AN OBSERVABILITY PROBLEM FOR A CLASS OF UNCERTAIN–PARAMETER LINEAR DYNAMIC SYSTEMS

KRZYSZTOF OPRZĘDKIEWICZ

AGH University of Science and Technology
al. A. Mickiewicza 30, 30–059 Cracow, Poland
e-mail: kop@uci.agh.edu.pl

An observability problem for a class of linear, uncertain-parameter, time-invariant dynamic SISO systems is discussed. The class of systems under consideration is described by a finite dimensional state-space equation with an interval diagonal state matrix, known control and output matrices and a two-dimensional uncertain parameter space. For the system considered a simple geometric interpretation of the system spectrum can be given. The geometric interpretation of the system spectrum is the base for defining observability and non-observability areas for the discussed system. The duality principle allows us to test observablity using controllability criteria. For the uncertain-parameter system considered, some controllability criteria presented in the author's previous papers are used. The results are illustrated with numerical examples.

Keywords: linear uncertain-parameter dynamic systems, observability

1. Introduction

In most real situations, while building a mathematical model for a real system, we are not able to perform an accurate identification of model parameters. The reasons behind this situation are well known. Mathematical models with unknown, interval or uncertain parameters describe a very broad class of control systems. In order to steer these objects, special versions of control algorithms should be used. Systems with uncertain parameters can be analysed using various approaches. One is the application of interval analysis.

Most papers are concentrated on stability analysis of interval dynamic systems, because the stability of a control system is a fundamental issue. But not only the satisfaction of stability conditions guarantees a proper construction and work of the control system. Another important feature, which often has be to be considered during control system construction, is observability. Observability always has to be tested during the construction of a state observer. For LQG purposes, an observability problem may be continuously monitored by on-line identification, e. g., with the Instrumental Variable Method.

In this paper observability analysis for a class of uncertain-parameter dynamic systems will be presented. The system under consideration is described by a linear, finite-dimensional, time-invariant state-space equation with an uncertain diagonal state matrix and known control and output matrices. As an example of this class of systems, an interval parabolic system, described by an abstract state-space equation in the Hilbert space, can be given (Oprzędkiewicz, 2003, Example 3).

2. Uncertain–Parameter Linear Dynamic System with a Diagonal State Matrix and a Two–Dimensional Uncertain Parameter Space

The definiton of the interval dynamic system can be found in many papers, (Jakubowska, 1999). In this paper we shall deal with the SISO uncertain-parameter time-invariant linear dynamic system described by the following state-space equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}, y(t) \in \mathbb{R}$. Denote by q the vector of uncertain model parameters. It can be expressed as $q = [q_1, q_2]^T$, $q \in Q$, where Q denotes the whole set of uncertain parameters. The elements of the vector q can be described as the following interval numbers:

$$q_1 = [\underline{q_1}, \overline{q_1}], \tag{2}$$

$$q_2 = [\underline{q_2}, \overline{q_2}]. \tag{3}$$

The set of uncertain model parameters Q has four vertices, defined as follows:

$$\begin{cases} q_{ll} &= [\underline{q_1}, \underline{q_2}], \\ q_{lh} &= [\underline{q_1}, \overline{q_2}], \\ q_{hl} &= [\overline{q_1}, \underline{q_2}], \\ q_{hh} &= [\overline{q_1}, \overline{q_2}]. \end{cases} \tag{4}$$

As an example of a real system described by (1)–(4), the interval parabolic system discussed in (Oprzędkiewicz, 2003), Example 3, can be given.

Assume that the eigenvalues λ_i of the state matrix A are linear functions of uncertain model parameters. This can be expressed as

$$\lambda_i(q) = c_{1_i} q_1 + c_{2_i} q_2 + d_i, \tag{5}$$

where c_{1_i}, c_{2_i} and d_i denote known real constants. The above implies that the state matrix A is an interval matrix (for a definition of the interval matrix, see, e.g., (Białas, 2002; Jakubowska, 1999)), dependent on uncertain model parameters. In the case considered the matrix A has the following form:

$$A = \begin{bmatrix} \lambda_1 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}_{n \times n}, \tag{6}$$

where λ_i is defined by (5). The interval state matrix A has four vertex matrices, which can be expressed as follows:

$$\begin{cases} A_{ll}(q) &= A(q_{ll}), \\ A_{lh}(q) &= A(q_{lh}), \\ A_{hl}(q) &= A(q_{hl}), \\ A_{hh}(q) &= A(q_{hh}), \end{cases} \tag{7}$$

where q_{ll}, q_{lh}, q_{hl} and q_{hh} denote the vertex vectors defined by (4). Now, we have to make elementary assumptions about the control matrix B and the output matrix C. We assume that the elements of these matrices are known real numbers and they are non-zero. These matrices have the following form:

$$B = [b_1, b_2, \ldots, b_n]^T, \tag{8}$$

where $b_i \neq 0$ for $i = 1, \ldots, n$,

$$C = [c_1, c_2, \ldots, c_n], \tag{9}$$

where $c_i \neq 0$ for $i = 1, \ldots, n$.

3. Geometric Interpretation of the Interval Spectrum

In the case considered the dimension of the uncertain parameter space equals 2. This fact allows us to formulate a simple geometric interpretation of the system spectrum in $\mathbb{R}^3$ (Oprzędkiewicz, 2003). At the beginning, note that the set of uncertain parameters Q in the plane of uncertain parameters q_1 and q_2 constitutes the rectangle

$$Q = \{q \in \mathbb{R}^2 : \underline{q_1} \le q_1 \le \overline{q_1}, \underline{q_2} \le q_2 \le \overline{q_2}\}. \tag{10}$$

The vertices of the rectangle Q have coordinates defined by (4). Furthermore, each eigenvalue λ_i of the interval diagonal state matrix A can be interpreted as a tetrahedron in $\mathbb{R}^3$:

$$\Lambda_i = \Big\{ \lambda_i(q) : q = [q_1, q_2]^T \in Q, \quad \lambda_i = c_{i_1} q_1 + c_{i_2} q_2 + d_i \Big\}, \tag{11}$$

where Q denotes the set of uncertain model parameters defined by (10). Consequently, the interval spectrum of the system considered can be defined as a set of tetrahedrons (11) in $\mathbb{R}^3$:

$$\Lambda(q) = \bigcup_{i=1}^{n} \Lambda_i. \tag{12}$$

4. Observability Problem for the System in Question

Observability for a system with uncertain parameters can be formulated in much the same way as in the case of a system with known parameters.

Definition 1. (*Observability of an uncertain-parameter system)* An uncertain-parameter finite-dimensional linear dynamic system described by the interval model (1)–(9) is *controllable* if and only if it is controllable for each vector $q \in Q$.

If system observability is equivalent to that of the pair of matrices (C, A), then the following definition can be given:

Definition 2. *(Observability of the pair $(C, A(q))$)* A pair of matrices $(C, A(q))$ (here C is a real matrix and $A(q)$ is an interval matrix) is *observable* if and only if it is observable for each vector $q \in Q$.

The definitions formulated above are very general and do not allow us to simply test the observability of an uncertain system with observability criteria for systems with known parameters. Fortunately, we can expect that the duality principle holds for the discussed uncertain-parameter system. Consequently, the controllability conditions formulated for the discussed uncertain-parameter system (Oprzędkiewicz, 2004) can be used for testing observability. A further analysis of the observability problem is dual to controllability analysis for the discussed system. It will be presented in the sequel.

5. Observability and Non-Observability Areas and Their Geometric Interpretation

Observability analysis for the uncertain system discussed in the previous sections will be continued after formulating additional assumptions and presenting some new ideas.

At the beginning, note that observability for an uncertain-parameter system is not a synonymous idea, because in the domain of uncertain parameters we can expect subareas where the system will not be observable and subareas where it will be observable. That is why we should now define the notions of *observability* and *non-observability* areas.

Definition 3. The set of uncertain system parameters for which the system is observable is called the *observability area* and will be denoted by Q_o.

Definition 4. The set of uncertain system parameters for which the system is not observable is called the *non-observability area* and will be denoted by Q_{no}.

The observability and non-observability areas have the following elementary properties:

$$Q_o \cap Q_{no} = Q, \quad Q_o \cup Q_{no} = 0. \tag{13}$$

Observability analysis for the discussed uncertain-parameter system will consist in a simple determination of observability or non-observability areas inside the area of uncertain model parameters Q.

To determine these areas in accordance with the duality principle, the approach presented in (Oprzędkiewicz, 2004) will be used.

