned} \frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(x - \tilde{x}) = \frac{\mathrm{d}}{\mathrm{d}t}(x - q - MCx) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}((I - MC)x - q) \\ &= (I - MC)(Ax + B_1u_1 + B_2u_2) \\ &\quad - (I - MC)\Big(Aq + AMy + B_1u_1 \\ &\quad + L(y - Cq - CMy)\Big) \\ &= (I - MC)(Ax + B_1u_1) + (I - MC)B_2u_2 \\ &\quad - (I - MC)\Big(Aq + AMCx + B_1u_1 \\ &\quad + L(Cx - Cq - CMCx)\Big) \\ &= (I - MC)(A - LC)(x - q - MCx) \\ &= (I - MC)\,(A - LC)\,e. \end{aligned} \tag{9}$$

Our objective is to specify M and L and a set of initial conditions so that $e(t) \to \mathbf{0}$ as $t \to \infty$. A particular class of solutions to $(I - MC)B_2 = O$ is given by

$$M = B_2\Big((CB_2)^\dagger + H_0\Big(I_p - (CB_2)(CB_2)^\dagger\Big)\Big),$$

where the superscript $\dagger$ denotes the Moore-Penrose pseudo-inverse operation and $H_0 \in \mathbb{R}^{m_2 \times p}$ is a design parameter matrix. (See, for example, (Kaczorek, 1998, Section 1.5) for more information on pseudo-inverse matrices). Because, by assumption, rank $(CB_2) =$ rank B_2 and B_2 has a full rank, we have $(CB_2)^\dagger(CB_2) = I_{m_2}$. If CB_2 is a square matrix, then CB_2 is invertible by assumption and the above M reduces to $B_2(CB_2)^{-1}$. Furthermore, it is easy to check that for the above class of M, the product MC is a projection (not necessarily orthogonal):

$$(MC)^2 = MC.$$

It follows that

$$\tilde{P} = I - MC$$

is also a projection.

To proceed further, we need the following lemma:

Lemma 1. *Let $\tilde{P} : \mathbb{R}^n \to \mathbb{R}^n$ be a projection, that is, $\tilde{P}^2 = \tilde{P}$, and let rank $\tilde{P} = n - m_2$. Then $\tilde{P}$ has $(n - m_2)$ eigenvalues equal to 1 while the remaining m_2 eigenvalues are equal to 0 and there is a basis of $\mathbb{R}^n$ in which the matrix $\tilde{P}$ relative to this basis has the form*

$$P = \begin{bmatrix} I_{n-m_2} & O \\ O & O \end{bmatrix},$$

that is, there is an invertible matrix Q whose columns are eigenvectors of $\tilde{P}$ such that

$$Q^{-1}\tilde{P}Q = P = \begin{bmatrix} I_{n-m_2} & O \\ O & O \end{bmatrix}.$$

Proof. See (Smith, 1984, pp. 156–158 and pp. 194–195). ■

4. Constructing the Full–Order Observer Using the Projection Operator Approach

We begin this section by introducing the following coordinate transformation:

$$\tilde{e} = Q^{-1}e, \tag{10}$$

where the transformation matrix Q is obtained, using Lemma 1, from the representation of the projection operator $\tilde{P}$ in the form

$$\tilde{P} = QPQ^{-1}. \tag{11}$$

Applying the coordinate transformation (10) to the error equation (9) gives

$$\begin{aligned}\dot{\tilde{e}} &= PQ^{-1}(A - LC)Q\tilde{e} \\ &= P\left(Q^{-1}AQ - \left(Q^{-1}L\right)(CQ)\right)\tilde{e}.\end{aligned} \tag{12}$$

Let

$$\tilde{A} = Q^{-1}AQ = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{21} \end{bmatrix},$$

$$\tilde{L} = Q^{-1}L = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix},$$

$$\tilde{C} = CQ = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}, \tag{13}$$

where $\tilde{A}_{11} \in \mathbb{R}^{(n-m_2)\times(n-m_2)}$, $\tilde{L}_1 \in \mathbb{R}^{(n-m_2)\times p}$, $\tilde{C}_1 \in \mathbb{R}^{p\times(n-m_2)}$, and the remaining block submatrices are of appropriate dimensions. Using the above notation, we represent (12) in the form

$$\begin{aligned}\dot{\tilde{e}} &= P\left(\tilde{A} - \tilde{L}\tilde{C}\right)\tilde{e} = \begin{bmatrix} I_{n-m_2} & O \\ O & O \end{bmatrix} \\ &\quad \times \left(\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{21} \end{bmatrix} - \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix}\begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}\right)\tilde{e} \\ &= \begin{bmatrix} \tilde{A}_{11} - \tilde{L}_1\tilde{C}_1 & \tilde{A}_{12} - \tilde{L}_1\tilde{C}_2 \\ O & O \end{bmatrix}\tilde{e}.\end{aligned} \tag{14}$$

Let

$$\tilde{e} = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix}, \tag{15}$$

where $\tilde{e}_1 \in \mathbb{R}^{n-m_2}$. Note that $\dot{\tilde{e}}_2 = \mathbf{0}$. Hence if $\tilde{e}_2(0) = \mathbf{0}$, then $\tilde{e}_2 = \mathbf{0}$ for all $t \geq 0$. Thus if $\tilde{e}_2 = \mathbf{0}$, then

$$\dot{\tilde{e}}_1 = \left(\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1\right)\tilde{e}_1,$$

and so if $\tilde{e}_2 = \mathbf{0}$ and $\tilde{e}_1 \to \mathbf{0}$, then $\tilde{e} \to \mathbf{0}$. Obviously, $\tilde{e}_1 \to \mathbf{0}$ for arbitrary $\tilde{e}_1(0)$ if and only if the matrix $\left(\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1\right)$ is asymptotically stable.

We now give a condition on $q(0)$ that guarantees that $\tilde{e}_2 = \mathbf{0}$. We have

$$\begin{aligned}MC &= I - \tilde{P} \\ &= Q(I - P)Q^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}\begin{bmatrix} \mathbf{0} \\ \tilde{e}_2 \end{bmatrix} &= (I_n - P)Q^{-1}e = Q^{-1}MCe \\ &= Q^{-1}MC(x - q - MCx) \\ &= Q^{-1}(MCx - MCq - (MC)^2x) \\ &= -Q^{-1}MCq.\end{aligned} \tag{16}$$

Therefore $\tilde{e}_2(0) = \mathbf{0}$ if and only if $MCq(0) = \mathbf{0}$, which is equivalent to

$$q(0) = (I - MC)v$$

for arbitrary $v \in \mathbb{R}^n$. In particular, $q(0) = \mathbf{0}$ satisfies the above condition.

In summary, we proved the following theorem:

Theorem 1. *If the following conditions are satisfied:*

1. rank $(CB_2) = \operatorname{rank} B_2$*;*
2. *the pair* $(\tilde{A}_{11}, \tilde{C}_1)$ *defined in (13) is detectable;*
3. $q(0) = (I - MC)v$ *for arbitrary* $v \in \mathbb{R}^n$*,*

then there exists a gain matrix L *such that the estimation error,* $e = x - \tilde{x}$*, of the full-order observer given by*

$$\begin{aligned}\dot{q} &= (I - MC)(Aq + AMy + B_1u_1 \\ &\quad + L(y - Cq - CMy)), \\ \tilde{x} &= q + My\end{aligned}$$

converges to $\mathbf{0}$ *as* $t \to \infty$*.*

Theorem 2. *The second condition of Theorem 1, which states that the pair* $(\tilde{A}_{11}, \tilde{C}_1)$ *defined in (13) is detectable, is equivalent to*

$$\operatorname{rank}\begin{bmatrix} sI_n - A & B_2 \\ C & O \end{bmatrix} = n + m_2$$

for all s *such that* $\operatorname{Re}(s) \geq 0$*.*

Proof. We begin the proof by considering the projection matrix, $\tilde{P} = I - MC$, where

$$M = B_2\left((CB_2)^{\dagger} + H_0\left(I_p - (CB_2)(CB_2)^{\dagger}\right)\right).$$

To simplify the further analysis, we let $\boldsymbol{F} = (\boldsymbol{C}\boldsymbol{B}_2)^{\dagger} + \boldsymbol{H}_0\left(\boldsymbol{I}_p - (\boldsymbol{C}\boldsymbol{B}_2)(\boldsymbol{C}\boldsymbol{B}_2)^{\dagger}\right)$ and $\boldsymbol{S} = \boldsymbol{F}\boldsymbol{C}$. Note that

$$\boldsymbol{S}\boldsymbol{B}_2 = \boldsymbol{I}_{m_2}. \tag{17}$$

Thus, $\operatorname{rank}\boldsymbol{S} = m_2$, and so we can find a full rank $n \times (n - m_2)$ matrix $\boldsymbol{W}$ such that

$$\boldsymbol{S}\boldsymbol{W} = \boldsymbol{O}. \tag{18}$$

Combining (17) and (18), we conclude that $[\boldsymbol{W}\ \ \boldsymbol{B}_2]$ is invertible. Let

$$\begin{bmatrix} \boldsymbol{W} & \boldsymbol{B}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{N} \end{bmatrix},$$

where $\boldsymbol{W}^g$ is $(n - m_2) \times n$ and $\boldsymbol{N}$ is $m_2 \times n$. Then

$$\boldsymbol{N}\begin{bmatrix} \boldsymbol{W} & \boldsymbol{B}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_{m_2} \end{bmatrix} = \boldsymbol{S}\begin{bmatrix} \boldsymbol{W} & \boldsymbol{B}_2 \end{bmatrix}.$$

Since $[\boldsymbol{W}\ \ \boldsymbol{B}_2]$ has a full rank, we conclude that $\boldsymbol{N} = \boldsymbol{S}$.