First note that, according to the duality principle, the observability test requires that of the controllability of the pair (A^T, C^T). In our case, in accordance with (6) and (9), these matrices are

$$A^T = A, \quad C^T = [c_1, \ldots, c_n]^T. \tag{14}$$

Next we construct the controllability matrix S_o for the pair (A^T, C^T). It has the following general form:

$$S_o = \left[C^T | A^T C^T | \ldots (A^T)^{n-1} C^T \right]. \tag{15}$$

It is easy to see that $S_o = R^T$. This implies that the approach proposed in (Oprzędkiewicz, 2004) can be also applied to S_o or R. In our further deliberations we shall deal with the matrix S_o.

Furthermore, we can remark that if the state matrix A is a function of the vector of uncertain parameters q, then the matrix S_o is a function of this vector, too. This can be expressed as $S_o = S_o(q)$. It is easy to notice that at each point from the observability area $\operatorname{rank}(S_o(q)) = n$ and at each point from the non-observability area $\operatorname{rank}(S_o(q)) \neq n$.

In the case of a SISO system with known parameters, the observability test consists in rank checking for the matrix $S_o(q)$. In the case considered, this matrix is a square matrix of size $n \times n$. It is known that the rank of this matrix equals n if and only if all rows and all columns of this matrix are linearly independent. In the case of the system considered this condition can be interpreted as follows: for the observability area all rows and all columns of the matrix $S_o(q)$ are linearly independent, while for the non-observability area there exist rows or columns which are linearly dependent. For the discussed system, the observability matrix $S_o(q)$ has the following form:

$$S_o(q) = \begin{bmatrix} c_1 & \lambda_1(q)c_1 & \cdots & \lambda_1^{n-1}(q)c_1 \\ \vdots & \vdots & \ldots & \vdots \\ c_n & \lambda_n c_n(q) & \cdots & \lambda_n^{n-1}(q)c_n \end{bmatrix}, \tag{16}$$

where $\lambda_1(q),\ldots, \lambda_n(q)$ are the interval eigenvalues of the state matrix defined by (5) and described by (12). The i-th row of the observability matrix $S_o(q)$ expressed by (16) has the following form:

$$S_{oi}(q) = c_i \left[\begin{array}{ccccc} 1 & \lambda_i(q) & {\lambda_i}^2(q) & \cdots & {\lambda_i}^{n-1}(q) \end{array} \right]. \tag{17}$$

In the case of a SISO system with known parameters it is well known that the rank of the matrix $S_o(q)$ is less than n if and only if the system's spectrum consists of two (or more) different eigenvalues which are equal.

This simple condition was used for the determination of non-controllability areas inside the set of uncertain parameters Q (Oprzędkiewicz, 2004) and it can be analogously applied now. First, the following remarks can be given:

1. In the case of the discussed uncertain-parameter system there might exist vectors q such that the eigenvalues $\lambda_i(q)$ and $\lambda_j(q)$ will be equal. These vectors form the non-observability area Q_{no}.

2. There might exist vectors q such that the eigenvalues will not be equal. These vectors form the observability area Q_o.

More formally, the non-observability area can be defined as follows:

$$Q_{no} = \{q \in Q : \lambda_i(q) = \lambda_j(q), \quad i, j = 1, \dots, n, i \neq j\}, \tag{18}$$

and, analogously, we can express the observability area as

$$Q_o = \{q \in Q : \lambda_i(q) \neq \lambda_j(q), \quad i, j = 1, \dots, n, i \neq j\}. \tag{19}$$

Interval eigenvalues are the linear functions of uncertain parameters described by (5). Denote by λ_{ij} the common part of the eigenvalues λ_i and λ_j in the space $\mathbb{R}^3$:

$$\lambda_{ij} = \Lambda_i \cap \Lambda_j. \tag{20}$$

Remember that interval eigenvalues form tetrahedrons in $\mathbb{R}^3$. This implies that the set λ_{ij} described by (20) may have various forms: an empty set, a single point, a segment or a tetrahedron.