Since

$$\begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{S} \end{bmatrix}\begin{bmatrix} \boldsymbol{W} & \boldsymbol{B}_2 \end{bmatrix} = \boldsymbol{I}_n,$$

we have

$$\boldsymbol{W}^g\boldsymbol{W} = \boldsymbol{I}_{n-m_2} \quad \text{and} \quad \boldsymbol{W}^g\boldsymbol{B}_2 = \boldsymbol{O}.$$

Let $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{B}_2 \end{bmatrix}$. Then

$$\begin{aligned}
\boldsymbol{Q}^{-1}\tilde{\boldsymbol{P}}\boldsymbol{Q} &= \begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{S} \end{bmatrix}(\boldsymbol{I}_n - \boldsymbol{B}\boldsymbol{S})\begin{bmatrix} \boldsymbol{W} & \boldsymbol{B}_2 \end{bmatrix} \\
&= \boldsymbol{I}_n - \begin{bmatrix} \boldsymbol{W}^g\boldsymbol{B}_2\boldsymbol{S} \\ \boldsymbol{S}\boldsymbol{B}_2\boldsymbol{S} \end{bmatrix}\begin{bmatrix} \boldsymbol{W} & \boldsymbol{B}_2 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_2} \end{bmatrix} \\
&\quad - \begin{bmatrix} \boldsymbol{W}^g\boldsymbol{B}_2\boldsymbol{S}\boldsymbol{W} & \boldsymbol{W}^g\boldsymbol{B}_2\boldsymbol{S}\boldsymbol{B}_2 \\ \boldsymbol{S}\boldsymbol{B}_2\boldsymbol{S}\boldsymbol{W} & \boldsymbol{S}\boldsymbol{B}_2\boldsymbol{S}\boldsymbol{B}_2 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_2} \end{bmatrix} - \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{m_2} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{I}_{n-m_2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}.
\end{aligned} \tag{19}$$

We now apply the following coordinate transformation to the system modeled by (1) and (2):

$$\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{S} \end{bmatrix}\boldsymbol{x} = \boldsymbol{Q}^{-1}\boldsymbol{x}.$$

Then

$$\begin{aligned}
\begin{bmatrix} \dot{\boldsymbol{z}} \\ \dot{\boldsymbol{\sigma}} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{W}^g\boldsymbol{A}\boldsymbol{W} & \boldsymbol{W}^g\boldsymbol{A}\boldsymbol{B}_2 \\ \boldsymbol{S}\boldsymbol{A}\boldsymbol{W} & \boldsymbol{S}\boldsymbol{A}\boldsymbol{B}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\sigma} \end{bmatrix} \\
&\quad + \boldsymbol{Q}^{-1}\boldsymbol{B}_1\boldsymbol{u}_1 + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{I}_{m_2} \end{bmatrix}\boldsymbol{u}_2 \\
&= \begin{bmatrix} \tilde{\boldsymbol{A}}_{11} & \tilde{\boldsymbol{A}}_{12} \\ \tilde{\boldsymbol{A}}_{21} & \tilde{\boldsymbol{A}}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\sigma} \end{bmatrix} + \boldsymbol{Q}^{-1}\boldsymbol{B}_1\boldsymbol{u}_1 \\
&\quad + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{I}_{m_2} \end{bmatrix}\boldsymbol{u}_2, \\
\boldsymbol{y} &= \begin{bmatrix} \boldsymbol{C}\boldsymbol{W} & \boldsymbol{C}\boldsymbol{B}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\sigma} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{C}}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\sigma} \end{bmatrix}.
\end{aligned}$$

If the trajectory of the system described by the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{F}\boldsymbol{C})$ resides in the null space of $\boldsymbol{S}$, then such a motion is described by

$$\dot{\boldsymbol{z}} = \tilde{\boldsymbol{A}}_{11}\boldsymbol{z}.$$

It follows from (Żak, 2003, pp. 328, 329) that the poles of the above system are the zeros of the system described by the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{F}\boldsymbol{C})$, which are the complex numbers s for which the system matrix

$$\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B}_2 \\ \boldsymbol{F}\boldsymbol{C} & \boldsymbol{O} \end{bmatrix}$$

loses its full rank. On the other hand, the zeros of the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ are also the zeros of the squared-down system $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{F}\boldsymbol{C})$, that is, the zeros of $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ form a subset of the set of the eigenvalues of $\tilde{\boldsymbol{A}}_{11}$. It is well known that zeros are invariant with respect to similarity transformations. Therefore,

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_{n-m_2} - \tilde{\boldsymbol{A}}_{11} & -\tilde{\boldsymbol{A}}_{12} & \boldsymbol{O} \\ -\tilde{\boldsymbol{A}}_{21} & s\boldsymbol{I}_{m_2} - \tilde{\boldsymbol{A}}_{22} & \boldsymbol{I}_{m_2} \\ \tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{C}}_2 & \boldsymbol{O} \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_{n-m_2} - \tilde{\boldsymbol{A}}_{11} \\ \tilde{\boldsymbol{C}}_1 \end{bmatrix} + 2m_2,
\end{aligned}$$

for $s \in \mathbb{C}$, which means that the zeros of the system $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ are in the open left-half plane if and only if the pair $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{C}}_1)$ is detectable. ■

5. Reduced–Order Unknown Input Observer

The error dynamics of the full-order observer that we analyzed above are given by (14):

$$\dot{\tilde{e}} = \begin{bmatrix} \tilde{A}_{11} - \tilde{L}_1\tilde{C}_1 & \tilde{A}_{12} - \tilde{L}_1\tilde{C}_2 \\ O & O \end{bmatrix} \tilde{e}.$$

The reader may have noticed that since we choose the initial condition for q to force $\tilde{e}_2(t) = 0$ for $t \geq 0$, the dynamics of the error are completely determined by the dynamics of $\tilde{e}_1$, which are given by

$$\dot{\tilde{e}}_1 = \left(\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1\right)\tilde{e}_1, \tag{20}$$

an $(n - m_2)$-dimensional system. This motivates us to apply the transformation from e into $\tilde{e}$ to q:

$$\tilde{q} = Q^{-1}q.$$

From (7) and $I - MC = QPQ^{-1}$, we obtain

$$\begin{aligned}
\dot{\tilde{q}} &= P\Big(Q^{-1}AQ\tilde{q} + Q^{-1}AMy + Q^{-1}B_1u_1 \\
&\quad + Q^{-1}L(y - CQ\tilde{q} - CMy)\Big) \\
&= PQ^{-1}AQ\tilde{q} + PQ^{-1}AMy + PQ^{-1}B_1u_1 \\
&\quad + PQ^{-1}L(y - CQ\tilde{q} - CMy) \\
&= P\left(Q^{-1}AQ - Q^{-1}LCQ\right)\tilde{q} \\
&\quad + P\left(Q^{-1}AM + Q^{-1}L - Q^{-1}LCM\right)y \\
&\quad + PQ^{-1}B_1u_1.
\end{aligned} \tag{21}$$

Using the notation defined in (13), we have

$$\begin{aligned}
\dot{\tilde{q}} &= P\left(\tilde{A} - \tilde{L}\tilde{C}\right)\tilde{q} + PQ^{-1} \\
&\quad \times \Big[\Big(AM + Q\tilde{L}(I_p - CM)\Big)y + B_1u_1\Big].
\end{aligned}$$

Let

$$\tilde{q} = \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix},$$

where $\tilde{q}_1 \in \mathbb{R}^{n-m_2}$ and $\tilde{q}_2 \in \mathbb{R}^{m_2}$. Since

$$P = \begin{bmatrix} I_{n-m_2} & O \\ O & O \end{bmatrix},$$

we have $\dot{\tilde{q}}_2(t) = 0$. Therefore, setting $\tilde{q}_2(0) = 0$ ensures that $\tilde{q}_2(t) = 0$ for $t \geq 0$. We thus can remove m_2 observer states from observer dynamics. Let $\tilde{G} = AM + Q\tilde{L}(I_p - CM)$. Then the resulting reduced-order observer takes the form

$$\begin{aligned}
\dot{\tilde{q}}_1 &= (\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1)\tilde{q}_1 + \begin{bmatrix} I_{n-m_2} & O_{m_2} \end{bmatrix} \\
&\quad \times Q^{-1}\left(\tilde{G}y + B_1u_1\right), \quad \tilde{q}_1(0) = 0, \\
\tilde{x} &= Q\begin{bmatrix} I_{n-m_2} \\ O_{m_2\times(n-m_2)} \end{bmatrix}\tilde{q}_1 + My,
\end{aligned}$$

where the vector $\tilde{x}$ is the estimate of the plant state x.

We now summarize the above deliberations in the form of the following design algorithm:

Reduced-Order Unknown Input Observer Design Algorithm

For a given quadruple of matrices (A, B_1, B_2, C), modeling the plant, do as follows:

1. Check that rank $(CB_2) =$ rank B_2.
   If rank $(CB_2) <$ rank B_2, STOP. The observer does not exist.

2. Compute

   $$M = B_2\left((CB_2)^\dagger + H_0\left(I_p - (CB_2)(CB_2)^\dagger\right)\right),$$

   where the superscript $\dagger$ denotes the Moore-Penrose pseudo-inverse operation and $H_0 \in \mathbb{R}^{m_2\times p}$ is a design parameter matrix.

3. Compute the projector

   $$\tilde{P} = I_n - MC.$$

4. Represent $\tilde{P}$ as

   $$\tilde{P} = QPQ^{-1},$$

   where

   $$P = \begin{bmatrix} I_{n-m_2} & O \\ O & O \end{bmatrix}.$$

5. Compute

   $$\tilde{A} = Q^{-1}AQ = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$$

   and $\quad \tilde{C} = CQ = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix},$

   where $\tilde{A}_{11} \in \mathbb{R}^{(n-m_2)\times(n-m_2)}$ and $\tilde{C}_1 \in \mathbb{R}^{p\times(n-m_2)}$

6. Check the detectability of the pair $(\tilde{A}_{11}, \tilde{C}_1)$.
   If the pair $(\tilde{A}_{11}, \tilde{C}_1)$ is not detectable, STOP. The observer does not exist.
   Note that if the matrix $\tilde{A}_{11}$ is asymptotically stable, then the pair $(\tilde{A}_{11}, \tilde{C}_1)$ is detectable for an arbitrary matrix $\tilde{C}_1$.

7. If there are eigenvalues of $\tilde{A}_{11}$ that are not asymptotically stable, construct $\tilde{L}_1$ so that the matrix $(\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1)$ has its eigenvalues in locations as close to the desired eigenvalues as possible.

8. Form
$$\tilde{L} = \begin{bmatrix} \tilde{L}_1 \\ O_{m_2\times p} \end{bmatrix},$$
   where $O_{m_2\times p}$ is an $m_2 \times p$ zero matrix.

9. Compute the matrix
$$\tilde{G} = AM + Q\tilde{L}(I_p - CM).$$

10. Construct the observer
$$\dot{\tilde{q}}_1 = (\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1)\tilde{q}_1 + \begin{bmatrix} I_{n-m_2} & O_{m_2} \end{bmatrix} \times Q^{-1}\left(\tilde{G}y + B_1u_1\right), \quad \tilde{q}_1(0) = \mathbf{0},$$
$$\tilde{x} = Q\begin{bmatrix} I_{n-m_2} \\ O_{m_2\times(n-m_2)} \end{bmatrix}\tilde{q}_1 + My.$$
   The vector $\tilde{x}$ is the estimate of the state x.

Example 2. We consider the fifth-order lateral axis model of an L-1011 fixed-wing aircraft, with actuator dynamics neglected, at cruise flight conditions. This model can be found in the book (Edwards and Spurgeon, 1998, pp. 122, 123 and 179, 180). We assume that the inputs to the system are unknown and there are no known inputs. We have

$$A = \begin{bmatrix} 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.1540 & -0.0042 & 1.5400 & 0.0000 \\ 0.0000 & 0.2490 & -1.0000 & -5.2000 & 0.0000 \\ 0.0386 & -0.9960 & -0.0003 & -0.1170 & 0.0000 \\ 0.0000 & 0.5000 & 0.0000 & 0.0000 & -0.5000 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} 0.0000 & 0.0000 \\ -0.7440 & -0.0320 \\ 0.3370 & -1.1200 \\ 0.0200 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $u_2 = \begin{bmatrix} \cos(t) & \sin(t) \end{bmatrix}^T$. We first check that $\operatorname{rank}(CB_2) = \operatorname{rank} B_2$. We then compute the matrix M, where in this example we set $H_0 = O$,

$$M = B_2(CB_2)^\dagger = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.9993 & 0.0000 & -0.0265 & 0.0000 \\ 0.0000 & 1.0000 & 0.0008 & 0.0000 \\ -0.0265 & 0.0008 & 0.0007 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

Then the projector $\tilde{P}$ is

$$\tilde{P} = I_5 - MC = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0007 & 0.0000 & 0.0265 & 0.9993 \\ 0.0000 & -0.0000 & 0.0000 & -0.0008 & 0.0000 \\ 0.0000 & 0.0265 & -0.0008 & 0.9993 & -0.0265 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}.$$

We next compute Q such that

$$\tilde{P} = QPQ^{-1} = Q\begin{bmatrix} I_3 & O_{3\times 2} \\ O_{3\times 2}^T & O_{2\times 2} \end{bmatrix}Q^{-1}.$$

We have

$$Q = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7072 & 0.0265 & -0.1162 & -0.1162 \\ 0.0000 & 0.0000 & -0.0008 & 0.7346 & 0.7346 \\ 0.0000 & 0.0143 & 0.9996 & 0.0036 & 0.0036 \\ 0.0000 & 0.7068 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

Hence

$$\tilde{A}_{11} = \begin{bmatrix} 0.0000 & 0.0000 & -0.0008 \\ -0.0014 & -0.0737 & 1.0151 \\ 0.0386 & -0.7058 & -0.1322 \end{bmatrix}$$

and

$$\tilde{C}_1 = \begin{bmatrix} 0.0000 & 0.0004 & 0.0265 \\ 0.0000 & 0.0000 & -0.0008 \\ 0.0000 & 0.0143 & 0.9996 \\ 1.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

The pair $(\tilde{A}_{11}, \tilde{C}_1)$ is detectable and the eigenvalues of $\tilde{A}_{11}$ are located at $0.0000, -0.1030 + 0.8459j, -0.1030 - 0.8459j$. We select the desired eigenvalues to be located at $-3, -4, -5$. The gain matrix $\tilde{L}_1$ such that $\operatorname{eig}(\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1) = \{-3, -4, -5\}$ is

$$\tilde{L}_1 = \begin{bmatrix} 0.0015 & 0.0000 & 0.0577 & 4.0069 \\ -0.5115 & 0.0146 & -19.2737 & -3.2839 \\ 0.2139 & -0.0061 & 8.0602 & 0.7062 \end{bmatrix}.$$

We obtained the above gain matrix using MATLAB's command `place`. We next form the matrix $\tilde{\boldsymbol{L}}$ by adding two zero rows to $\tilde{\boldsymbol{L}}_1$ and compute $\begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{O}_{3\times 2} \end{bmatrix} \boldsymbol{Q}^{-1}\tilde{\boldsymbol{G}}$.

The reduced-order UIO has the form

$$\begin{aligned}
\dot{\tilde{\boldsymbol{q}}}_1 &= (\tilde{\boldsymbol{A}}_{11} - \tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{C}}_1)\tilde{\boldsymbol{q}}_1 + \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{O}_{3\times 2} \end{bmatrix} \boldsymbol{Q}^{-1}\tilde{\boldsymbol{G}}\boldsymbol{y} \\
&= \begin{bmatrix} -4.0069 & -0.0008 & -0.0585 \\ 3.2824 & 0.2021 & 20.2955 \\ -0.6676 & -0.8211 & -8.1953 \end{bmatrix} \tilde{\boldsymbol{q}}_1 \\
&\quad + \begin{bmatrix} 0.0016 & 1.0000 & 0.0585 & 4.0069 \\ -0.6633 & -0.2145 & -19.2698 & -3.2839 \\ -0.7784 & 0.0017 & 8.0865 & 0.7062 \end{bmatrix} \boldsymbol{y}
\end{aligned}$$

$$\begin{aligned}
\tilde{\boldsymbol{x}} &= \boldsymbol{Q} \begin{bmatrix} \boldsymbol{I}_3 \\ \boldsymbol{O}_{2\times 3} \end{bmatrix} \tilde{\boldsymbol{q}}_1 + \boldsymbol{M}\boldsymbol{y} \\
&= \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7072 & 0.0265 \\ 0.0000 & 0.0000 & -0.0008 \\ 0.0000 & 0.0143 & 0.9996 \\ 0.0000 & 0.7068 & 0.0000 \end{bmatrix} \tilde{\boldsymbol{q}}_1 \\
&\quad + \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.9993 & 0.0000 & -0.0265 & 0.0000 \\ 0.0000 & 1.0000 & 0.0008 & 0.0000 \\ -0.0265 & 0.0008 & 0.0007 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \boldsymbol{y}.
\end{aligned}$$

In Fig. 1, we show plots of system state variables and their estimates versus time. The initial conditions of the plant were selected randomly to be equal to

$$\boldsymbol{x}(0) = \begin{bmatrix} 0.3420 & 0.3200 & 0.0178 & -0.287 & -0.9497 \end{bmatrix}^T.$$