Next consider a projection of the set λ_{ij} onto the set of uncertain parameters Q. Denote by q_{ij} this projection. The elements of the set p_{ij} must meet the following condition:

$$(c_{1_i} - c_{1_j})q_1 + (c_{2_i} - c_{2_j})q_2 = d_i - d_j, \tag{21}$$

where c_{1_i},c_{2_i},c_{1_j},c_{2_j},d_i,$d_j \in R$, $q = [q_1, q_2]^T$, $q \in Q$. The set q_{ij} describes the part of the non-observability area Q_{no} generated by two eigenvalues only. Then the whole non-observability area Q_{no} can be expressed as a sum of the sets q_{ij} for each pair i and j:

$$Q_{no} = \bigcup_{i,j=1 i\neq j}^{n} q_{ij}. \tag{22}$$

The above deliberations are illustrated in Fig. 1.

The relations (26) and (21) can be used for the determination of the non-observability area Q_{no} for a given uncertain parameter area Q. This reduces to the determination of the sets q_{ij} for all pairs of the eigenvalues λ_i and λ_j. The sum of all q_{ij} is the desired set Q_{no}.

The set q_{ij} determined by the solution of (21) has different forms depending on different values of the parameters c_{1_i},c_{2_i},c_{1_j},c_{2_j},d_i and d_j. This issue will be discussed below:

1. $c_{1_i} = c_{1_j}$, $c_{2_i} = c_{2_j}$ and $d_i = d_j$: the eigenvalues λ_i and λ_j are equal for each vector $q \in Q$. In this case $Q_{no} = Q$ and $Q_o = 0$.

2. $c_{1_i} = c_{1_j}$, $c_{2_i} = c_{2_j}$ and $d_i \neq d_j$: the eigenvalues λ_i and λ_j are "parallel" and q_{ij} is an empty set for each vector $q \in \mathbb{R}^2$.

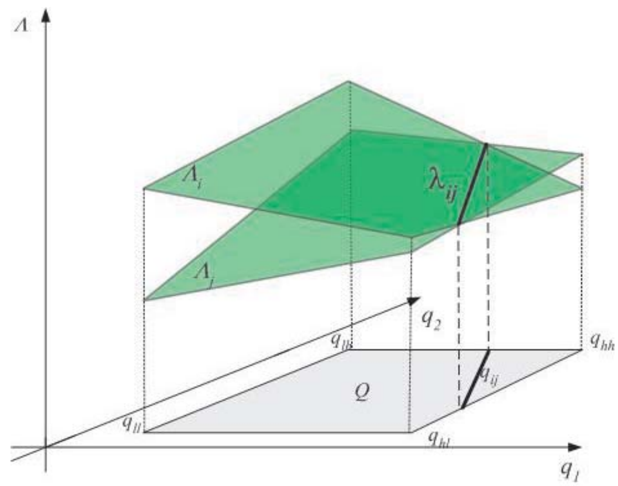

Fig. 1. Geometric interpretation of the non-observability area.

3. $c_{1_i} = c_{1_j}$ and $c_{2_i} \neq c_{2_j}$: the set q_{ij} has the form of a sector parallel to the q_1-axis, which can be expressed as follows:

$$q_{ij} = \Big\{q \in Q : q_2 = \frac{d_i - d_j}{c_{2_i} - c_{2_j}}\Big\}. \tag{23}$$

4. $c_{1_i} \neq c_{1_j}$ and $c_{2_i} = c_{2_j}$: the set q_{ij} has the form of a sector parallel to the q_2-axis, which can be expressed as follows:

$$q_{ij} = \Big\{q \in Q : q_2 = \frac{d_i - d_j}{c_{1_j} - c_{1_i}}\Big\}. \tag{24}$$

5. $c_{1_i} \neq c_{1_j}$, $c_{2_i} \neq c_{2_j}$, $c_{1_i} \neq 0$, $c_{2_i} \neq 0$: the set q_{ij} has the form of the following sector:

$$q_{ij} = \Big\{q \in Q : q_2 = q_1 \frac{c_{1_i} - c_{1_j}}{c_{2_i} - c_{2_j}} + \frac{d_j - d_i}{c_{2_i} - c_{2_j}}\Big\}. \tag{25}$$

Here we can immediately notice that the set q_{ij} described by (23), (24) or (25) can have different forms:

- An empty set—this solution occurs if the solution of (21) is in the exterior of the set Q.
- A single point—it occurs if two eigenvalues have a common vertex.
- A sector—this is the most typical situation, when part of the solution of (21) lies inside the area Q.