The initial conditions of the observer were set to zero. We note that the plots of the state variable x_3 and its estimate are undistinguishable because the estimate of x_3 is almost the same as y_2, which is equal to x_3. ■

6. Relation with Other Unknown Input Observer Architectures

In this paper, we concentrated on the analysis and design of full-order observers that can be used to construct reduced-order observers. Our analysis can be extended to cover the case

$$\begin{aligned}
\dot{\boldsymbol{q}} &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{y}),
\end{aligned} \tag{22}$$

where the term $\boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\boldsymbol{q} - \boldsymbol{C}\boldsymbol{M}\boldsymbol{y})$ is not premultiplied by $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})$. However, this case leads to the observer analyzed in (Chen *et al.*, 1996; Chen and Patton, 1999) even though the approach adopted there is quite different. Indeed, we can equivalently represent the dynamics of the proposed full-order observer as follows:

$$\begin{aligned}
\dot{\boldsymbol{q}} &= \Big((\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\,\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} \Big)\boldsymbol{q} \\
&\quad + \Big(\Big[(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\,\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} \Big] \boldsymbol{M} + \boldsymbol{L} \Big)\boldsymbol{y} \\
&\quad + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\,\boldsymbol{B}_1\boldsymbol{u}_1 \\
&= (\boldsymbol{T}\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{q} + \boldsymbol{K}\boldsymbol{y} + \boldsymbol{T}\boldsymbol{B}_1\boldsymbol{u}_1, \\
\tilde{\boldsymbol{x}} &= \boldsymbol{q} + \boldsymbol{M}\boldsymbol{y},
\end{aligned}$$

where, using the notation similar to that in (Chen *et al.*, 1996; Chen and Patton, 1999),

$$\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}, \qquad \boldsymbol{K}_1 = \boldsymbol{L},$$

$$\boldsymbol{K}_2 = [\boldsymbol{T}\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}]\,\boldsymbol{M}, \quad \boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2.$$

In addition to that, the conditions for the existence of the full-order observer presented in (Chen *et al.*, 1996; Chen and Patton, 1999) and our observers are equivalent.

The observer given by (22) is also the same as the one proposed by Yang and Wilde (1988) and further analyzed by Darouach *et al.* (1994). The connections are as follows: (i) $\boldsymbol{M}$ is called $-\boldsymbol{E}$ in (Darouach *et al.*, 1994; Yang and Wilde, 1988), (ii) $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})$ corresponds to $\boldsymbol{P}$ there, (iii) $\boldsymbol{B}_1$ is $\boldsymbol{B}$ and $\boldsymbol{B}_2$ is $\boldsymbol{D}$ in (Darouach *et al.*, 1994; Yang and Wilde, 1988), iv) $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})$ corresponds to $\boldsymbol{N}$.

We now compare the reduced-order UIO proposed by Hou and Müller (1992) with our reduced-order UIO. Somewhat similar approach is proposed by Kudva *et al.* (1980). Hou and Müller first transform the system (3) into the form

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{I}_{m_2} \end{bmatrix} \boldsymbol{u}_2 \\
&= \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{B}_{12} \end{bmatrix} \boldsymbol{u}_1 + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{I}_{m_2} \end{bmatrix} \boldsymbol{u}_2.
\end{aligned}$$

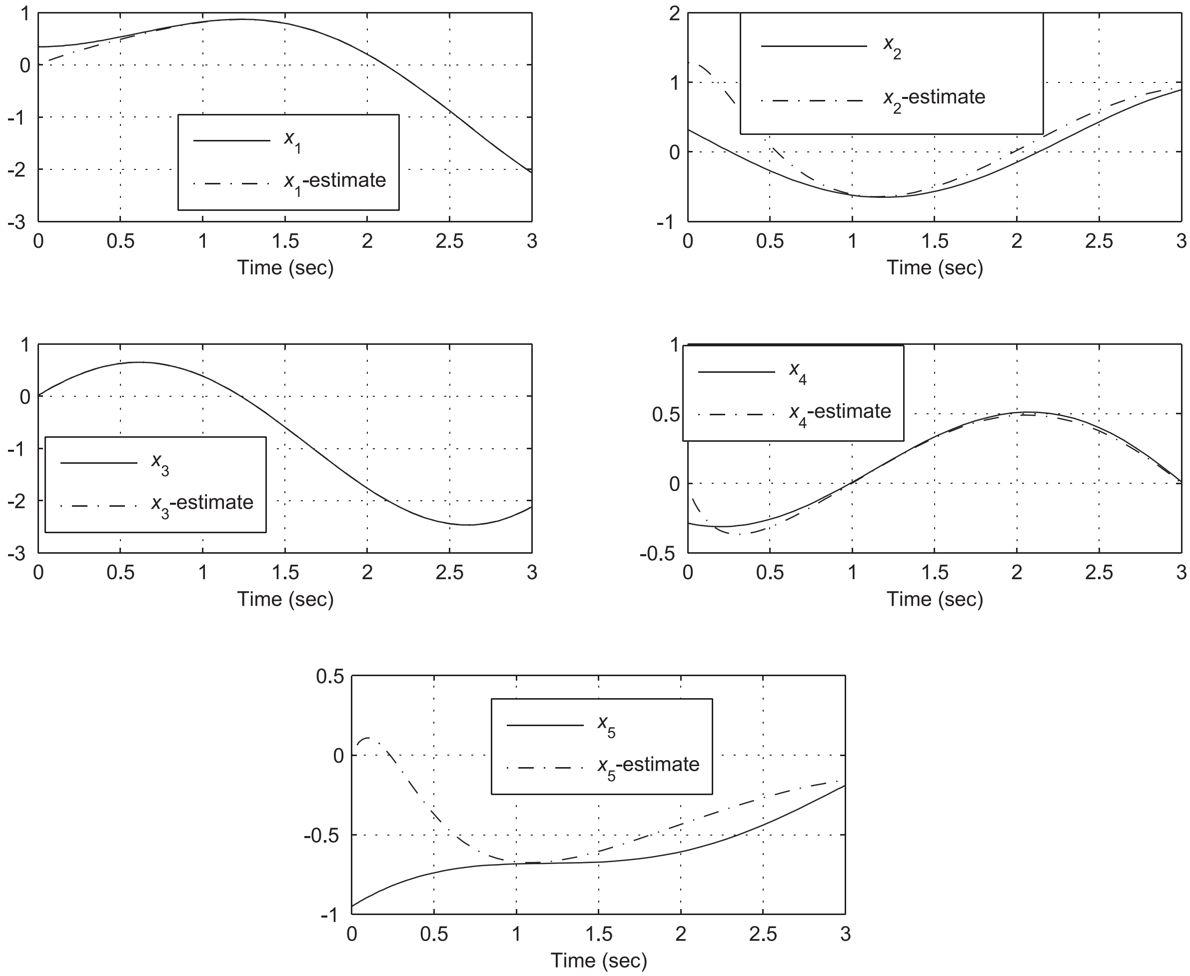

Fig. 1. Plots of x_i's and their estimates versus time for Example 2.

Note that $\boldsymbol{x}_1$ in the new coordinates is independent of $\boldsymbol{u}_2$ and we have

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{x}_2 + \boldsymbol{B}_{11}\boldsymbol{u}_1. \tag{23}$$

Let

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix},$$

where $\boldsymbol{C}_2 \in \mathbb{R}^{p\times m_2}$. Because, by assumption, $\text{rank}(\boldsymbol{C}\boldsymbol{B}_2) = \text{rank}\boldsymbol{B}_2 = m$, the submatrix $\boldsymbol{C}_2$ has a left inverse, $\boldsymbol{C}_2^\dagger$. Hence we can compute

$$\boldsymbol{x}_2 = -\boldsymbol{C}_2^\dagger \boldsymbol{C}_1 \boldsymbol{x}_1 + \boldsymbol{C}_2^\dagger \boldsymbol{y}. \tag{24}$$

Substituting the above into (23) gives

$$\dot{\boldsymbol{x}}_1 = \left(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{C}_2^\dagger \boldsymbol{C}_1\right)\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{C}_2^\dagger \boldsymbol{y} + \boldsymbol{B}_{11}\boldsymbol{u}_1.$$

Hou and Müller (1992) propose now to construct an observer for $\boldsymbol{x}_1$ using only known signals and then substitute the estimate of $\boldsymbol{x}_1$ into (24) to obtain an estimate of $\boldsymbol{x}_2$. Thus, the resulting architecture of the reduced-order UIO proposed by Hou and Müller, as well as their approach, differs from our design. Yet another approach to constructing reduced-order UIOs can be found in (Hui and Żak, 1993).

7. Sliding Mode Observer Design for Systems with Unknown Inputs

In this approach, we assume that $\boldsymbol{u}_2$ is bounded, that is, there exists a nonnegative real number, ρ, such that

$$\|\boldsymbol{u}_2(t)\| \le \rho \quad \text{for all } t.$$

Let $\hat{\boldsymbol{x}}$ be an estimate of $\boldsymbol{x}$. Let $\boldsymbol{e}$ denote the estimation error, that is,

$$\boldsymbol{e}(t) = \hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t).$$

The observability of $(\boldsymbol{A}, \boldsymbol{C})$ implies the existence of a matrix $\boldsymbol{L} \in \mathbb{R}^{n\times p}$ such that the matrix $(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})$ has prescribed (symmetric with respect to the real axis) eigenvalues in the open left-half plane. Because $(\boldsymbol{A}-\boldsymbol{LC})$ is asymptotically stable, for any $\boldsymbol{Q}=\boldsymbol{Q}^T>0$, there is a unique $\boldsymbol{P}=\boldsymbol{P}^T>0$ such that

$$(\boldsymbol{A}-\boldsymbol{LC})^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{LC})=-\boldsymbol{Q}. \tag{25}$$

We choose $\boldsymbol{Q}$, if possible, so that for some $\boldsymbol{F}\in\mathbb{R}^{m_2\times p}$,

$$\boldsymbol{FC}=\boldsymbol{B}_2^T\boldsymbol{P}. \tag{26}$$

We need this technical condition to ensure the realizablity of the observer.

To proceed, we define the vector function

$$\boldsymbol{E}(\boldsymbol{e},\eta)=\begin{cases}\eta\dfrac{\boldsymbol{FCe}}{\|\boldsymbol{FCe}\|_2} & \text{for} \quad \boldsymbol{FCe}\neq\boldsymbol{0},\\ \boldsymbol{r}\in\mathbb{R}^{m_2},\ \|\boldsymbol{r}\|_2\le\eta & \text{for} \quad \boldsymbol{FCe}=\boldsymbol{0},\end{cases}$$

where $\eta\ge\rho$ is a design parameter. In the case of single-input single-output plant, we can write

$$\boldsymbol{E}(\boldsymbol{e},\eta)=\eta\,\mathrm{sign}\,(\boldsymbol{FCe}).$$

We note that

$$\boldsymbol{Ce}=\boldsymbol{C}(\hat{\boldsymbol{x}}-\boldsymbol{x})=\hat{\boldsymbol{y}}-\boldsymbol{y}.$$

The vector function $\boldsymbol{E}(\boldsymbol{e},\eta)$ is an essential ingredient of the sliding mode observer that we present next. When implementing the function $\boldsymbol{E}$, we use the output measurements $\hat{\boldsymbol{y}}$ and $\boldsymbol{y}$, that is, instead of using $E(\boldsymbol{e},\eta)$, we utilize

$$\boldsymbol{E}(\hat{\boldsymbol{y}},\boldsymbol{y},\eta)=\begin{cases}\eta\dfrac{\boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y})}{\|\boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y})\|_2} & \text{for} \quad \boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y})\neq\boldsymbol{0},\\ \boldsymbol{r}\in\mathbb{R}^{m_2},\ \|\boldsymbol{r}\|_2\le\eta & \text{for} \quad \boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y})=\boldsymbol{0}.\end{cases}$$

Hence for the case of a single-input single-output plant, we have

$$\boldsymbol{E}(\hat{y},y,\eta)=\eta\,\mathrm{sign}\,(F(\hat{y}-y)).$$

Using arguments similar to those found in (Walcott and Żak, 1987), we can show that the state $\hat{\boldsymbol{x}}$ of the dynamical system

$$\dot{\hat{\boldsymbol{x}}}=\boldsymbol{A}\hat{\boldsymbol{x}}+\boldsymbol{B}_1\boldsymbol{u}_1+\boldsymbol{L}(\boldsymbol{y}-\hat{\boldsymbol{y}})-\boldsymbol{B}_2\boldsymbol{E}(\hat{\boldsymbol{y}},\boldsymbol{y},\eta) \tag{27}$$

for $\eta\ge\rho$ is an asymptotic estimate of the state $\boldsymbol{x}$ of the system described by (1) and (2), that is,

$$\lim_{t\to\infty}\boldsymbol{e}(t)=\lim_{t\to\infty}(\hat{\boldsymbol{x}}(t)-\boldsymbol{x}(t))=\boldsymbol{0}.$$

To prove the above statement using Lyapunov's type of arguments, first represent (27) as

$$\dot{\hat{\boldsymbol{x}}}=(\boldsymbol{A}-\boldsymbol{LC})\hat{\boldsymbol{x}}+\boldsymbol{Ly}+\boldsymbol{B}_1\boldsymbol{u}_1-\boldsymbol{B}_2\boldsymbol{E}(\boldsymbol{e},\eta).$$

Then construct the differential equation describing the dynamics of the estimation error $\boldsymbol{e}$,

$$\dot{\boldsymbol{e}}=\dot{\hat{\boldsymbol{x}}}-\dot{\boldsymbol{x}}=(\boldsymbol{A}-\boldsymbol{LC})\boldsymbol{e}-\boldsymbol{B}_2\boldsymbol{u}_2-\boldsymbol{B}_2\boldsymbol{E}(\boldsymbol{e},\eta), \tag{28}$$

and show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}^T\boldsymbol{Pe}\right)=-\boldsymbol{e}^T\boldsymbol{Qe}<0,$$

which implies

$$\lim_{t\to\infty}\boldsymbol{e}(t)=\boldsymbol{0}.$$

It follows from the above that the estimation error is insensitive to the uncertainty modeled by the term $\boldsymbol{B}_2\boldsymbol{u}_2$. In summary, the design of the observer proposed by Walcott and Żak (1987) for a system modeled by the quadruple $(\boldsymbol{A},\boldsymbol{B}_1,\boldsymbol{B}_2,\boldsymbol{C})$ can be thought of as finding a pair of matrices $(\boldsymbol{P},\boldsymbol{F})$ satisfying (25) and (26) for some $\boldsymbol{L}$ and $\boldsymbol{Q}$. Edwards and Spurgeon (1998) (see also (Saif and Xiong, 2003)) present necessary and sufficient conditions for the existence of the above observer, which are

(i) rank $\boldsymbol{B}_2=\mathrm{rank}\,\boldsymbol{CB}_2=r$;

(ii) the system zeros of the triple $(\boldsymbol{A},\boldsymbol{B}_2,\boldsymbol{C})$ are in the open left-hand complex plane, that is,

$$\mathrm{rank}\begin{bmatrix} s\boldsymbol{I}_n-\boldsymbol{A} & \boldsymbol{B}_2\\ \boldsymbol{C} & \boldsymbol{O}\end{bmatrix}=n+r$$

for all s such that $\mathrm{Re}(s)\ge 0$.

It is interesting to note that the above conditions are also necessary and sufficient for the existence of the observers with unknown inputs of (Hui and Żak, 1993) as well as the unknown input observers (UIOs) analyzed by us in the previous sections.

8. Sliding Mode Observer Construction

We first present a lemma that will serve us as a platform for the design of the sliding mode observer for uncertain systems. The lemma is a minor modification of Lemma 1 of Corless and Tu (1998), who proved it constructively using a singular value decomposition approach. We offer a different constructive proof using the Q-R decomposition.

Lemma 2. *For a triple* $(\boldsymbol{A},\boldsymbol{B}_2,\boldsymbol{C})\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m_2}\times\mathbb{R}^{p\times n}$,

$$\mathrm{rank}\,\boldsymbol{B}_2=\mathrm{rank}\,(\boldsymbol{CB}_2)=r, \tag{29}$$

if and only if there exist nonsingular matrices $\boldsymbol{T}$ *and* $\boldsymbol{S}$ *such that*

$$\boldsymbol{TAT}^{-1}=\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix},\quad \boldsymbol{TB}_2=\begin{bmatrix}\boldsymbol{B}_{2_1}\\ \boldsymbol{O}\end{bmatrix},$$

$$\boldsymbol{SCT}^{-1}=\begin{bmatrix}\boldsymbol{I}_r & \boldsymbol{O}\\ \boldsymbol{O} & C_{22}\end{bmatrix}, \tag{30}$$

where $\boldsymbol{A}_{11} \in \mathbb{R}^{r\times r}$, $\boldsymbol{A}_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, $\boldsymbol{B}_{2_1} \in \mathbb{R}^{r\times m_2}$, *rank* $\boldsymbol{B}_{2_1} = r$, *and* $\boldsymbol{C}_{22} \in \mathbb{R}^{(p-r)\times(n-r)}$.