From the above discussion we can deduce a simple method for determining non-observability areas. To localize them, we need to find the common parts of all interval eigenvalues. The vectors q for which the eigenvalues have common parts form the non-observability area. This assignment can be performed using analytical or simulation methods. This issue will be illustrated with examples in Section 7. In the next section, simple observability criteria for the interval system will be proposed.

6. Observability Conditions

The discussion in the previous section allows us to formulate observability criteria for the uncertain-parameter dynamic system considered.

Proposition 1. (Necessary and sufficient cotrollability condition for the entire area Q) *Consider the linear uncertain-parameter dynamic SISO system described by (1), (5), (6) and (10). The following expressions are equivalent:*

- *The uncertain-parameter linear dynamic system described by the state equation (1) is controllable in the whole uncertain-paramater area Q.*
- *The non-observability subspace Q_{nc} is an empty set and the observability subspace Q_o is equal to the set Q.*
- *Equation (21) does not have any solution inside the set of uncertain parameters Q.*
- *There are no interval eigenvalues which have a common part for each vector $q \in Q$.*

The proof of the above proposition directly follows from the deliberations of Section 6. The above proposition formulates observability conditions for the whole area of uncertain parameters Q. This issue is important, but the analysis of the problem considered would be by far incomplete if it was finished at this moment. In the case of the discussed uncertain-parameter system, a complete analysis requires formulating a simple condition for the non-observability area Q_{no}.

Proposition 2. (Necessary and sufficient condition for the existence of non-observability areas inside Q) *Consider the system defined in Proposition 1. The following expressions are equivalent:*

- *There are vectors $q \in Q$ for which the system is not controllable.*
- *A solution to (21) exists which lies inside the uncertain-parameter area Q, defined by (23), (24) or (25).*
- *There are at least two different interval eigenvalues λ_i and λ_j defined by (5) such that their common part λ_{ij} defined by (20) is a non-empty set.*

The above condition collects all existence conditions for the non-observability area for the uncertain-parameter linear dynamic system considered. A particular case of the above deliberations covers the situation when there exist multiple eigenvalues. This situation implies the non-observability of the system considered in the whole area Q:

Proposition 3. (Necessary and sufficient condition for the non-observability of the whole area Q) *Consider the system discussed in the previous propositions. The following expressions are equivalent:*

- *The system is uncontrollable in the whole area of uncertain parameters Q.*
- *There exist different interval eigenvalues λ_i and λ_j such that their parameters c_{1_i},c_{2_i},c_{1_j},c_{2_j},d_i and d_j meet the following conditions:*
$$c_{1_i} = c_{1_j}, \quad c_{2_i} = c_{2_j}, \quad d_i = d_j.$$
- *The non-observability area Q_{no} is equal to the whole set Q.*

The conditions formulated above describe all the possibilities for observability in the discussed case: the system observable in the whole area Q, the system observable in a part of the area Q and the system unobservable in the whole area Q.

7. Examples

Example 1. Consider the linear, time invariant SISO system with the diagonal interval state matrix, described by (1) with the following matrices A, B and C:

$$A(q) = \begin{bmatrix} \lambda_1(q) & 0 & 0 \\ 0 & \lambda_2(q) & 0 \\ 0 & 0 & \lambda_3(q) \end{bmatrix},$$

$$B = \begin{bmatrix} 1.2 \\ -2.0 \\ 3.0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.0 & -1.2 & 2.5 \end{bmatrix}.$$

The uncertain parameters of the system are described by (3) and have the following form:

$$q_1 = [1.0, 5.2], \quad q_2 = [0.7, 1.2].$$

Then the set of uncertain parameters in (10) is

$$Q = \{q \in \mathbb{R}^2 : 1.0 \le q_1 \le 5.2, 0.7 \le q_2 \le 1.2\}.$$

Interval eigenvalues are linear functions of the uncertain parameters and are described by (5). The coefficient values of (5) for $i = 1, 2, 3$ are shown in Table 1.

Table 1. Parameters c_{1_i} and c_{2_i}.

i	c_{1_i}	c_{2_i}	d_i
1	2.0	0.5	2.0
2	2.0	-0.2	3.12
3	1.9	-0.1	2.5

To determine the non-observability area, we have to find the sets q_{ij} for $i, j = 1, 2, 3$ and $i \neq j$. To do it, we use (21). In our example they have the following forms for all pairs of i and j:

$$(c_{1_1} - c_{1_2})q_1 + (c_{2_1} - c_{2_2})q_2 + (d_1 - d_2) = 0, \quad (26)$$

$$(c_{1_1} - c_{1_3})q_1 + (c_{2_1} - c_{2_3})q_2 + (d_1 - d_3) = 0, \quad (27)$$

$$(c_{1_2} - c_{1_3})q_1 + (c_{2_2} - c_{2_3})q_2 + (d_2 - d_3) = 0. \quad (28)$$

After inserting the coefficients from Table 1 into (26)–(28) and performing elementary transformations, we obtain the following sets q_{12}, q_{13} and q_{23}:

$$q_{12} = \{q \in Q : q_2 = 1.6\}, \quad (29)$$

$$q_{13} = \{q \in Q : q_2 = q_1 + 6.2\}, \quad (30)$$

$$q_{23} = \{q \in Q : q_2 = -(1/6)q_1 + 5/6\}. \quad (31)$$

It is easy to see that the sets q_{12}, q_{13} and q_{23} described by (29)–(31) are empty sets, because all the solutions to (21) are located outside the set Q. The general conclusion from the above deliberations is that the uncertain parameter system deliberations is observable in the whole area Q. ♦

Example 2. Consider the system from the previous example. Assume that the uncertain parameters of the system are described by (3) and in the case considered they are

$$q_1 = [1.0, 5.2], \quad q_2 = [0.2, 1.8].$$

Then the set of uncertain parameters in (10) is

$$Q = \{q \in \mathbb{R}^2 : 1.0 \leq q_1 \leq 5.2, 0.2 \leq q_2 \leq 1.8\}.$$

Furthermore, assume that the coefficients of interval eigenvalues are the same as in the previous example (see Table 1). The further analysis will proceed in much the same way as above. The sets q_{11}, q_{12} and q_{23} are defined by (29)–(31) too, but the set Q is different, as in Example 1. This implies that the sets q_{12} and q_{23} are not empty sets and the set q_{13} is an empty set. Then the non-observability area is determined in accordance with (26) as

$$Q_{no} = q_{12} \cup q_{23},$$

where q_{12} and q_{23} are described by (29) and (31), respectively. The geometric interpretation of the above is shown in Fig. 2, and the area of uncertain parameters Q with the non-observability area is shown in Fig. 3.

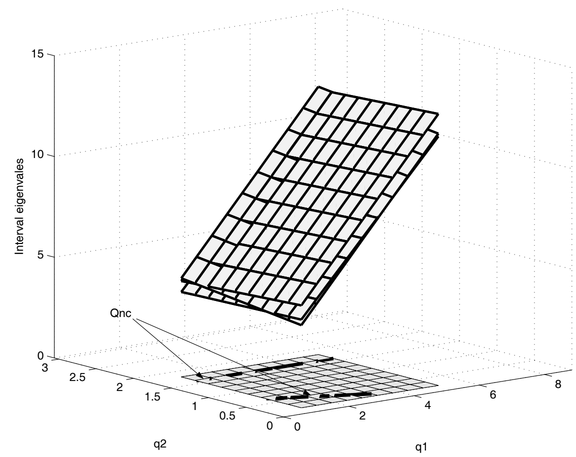

Fig. 2. Interval eigenvalues in $\mathbb{R}^3$ and the non-observability area.

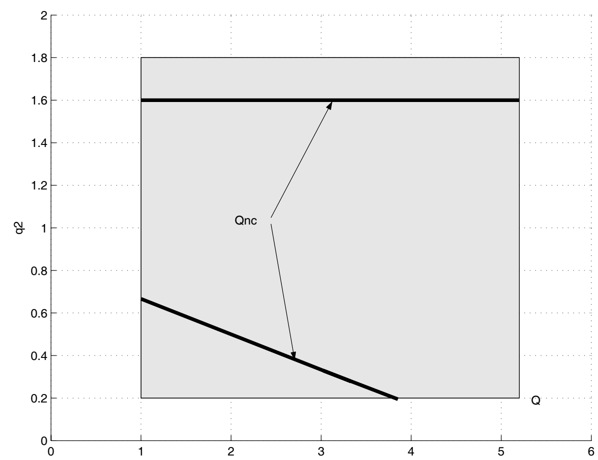

Fig. 3. Non-observability area inside the area of uncertain parameters Q for Example 2.