Proof. (Necessity) The proof is constructive. Using the Q-R decomposition applied to $\boldsymbol{B}_2$, we obtain

$$\boldsymbol{B}_2 = \boldsymbol{Q}_{B_2}\boldsymbol{R}_{B_2},$$

where $\boldsymbol{Q}_{B_2} \in \mathbb{R}^{n\times n}$ is a unitary matrix and the matrix $\boldsymbol{R}_{B_2} \in \mathbb{R}^{n\times m_2}$ is upper triangular, where

$$\operatorname{rank} \boldsymbol{R}_{B_2} = r.$$

Let $\boldsymbol{T}_1 = \boldsymbol{Q}_{B_2}^{-1}$. Then we obtain

$$\boldsymbol{T}_1\boldsymbol{B}_2 = \begin{bmatrix} \tilde{\boldsymbol{B}}_{2_1} \\ \boldsymbol{O} \end{bmatrix},$$

where $\tilde{\boldsymbol{G}}_1 \in \mathbb{R}^{r\times m_2}$. We next partition the matrix $\boldsymbol{C}\boldsymbol{T}_1^{-1}$ as follows:

$$\boldsymbol{C}\boldsymbol{T}_1^{-1} = \begin{bmatrix} \tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{C}}_2 \end{bmatrix},$$

where $\tilde{\boldsymbol{C}}_1 \in \mathbb{R}^{p\times r}$. Note that

$$\boldsymbol{C}\boldsymbol{B}_2 = \left(\boldsymbol{C}\boldsymbol{T}_1^{-1}\right)(\boldsymbol{T}_1\boldsymbol{B}_2) = \tilde{\boldsymbol{C}}_1\tilde{\boldsymbol{B}}_{2_1}.$$

By the hypothesis of the lemma, $\operatorname{rank} \boldsymbol{B}_2 = \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = r$. Hence

$$\operatorname{rank} \tilde{\boldsymbol{C}}_1 = r.$$

Applying the Q-R decomposition to $\tilde{\boldsymbol{C}}_1$ yields

$$\tilde{\boldsymbol{C}}_1 = \boldsymbol{Q}_{\tilde{C}_1}\boldsymbol{R}_{\tilde{C}_1},$$

where

$$\boldsymbol{R}_{\tilde{C}_1} = \begin{bmatrix} \boldsymbol{C}_{11} \\ \boldsymbol{O} \end{bmatrix} \quad \text{and} \quad \det \boldsymbol{C}_{11} \neq 0.$$

Note that $\boldsymbol{C}_{11} \in \mathbb{R}^{r\times r}$. Let $\boldsymbol{S} = \boldsymbol{Q}_{\tilde{C}_1}^{-1}$. Then

$$\boldsymbol{S}\boldsymbol{C}\boldsymbol{T}_1^{-1} = \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{O} & \boldsymbol{C}_{22} \end{bmatrix}.$$

Postmultiplying $\boldsymbol{S}\boldsymbol{C}\boldsymbol{T}_1^{-1}$ by

$$\boldsymbol{T}_2^{-1} = \begin{bmatrix} \boldsymbol{C}_{11}^{-1} & -\boldsymbol{C}_{11}^{-1}\boldsymbol{C}_{12} \\ \boldsymbol{O} & \boldsymbol{I}_{n-r} \end{bmatrix}$$

gives

$$\boldsymbol{S}\boldsymbol{C}\boldsymbol{T}_1^{-1}\boldsymbol{T}_2^{-1} = \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{O} & \boldsymbol{C}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{C}_{11}^{-1} & -\boldsymbol{C}_{11}^{-1}\boldsymbol{C}_{12} \\ \boldsymbol{O} & \boldsymbol{I}_{n-r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{22} \end{bmatrix}.$$

We then have $\boldsymbol{T} = \boldsymbol{T}_2\boldsymbol{T}_1$.

(Sufficiency) By inspection. ∎

Notice that the systems

$$\left.\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_2\boldsymbol{u}_2, \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\tilde{\boldsymbol{x}} + \boldsymbol{T}\boldsymbol{B}_2\boldsymbol{u}_2, \\ \tilde{\boldsymbol{y}} &= \boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{-1}\boldsymbol{x} \end{aligned}\right\}$$

have the same system zeros, that is, their system matrices have the same rank for all $s \in \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. One can prove the above statement by applying Sylvester's inequalities (see, for example, Gantmacher, (1990, pp. 65, 66)) to the right-hand side of the following relation between the system matrices of the above models:

$$\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} & \boldsymbol{T}\boldsymbol{B}_2 \\ \boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{-1} & \boldsymbol{O} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{S} \end{bmatrix}\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{T}^{-1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_q \end{bmatrix}. \tag{31}$$

Lemma 3. *Assume that* $\operatorname{rank} \boldsymbol{B}_2 = \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = r$. *Then, the pair* $(\boldsymbol{A}_{22}, \boldsymbol{C}_{22})$ *is detectable if and only if*

$$\operatorname{rank} \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} = n + r \tag{32}$$

for all s *such that* $\operatorname{Re}(s) \geq 0$.

Proof. By assumption, $\operatorname{rank} \boldsymbol{B}_2 = \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = r$. By Lemma 2, the above condition is equivalent to the existence of nonsingular matrices $\boldsymbol{T}$ and $\boldsymbol{S}$ such that

$$\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}, \quad \boldsymbol{T}\boldsymbol{B}_2 = \begin{bmatrix} \boldsymbol{B}_{2_1} \\ \boldsymbol{O} \end{bmatrix},$$

$$\boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{22} \end{bmatrix},$$

where $\boldsymbol{B}_{2_1} \in \mathbb{R}^{r\times m_2}$ and $\operatorname{rank} \boldsymbol{B}_{2_1} = r$. Then, for any $s \in \mathbb{C}$,

$$\operatorname{rank} \begin{bmatrix} s\boldsymbol{I}_r - \boldsymbol{A}_{11} & -\boldsymbol{A}_{12} & \boldsymbol{B}_{2_1} \\ -\boldsymbol{A}_{21} & s\boldsymbol{I}_{n-r} - \boldsymbol{A}_{22} & \boldsymbol{O} \\ \boldsymbol{I}_r & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{22} & \boldsymbol{O} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -\boldsymbol{A}_{12} & \boldsymbol{B}_{2_1} \\ s\boldsymbol{I}_{n-r} - \boldsymbol{A}_{22} & \boldsymbol{O} \\ \boldsymbol{C}_{22} & \boldsymbol{O} \end{bmatrix} + r = \operatorname{rank} \begin{bmatrix} s\boldsymbol{I}_{n-r} - \boldsymbol{A}_{22} \\ \boldsymbol{C}_{22} \end{bmatrix} + 2r.$$

It follows from the above that the pair $(\boldsymbol{A}_{22}, \boldsymbol{C}_{22})$ is detectable if and only if the rank condition (32) holds.

■

Note that if $m_2 = p = r$, then

$$\operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} = n + r$$

for all s such that $\operatorname{Re}(s) \geq 0$ if and only if the matrix $\boldsymbol{A}_{22}$ is asymptotically stable.

The following theorem appears in (Corless and Tu 1998, Lem. 3, p. 760). A related result was obtained by (Edwards and Spurgeon 1998, Prop. 6.2, p. 138). An algorithm for constructing matrices $\boldsymbol{L}$, $\boldsymbol{F}$ and $\boldsymbol{P}$ that are essential ingredients of the sliding mode observer for uncertain systems is contained in the proof of the theorem.

Theorem 3. *There exists a triple of matrices* $(\boldsymbol{L}, \boldsymbol{F}, \boldsymbol{P}) \in \mathbb{R}^{n\times p} \times \mathbb{R}^{m_2\times p} \times \mathbb{R}^{n\times n}$ *such that*

$$(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}) < 0 \tag{33}$$

and

$$\boldsymbol{F}\boldsymbol{C} = \boldsymbol{B}_2^T\boldsymbol{P} \tag{34}$$

if and only if

(i) $\operatorname{rank}\boldsymbol{B}_2 = \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = r$*;*

(ii) the system zeros of the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ *are in the open left-hand complex plane, that is,*

$$\operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B}_2 \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} = n + r$$

for all s *such that* $\operatorname{Re}(s) \geq 0$.

Proof. (Sufficiency) We follow the arguments of Corless and Tu (1998). By Lemma 2, the condition $\operatorname{rank}\boldsymbol{B}_2 = \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = r$ is equivalent to the existence of nonsingular matrices $\boldsymbol{T}$ and $\boldsymbol{S}$ such that

$$\hat{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix},$$

$$\hat{\boldsymbol{B}}_2 = \boldsymbol{T}\boldsymbol{G} = \begin{bmatrix} \boldsymbol{B}_{2_1} \\ \boldsymbol{O} \end{bmatrix},$$

$$\hat{\boldsymbol{C}} = \boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{22} \end{bmatrix},$$

where $\boldsymbol{B}_{2_1} \in \mathbb{R}^{r\times m_2}$ and $\operatorname{rank}\boldsymbol{B}_{2_1} = r$. Let

$$\hat{\boldsymbol{P}} = \boldsymbol{T}^{-T}\boldsymbol{P}\boldsymbol{T}^{-1}, \quad \hat{\boldsymbol{L}} = \boldsymbol{T}\boldsymbol{L}\boldsymbol{S}^{-1}, \quad \text{and} \quad \hat{\boldsymbol{F}} = \boldsymbol{F}\boldsymbol{S}^{-1}. \tag{35}$$

To proceed, note that condition (ii) is equivalent to the existence of a matrix $\boldsymbol{L}_{22}$ such that the eigenvalues of $(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22})$ are all in the open left-half complex plane. Then, for any symmetric positive definite $\boldsymbol{Q}_{22}$, the symmetric solution $\boldsymbol{P}_{22}$ to the Lyapunov matrix equation,

$$(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22})^T\boldsymbol{P}_{22} + \boldsymbol{P}_{22}(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22}) = -\boldsymbol{Q}_{22},$$

is also positive definite. Let

$$\hat{\boldsymbol{L}} = \begin{bmatrix} \kappa\boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L}_{22} \end{bmatrix},$$

where $\kappa > 0$ is a design parameter whose lower bound is determined in the following deliberations. We have

$$\begin{aligned} \hat{\boldsymbol{A}} - \hat{\boldsymbol{L}}\hat{\boldsymbol{C}} &= \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \\ &\quad - \begin{bmatrix} \kappa\boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}_{11} - \kappa\boldsymbol{I}_r & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22} \end{bmatrix}. \end{aligned}$$