The conclusion from this example is that here the area of uncertain parameters Q consists of the non-observability subarea Q_{no}.

A more general conclusion from both the examples is that the expanding of the uncertain-parameter area Q may cause a non-observability area appear inside the area Q. ♦

Example 3. In the third example the observability problem for the interval parabolic system discussed in (Oprzędkiewicz, 2003) will be presented. The system under consideration is a heat conduction process shown in Fig. 4. Its main part is a thin copper rod with an electric heater at one end and a temperature sensor at the other end. The input $u(t)$ and the output $y(t)$ of this system are electric signals. The length of the heater equals x_u and the length of the temperature sensor is $\Delta x = x_2 - x_1$.

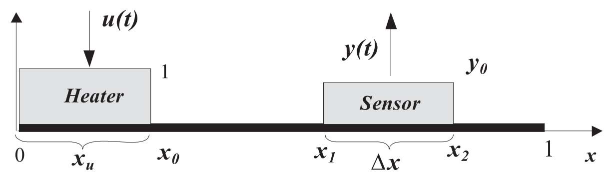

Fig. 4. Real experimental heat system.

A fundamental mathematical model describing heat conduction is a partial differential equation of the parabolic type with homogeneous Neuman boundary conditions at both ends, a homogeneous initial condition, heat exchange along the length of the rod and distributed control and observation. This equation has the following form:

$$\begin{cases} \dfrac{\partial Q(x,t)}{\partial t} = a\dfrac{\partial^2 Q(x,t)}{\partial x^2} - R_a Q(x,t) + b(x)u(t), \\ \qquad 0 \le x \le 1, \quad t \ge 0, \\ \dfrac{\partial Q(0,t)}{\partial x} = 0, \quad t \ge 0, \\ \dfrac{\partial Q(1,t)}{\partial x} = 0, \quad t \ge 0, \\ \dfrac{\partial Q(x,0)}{\partial t} = 0, \quad 0 \le x \le 1, \\ y(t) = y_0 \displaystyle\int_0^1 Q(x,t)c(x)\,\mathrm{d}x, \end{cases} \tag{32}$$

where $Q(x,t)$ denotes temperature at the moment t and the point x, R_a, and a denote uncertain coefficients of heat conduction and heat exchange, respectively, $b(x)$ denotes the control function, $c(x)$ is an observation function and y_0 denotes the steady-state gain of the system.

The heat equation (32) can be expressed as an equivalent abstract initial problem in the Hilbert space $X = L^2(0,1)$ with a standard scalar product. In the case considered the abstract form of the heat equation (32) is as follows (Oprzędkiewicz, 2003):

$$\begin{cases} \dot{Q}(t) = AQ(t) + bu(t), \\ Q(0) = 0, \\ y(t) = CQ(t), \end{cases} \tag{33}$$

where

$$\begin{aligned} AQ &= aQ'' - R_a Q, D(A) \\ &= \left\{u \in H^2(0,1) : Q'(0) = 0, Q'(1) = 0\right\}, \\ &\quad a, R_a > 0, \\ H^2(0,1) &= \left\{u \in L^2(0,1) : u', u'' \in L^2(0,1)\right\}, \\ CQ(t) &= \langle c, Q(t)\rangle, \\ Bu(t) &= bu(t), \\ \langle u, v\rangle &= \int_0^1 u(x)v(x)\mathrm{d}x \end{aligned}$$

(the standard scalar product).

The following set of eigenvectors for the state operator A forms the orthonormal basis of the state space:

$$h_i = \begin{cases} 0, & i = 0, \\ \sqrt{2}\cos(i\pi x), & i = 1, 2, \dots. \end{cases} \tag{34}$$

The coefficients of (1) expressing the interval eigenvalues of the discussed system are

$$c_{1_i} = -i^2\pi^2, \quad c_{2_i} = -1, \quad d_i = 0. \tag{35}$$

The discrete spectrum of the state operator A is a set of single, real eigenvalues, which are expressed as follows:

$$\lambda_i = -a\pi^2 i^2 - R_a, \quad i = 0, 1, 2, 3, \dots. \tag{36}$$

In the state-space basis defined by the set of eigenvectors (34), the operators A, B and C have the following matrix representation:

$$A = \operatorname{diag}\{\lambda_0, \lambda_1, \lambda_2, \dots\}, \tag{37}$$

$$B = [b_0, b_1, b_2, \dots]^T, \tag{38}$$

where $b_i = \langle b, h_i\rangle$, $b(x)$ denotes the control function,

$$b(x) = \begin{cases} 1, & x \in [0, x_0], \\ 0, & x \notin [0, x_0], \end{cases} \tag{39}$$

$$C = [c_0, c_1, c_2, \dots], \tag{40}$$

where $c_i = \langle c, h_i\rangle$, $c(x)$ denotes the output of the sensor,

$$c(x) = \begin{cases} y_0, & x \in [x_1, x_2], \\ 0, & x \notin [x_1, x_2]. \end{cases} \tag{41}$$

The general equation (21) along with (35) gives

$$\pi^2 \left(j^2 - i^2\right) a = 0. \tag{42}$$

From (42) we can immediately deduce that observability for the discussed interval parabolic system is not determined by the uncertain parameters a and R_a. ♦

8. Conclusions

The main conclusions can be formulated as follows:

- The most characteristic feature of the discussed uncertain-parameter dynamic system is that inside the area of uncertain parameters there might exist subareas where the system is observable and subareas where the system is not observable.
- The assumptions about the type of system uncertainty allows us to formulate a simple method of determining non-observability areas,
- From the presented examples we can deduce that extending the area of uncertain parameters may cause the non-observability area appear inside the set of uncertain parameters Q. This observation can be generalized: it can be expected that a suitable contraction of the area Q (e.g., by dynamic feedback) may remove non-observability areas.
- The presented results can be very useful in stability testing for linear interval parabolic systems with a two-dimensional space of uncertain parameters.
- The presented results can be thought of as a starting point for a further analysis of the observability problem for interval systems. As other problems appearing here, the following can be mentioned:
  - an extension to a general form of parameter uncertainty,
  - the formulation of observability conditions for a system with an interval control matrix,
  - observability analysis for uncertain-parameter MIMO systems.

References

Barnett S. (1992): *Matrices. Methods and Applications*. — Oxford: Clarendon Press.

Białas S. (2002): *Robust Stability of Polynomials and Matrices*. — Cracow: AGH University of Science and Technology Press, (in Polish).

Busłowicz M. (1997): *Stability of Linear Time Invariant Systems with Uncertain Parameters*. — Białystok: Technical University Press, (in Polish).

Busłowicz M. (2000): *Robust Stability of Dynamic Linear Time Invariant Systems with Delays*. — Warsaw-Białystok: Polish Academy of Sciences, The Committee of Automatics and Robotics, (in Polish).

Feintuch A. (1998): *Robust Control Theory in Hilbert Space*. — New York: Springer.

Jakubowska M. (1999): *Algorithms for checking stability of the interval matrix and their numerical realization*. — Automatyka, Vol. 3, No. 2, pp. 413–430, (in Polish).

Kalmikov S.A. , Sokin J.I. Juldasev Z. H. (1986): *Interval Analysis Methods*. — Moscow: Nauka, (in Russian).

Kharitonov W. L. (1978): *On the asymptotical stability of the equilibrium location for a system of linear differential equations*. — Diff. Uravnenya, Vol. 14, No. 11, pp. 2086–2088, (in Russian).

Klamka J. (1990): *Contollability of Dynamic Systems*. — Warsaw: Polish Scientific Publishers, (in Polish).

Mao X. (2002): *Exponential stability of stochastic delay interval systems with Markovian switching*. — IEEE Trans. Automat. Contr., Vol. 47, No. 10, pp. 1064–1612.

Mitkowski W. (1991): *Stabilisation of Dynamic Systems*. — Warsaw: Polish Scientific Publishers (in Polish).

Moore R. (1966): *Interval Analysis*. — Upper Saddle River, Englewood Cliffs: Prentice Hall.

Moore R. (1997): *Methods and Applications of Interval Analysis*. — Philadelphia: SIAM.

Oprzędkiewicz K. (2003): *The interval parabolic system*. — Arch. Contr. Sci., Vol. 13, No. 4, pp. 391–405.

Oprzędkiewicz K. (2004): *A controllability problem for a class of uncertain-parameters linear dynamic systems*. — Arch. Contr. Sci., Vol. 14 (L), No. 1, pp. 85–100.

Received: 5 October 2004
Revised: 22 January 2005
Re-revised: 14 March 2005