Let

$$\hat{\boldsymbol{P}} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{P}_{22} \end{bmatrix} \quad \text{and} \quad \hat{\boldsymbol{Q}} = \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{12}^T & \boldsymbol{Q}_{22} \end{bmatrix}.$$

Using the above, we obtain

$$\begin{aligned} -\hat{\boldsymbol{Q}} &= (\hat{\boldsymbol{A}} - \hat{\boldsymbol{L}}\hat{\boldsymbol{C}})^T\hat{\boldsymbol{P}} + \hat{\boldsymbol{P}}(\hat{\boldsymbol{A}} - \hat{\boldsymbol{L}}\hat{\boldsymbol{C}}) \\ &= \begin{bmatrix} \boldsymbol{A}_{11}^T - \kappa\boldsymbol{I}_r & \boldsymbol{A}_{21}^T \\ \boldsymbol{A}_{12}^T & (\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22})^T \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{P}_{22} \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{P}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{A}_{11} - \kappa\boldsymbol{I}_r & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}_{11}^T - \kappa\boldsymbol{I}_r & \boldsymbol{A}_{21}^T\boldsymbol{P}_{22} \\ \boldsymbol{A}_{12}^T & (\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22})^T\boldsymbol{P}_{22} \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{A}_{11} - \kappa\boldsymbol{I}_r & \boldsymbol{A}_{12} \\ \boldsymbol{P}_{22}\boldsymbol{A}_{21} & \boldsymbol{P}_{22}(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22}) \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}_{11}^T + \boldsymbol{A}_{11} - 2\kappa\boldsymbol{I}_r & \boldsymbol{A}_{21}^T\boldsymbol{P}_{22} + \boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^T + \boldsymbol{P}_{22}\boldsymbol{A}_{21} & -\boldsymbol{Q}_{22} \end{bmatrix} \\ &= \begin{bmatrix} -\boldsymbol{Q}_{11} & -\boldsymbol{Q}_{12} \\ -\boldsymbol{Q}_{12}^T & -\boldsymbol{Q}_{22} \end{bmatrix}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} \boldsymbol{Q}_{22} = -\Big(&(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22})^T\boldsymbol{P}_{22} \\ &+ \boldsymbol{P}_{22}(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22})\Big) \end{aligned}$$

is positive definite by construction. Our goal is to obtain a lower bound on the parameter κ that would yield a positive definite $\hat{\boldsymbol{Q}}$. Using the Schur complement of the positive definite $\boldsymbol{Q}_{22}$, we have that $\hat{\boldsymbol{Q}}$ is positive definite if and only if

$$\boldsymbol{Q}_{11} > \boldsymbol{Q}_{12}\boldsymbol{Q}_{22}^{-1}\boldsymbol{Q}_{12}^T.$$

Employing the above in (36), we obtain

$$2\kappa\boldsymbol{I}_r - \left(\boldsymbol{A}_{11}^T + \boldsymbol{A}_{11}\right) > \left(\boldsymbol{A}_{21}^T\boldsymbol{P}_{22} + \boldsymbol{A}_{12}\right) \times\boldsymbol{Q}_{22}^{-1}\left(\boldsymbol{A}_{12}^T + \boldsymbol{P}_{22}\boldsymbol{A}_{21}\right).$$

Hence, $\hat{\boldsymbol{Q}}$ is positive definite if

$$\kappa > \frac{1}{2}\lambda_{\max}\Big(\boldsymbol{A}_{11}^T + \boldsymbol{A}_{11} + \left(\boldsymbol{A}_{21}^T\boldsymbol{P}_{22} + \boldsymbol{A}_{12}\right) \times \boldsymbol{Q}_{22}^{-1}\left(\boldsymbol{A}_{12}^T + \boldsymbol{P}_{22}\boldsymbol{A}_{21}\right)\Big).$$

Let

$$\hat{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{B}_{2_1}^T & \boldsymbol{O} \end{bmatrix}.$$

Then, it is easy to see that

$$(\hat{\boldsymbol{A}} - \hat{\boldsymbol{L}}\hat{\boldsymbol{C}})^T\hat{\boldsymbol{P}} + \hat{\boldsymbol{P}}(\hat{\boldsymbol{A}} - \hat{\boldsymbol{L}}\hat{\boldsymbol{C}}) < 0 \tag{37}$$

and

$$\hat{\boldsymbol{F}}\hat{\boldsymbol{C}} = \hat{\boldsymbol{B}}_2^T\hat{\boldsymbol{P}}, \tag{38}$$

which are the conditions (33) and (34) in the new basis. Hence, the proof of the sufficiency conditions for the existence of the desired triple of matrices $(\boldsymbol{L}, \boldsymbol{F}, \boldsymbol{P})$ is complete.

(Necessity) See (Corless and Tu, 1998, p. 761). ■

We now summarize the above analysis in the form of the following design algorithm:

Sliding-Mode Observer Design Algorithm

Given a quadruple of matrices $(\boldsymbol{A}, \boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{C})$ modeling the plant, do the following:

1. Check that the rank condition, $\operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = \operatorname{rank}\boldsymbol{B}_2$, is satisfied.
   If $\operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) \neq \operatorname{rank}\boldsymbol{B}_2$, the sling-mode observer cannot be constructed, STOP.

2. Transform the triple $(\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{B}_2)$.
   Use the method of the proof of Lemma 2 to construct nonsingular matrices $\boldsymbol{T}$ and $\boldsymbol{S}$ and compute

$$\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}, \quad \boldsymbol{T}\boldsymbol{B}_2 = \begin{bmatrix} \boldsymbol{B}_{2_1} \\ \boldsymbol{O} \end{bmatrix},$$

$$\boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{22} \end{bmatrix},$$

   where $\boldsymbol{B}_{2_1} \in \mathbb{R}^{r\times m_2}$ and $\operatorname{rank}\boldsymbol{B}_{2_1} = r$.

3. Check the detectability of $(\boldsymbol{A}_{22}, \boldsymbol{C}_{22})$.
   If the pair $(\boldsymbol{A}_{22}, \boldsymbol{C}_{22})$ is not detectable, the sliding-mode observer cannot be constructed, STOP.

4. Construct a matrix $\boldsymbol{L}_{22}$ so that the eigenvalues of $(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22})$ are in the open left-half plane.

5. Choose a positive definite $\boldsymbol{Q}_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$ and solve for positive definite $\boldsymbol{P}_{22}$ the Lyapunov matrix equation,

$$(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22})^T\boldsymbol{P}_{22} + \boldsymbol{P}_{22}(\boldsymbol{A}_{22} - \boldsymbol{L}_{22}\boldsymbol{C}_{22}) = -\boldsymbol{Q}_{22}.$$

6. Choose κ that satisfies the condition

$$\kappa > \frac{1}{2}\lambda_{\max}\Big(\boldsymbol{A}_{11}^T + \boldsymbol{A}_{11} + \left(\boldsymbol{A}_{21}^T\boldsymbol{P}_{22} + \boldsymbol{A}_{12}\right) \times \boldsymbol{Q}_{22}^{-1}\left(\boldsymbol{A}_{12}^T + \boldsymbol{P}_{22}\boldsymbol{A}_{21}\right)\Big).$$

7. Construct

$$\hat{\boldsymbol{L}} = \begin{bmatrix} \kappa\boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L}_{22} \end{bmatrix}, \quad \hat{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{B}_{2_1}^T & \boldsymbol{O} \end{bmatrix}.$$

8. Compute

$$\boldsymbol{L} = \boldsymbol{T}^{-1}\hat{\boldsymbol{L}}\boldsymbol{S}, \quad \boldsymbol{F} = \hat{\boldsymbol{F}}\boldsymbol{S}.$$

9. Construct the observer

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{L}(\boldsymbol{y} - \hat{\boldsymbol{y}}) - \boldsymbol{B}_2\boldsymbol{E}(\hat{\boldsymbol{y}}, \boldsymbol{y}, \eta),$$

where

$$\boldsymbol{E}(\hat{\boldsymbol{y}}, \boldsymbol{y}, \eta) = \begin{cases} \eta\dfrac{\boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y})}{\|\boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y})\|_2} & \text{for} \quad \boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y}) \neq \boldsymbol{0}, \\ \boldsymbol{r} \in \mathbb{R}^q, \|\boldsymbol{r}\|_2 \leq \eta & \text{for} \quad \boldsymbol{F}(\hat{\boldsymbol{y}}-\boldsymbol{y}) = \boldsymbol{0}. \end{cases}$$

Example 3. We consider the same fifth-order lateral axis model of an L-1011 fixed-wing aircraft that we considered in Example 2. We have $\operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = \operatorname{rank}\boldsymbol{B}_2$. We use Lemma 2 to compute the transformation matrices $\boldsymbol{T}$ and $\boldsymbol{S}$,

$$\boldsymbol{T} = \begin{bmatrix} -0.0000 & -0.9106 & 0.4125 & 0.0245 & 0.9106 \\ 0.0000 & -0.4124 & -0.9110 & 0.0103 & 0.4124 \\ -0.9998 & -0.0005 & 0.0000 & -0.0179 & 0 \\ -0.0179 & 0.0265 & -0.0008 & 0.9995 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

and

$$S = \begin{bmatrix} -0.9106 & 0.4125 & 0.0245 & 0.0000 \\ -0.4124 & -0.9110 & 0.0103 & 0.0000 \\ 0.0265 & -0.0008 & 0.9996 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}.$$

We next transform the given model into the new coordinates and obtain the pair

$$A_{22} = \begin{bmatrix} 0.0006 & 0.0025 & 0.0178 \\ -0.0368 & -0.0993 & -0.9971 \\ -0.0002 & 0.0133 & -0.0004 \end{bmatrix}$$

and $$C_{22} = \begin{bmatrix} -0.0179 & 0.9998 & -0.0265 \\ -0.9998 & -0.0179 & 0.0000 \end{bmatrix}.$$

The pair (A_{22}, C_{22}) is detectable and we use MATLAB's `place` function to construct the gain matrix L_{22}, so that the eigenvalues of the matrix $(A_{22} - L_{22}C_{22})$ are located at $-3, -4, -5$, where

$$L_{22} = \begin{bmatrix} -0.1441 & -4.0022 \\ 7.5018 & 0.0744 \\ -14.9883 & -0.4586 \end{bmatrix}.$$

We then solve the Lyapunov matrix equation,

$$(A_{22} - L_{22}C_{22})^T P_{22} + P_{22}(A_{22} - L_{22}C_{22}) = -I_3,$$

to obtain

$$P_{22} = \begin{bmatrix} 0.1273 & -0.0445 & -0.0236 \\ -0.0445 & 0.9995 & 0.4735 \\ -0.0236 & 0.4735 & 0.3059 \end{bmatrix}.$$

After that we compute

$$\frac{1}{2}\lambda_{\max}\Big(A_{11}^T + A_{11} + \left(A_{21}^T P_{22} + A_{12}\right) \times Q_{22}^{-1}\left(A_{12}^T + P_{22}A_{21}\right)\Big) = 12.6790$$

and select

$$\kappa = 13.6790.$$

Finally we construct

$$L = T^{-1}\hat{L}S = \begin{bmatrix} 0.0003 & 0.0000 & 0.0100 & 4.0002 \\ 13.2773 & 0.0115 & -15.1363 & -0.4544 \\ 0.0001 & 13.6790 & 0.0044 & -0.0001 \\ -0.1532 & 0.0044 & 7.9048 & 0.1580 \\ -0.3976 & 0.0114 & -14.9831 & -0.4586 \end{bmatrix}$$

and

$$F = \hat{F}S = \begin{bmatrix} -0.7440 & 0.3370 & 0.0200 & 0.0000 \\ -0.0320 & -1.1200 & 0.0000 & 0.0000 \end{bmatrix}.$$

We selected $\eta = 7$. We then constructed the observer

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) - B_2 E(\hat{y}, y, \eta),$$

where

$$E(\hat{y}, y, \eta) = \begin{cases} \eta \dfrac{F(\hat{y} - y)}{\|F(\hat{y} - y)\|_2 + \mu} & \text{for} \quad F(\hat{y} - y) \neq 0 \\ r \in \mathbb{R}^q, \|r\|_2 \leq \eta & \text{for} \quad F(\hat{y} - y) = 0, \end{cases}$$

where $\mu = 0.0005$. The parameter μ was introduced to smooth out the discontinuity and facilitate the simulations. In Fig. 2, we show the plots of system state variables and their estimates versus time. The initial conditions are the same as in Example 2. ♦

For design methods of the Walcott-Żak sliding mode observer using linear matrix inequalities (LMIs), see (Choi and Ro, 2005; Xiang *et al.*, 2005).

9. Future Work

The effectiveness of unknown input observers (UIOs) in real-life applications needs to be investigated. A successful application of UIOs to a DC servo motor system was reported by Chang *et al.* (1997). On the the other hand, Millerioux and Daafouz (2004) proposed UIO architectures for switched linear discrete systems. Röbenack and Lynch (2004) presented a method of observer design for a class of nonlinear plants which yields almost linear observation error dynamics. This method looks like a promising tool to be used to extend our approach to a class of nonlinear plants. Another promising application of the proposed UIOs is in the area of fault detection and isolation—see, for example, (Edwards *et al.*, 2000) for an application of sliding mode observers for fault detection and isolation. In his practical guide for the selection and installation of observers in control systems, Ellis (2002, p. 3) writes: "Observers add complexity to the system and require computational resources. They may be less robust than physical sensors, especially when plant parameters change substantially during operation. Still, an observer applied with skill can bring substantial performance benefits and does so, in many cases, while reducing cost or increasing reliability." Examples of impressive applications of nonlinear observers to the control of electric machinery can be found in (Dawson *et al.*, 1998; Solsona and Valla, 2003; Utkin *et al.*, 1999). The above applications should

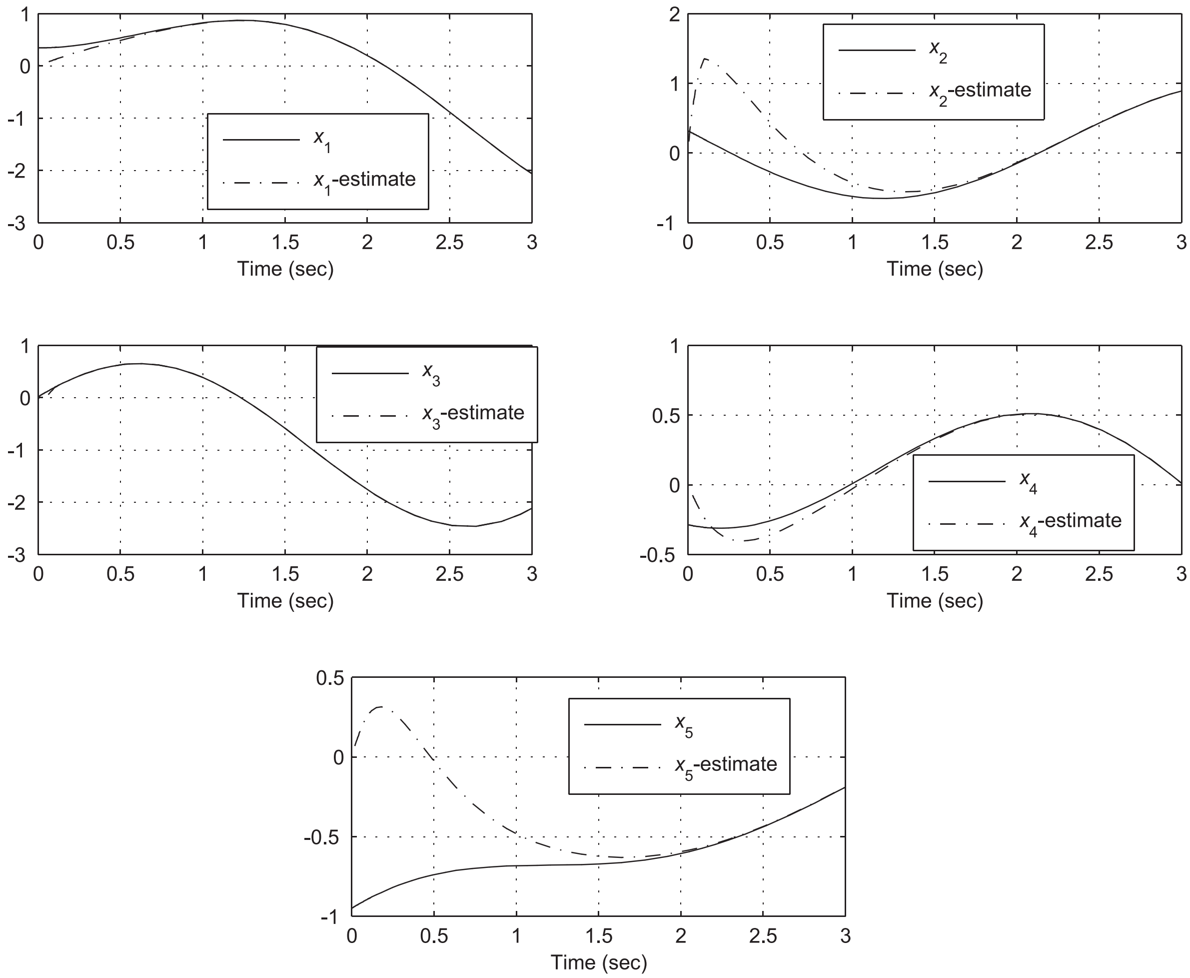

Fig. 2. Plots of x_is and their estimates versus time for Example 3.

serve as a motivation to generalize our techniques to nonlinear plants.

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