ON THE TWO-STEP ITERATIVE METHOD OF SOLVING FRICTIONAL CONTACT PROBLEMS IN ELASTICITY

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A class of contact problems with friction in elastostatics is considered. Under a certain restriction on the friction coefficient, the convergence of the two-step iterative method proposed by P.D. Panagiotopoulos is proved. Its applicability is discussed and compared with two other iterative methods, and the computed results are presented.

Keywords: contact problems with friction, iterative methods

1. Introduction

Unilateral contact problems in elasticity (Duvaut and Lions, 1976) have been extensively studied in the last two decades (Andersson and Klarbring, 2001; Angelov and Liolios, 2004; Căpătină and Cocu, 1991; Cocu, 1984; Demkowicz and Oden, 1982; Duvaut and Lions, 1976; Hlavaček *et al.*, 1988; Kikuchi and Oden, 1988; Klarbring *et al.*, 1989; Lee and Oden, 1993a,1993b; Nečas *et al.*, 1980; Oden and Carey, 1984; Panagiotopoulos, 1975; 1985; Rabier and Oden, 1987; 1988). Special attention has been paid to static contact problems with friction, describing one step (increment) in the actual quasistatic processes (Lee and Oden, 1993a; Oden and Carey, 1984). Since the problems are nonlinear, existence and uniqueness results were obtained using fixed point methods (Andersson and Klarbring, 2001; Căpătină and Cocu, 1991; Cocu, 1984; Demkowicz and Oden, 1982; Kikuchi and Oden, 1988; Klarbring *et al.*, 1989; Nečas *et al.*, 1980). Based on these methods, and the finite element method, various algorithms were proposed for solving the resulting finite dimensional systems of nonlinear equations (Angelov and Liolios, 2004; Căpătină and Cocu, 1991; Glowinski, 1984; Hlavaček *et al.*, 1988; Kikuchi and Oden, 1988; Lee and Oden, 1993a; 1993b).

In this work, we consider a class of contact problems with friction in elastostatics, falling in the category of contact problems with normal compliance (Andersson and Klarbring, 2001; Angelov and Liolios, 2004; Kikuchi and Oden, 1988; Klarbring *et al.*, 1989; Lee and Oden, 1993a; 1993b; Rabier and Oden, 1987; 1988). For the corresponding variational problems, existence and uniqueness results are briefly revisited. The convergence of the two–step iterative method proposed by Panagiotopoulos (1975) is proved under a certain restriction on the friction coefficient. The applicability of this method is further discussed and compared with two other iterative methods. An example is solved and the obtained results are presented.

2. Model Problem

Suppose that an elastic body, subjected to external forces, occupies an open, bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with a sufficiently smooth boundary $\Gamma = \Gamma_u \cup \Gamma_\sigma \cup \Gamma_c$, $\Gamma_u \cap \Gamma_\sigma = \emptyset$, $\Gamma_u \cap \Gamma_c = \emptyset$, $\Gamma_\sigma \cap \Gamma_c = \emptyset$, $\mathrm{meas}(\Gamma_u) > 0$. Γ_u and Γ_σ are the boundaries with prescribed displacements and tractions, respectively, Γ_c is the contact boundary. We denote by $\boldsymbol{x} = \{x_i\}$, $1 \le i \le n$ the Cartesian coordinates of the points of $\bar{\Omega} = \Omega \cup \Gamma$. Moreover, the standard indicial notation and summation convention are used.

Problem 1. Find the fields of displacements $\boldsymbol{u}(\boldsymbol{x}) = \{u_i(\boldsymbol{x})\}$ and stresses $\boldsymbol{\sigma}(\boldsymbol{x}) = \{\sigma_{ij}(\boldsymbol{x})\}$, $1 \le i, j \le n$ satisfying the following equations and relations:

– equations of equilibrium:

$$\sigma_{ij,j} + f_i = 0 \quad \text{in} \quad \Omega, \tag{1}$$

– constitutive relations:

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}, \tag{2}$$

– Cauchy conditions:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{3}$$

– boundary conditions:

$$u_i = 0 \quad \text{on} \quad \Gamma_u, \tag{4}$$

$$\sigma_{ij} n_j = F_i \quad \text{on} \quad \Gamma_\sigma, \tag{5}$$

$$\begin{aligned} &\text{if} \quad u_N \le 0 \quad \text{then} \quad \sigma_N = 0 \quad \text{and} \quad \boldsymbol{\sigma}_T = \mathbf{0}, \\ &\text{if} \quad u_N > 0 \quad \text{then} \quad \sigma_N + k_N(u_N) = 0 \quad \text{and} \\ &\text{if} \quad |\boldsymbol{\sigma}_T| < \mu|\sigma_N| \quad \text{then} \quad \boldsymbol{u}_T = \mathbf{0}, \\ &\text{if} \quad |\boldsymbol{\sigma}_T| = \mu|\sigma_N| \quad \text{then} \quad \exists \ \text{constant} \ \lambda \ge 0, \\ &\qquad \text{such that} \quad \boldsymbol{u}_T = -\lambda \boldsymbol{\sigma}_T \quad \text{on} \quad \Gamma_c. \end{aligned} \tag{6}$$

Here $\sigma_{ij} = \sigma_{ji}$, $\varepsilon_{ij} = \varepsilon_{ji}$ are the components of the stress and strain tensors; $E_{ijkl}(\boldsymbol{x})$ are the components of the elasticity tensor, satisfying the symmetry and ellipticity conditions

$$E_{ijkl} = E_{jikl} = E_{klij} = E_{ijlk}, \quad 1 \le i,j,k,l \le n, \tag{7}$$

$$E_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \ge \alpha \varepsilon_{ij}\varepsilon_{ij}, \quad \alpha > 0 \quad \text{constant}, \quad \forall \varepsilon_{ij}. \tag{8}$$

Here $\boldsymbol{f}(\boldsymbol{x}) = \{f_i(\boldsymbol{x})\}$ and $\boldsymbol{F}(\boldsymbol{x}) = \{F_i(\boldsymbol{x})\}$ are the vectors of volume and surface forces, $\boldsymbol{n} = \{n_i\}$ is the unit outward normal vector to Γ, and $\mu(\boldsymbol{x})$ is the friction coefficient,

$$\begin{aligned} u_N &= u_i n_i, & u_{T\,i} &= u_i - u_N n_i, \\ \sigma_N &= \sigma_{ij} n_j n_i, & \sigma_{T\,i} &= \sigma_{ij} n_j - \sigma_N n_i, \end{aligned} \tag{9}$$

are the components of the displacement and stress vectors on Γ. The function $k_N(u_N)$ is supposed to be nondecreasing, Lipschitz continuous and such that

$$\begin{aligned} k_N(v_N) &= 0 \ \text{for} \ u_N \le 0 \ \text{and} \\ k_N(v_N) &> 0 \ \text{for} \ u_N > 0. \end{aligned} \tag{10}$$

3. Variational Formulation

We introduce the Hilbert spaces

$$\boldsymbol{V} = \{\boldsymbol{v} : \ \boldsymbol{v} \in (H^1(\Omega))^n, \ \boldsymbol{v} = \mathbf{0} \ \text{on} \ \Gamma_u\},$$

$$\boldsymbol{H} \equiv (L_2(\Omega))^n$$

with inner products and norms:

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (u_i v_i + u_{i,j} v_{i,j}) \, \mathrm{d}x,$$

$$\|\boldsymbol{u}\|_1 = (\boldsymbol{u}, \boldsymbol{u})^{1/2}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},$$

$$(\boldsymbol{u}, \boldsymbol{v})_0 = \int_\Omega u_i v_i \, \mathrm{d}x,$$

$$\|\boldsymbol{u}\|_0 = (\boldsymbol{u}, \boldsymbol{u})_0^{1/2}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H},$$

respectively. We denote by $\boldsymbol{V}^*$ and $\boldsymbol{H}^*$ the dual spaces of $\boldsymbol{V}$ and $\boldsymbol{H}$, and by $\langle \cdot, \cdot \rangle_\Omega$ the duality pairing between $\boldsymbol{V}^*$ and $\boldsymbol{V}$. Then the following dense and continous embeddings hold:

$$\boldsymbol{V} \subset \boldsymbol{H} \equiv \boldsymbol{H}^* \subset \boldsymbol{V}^*.$$

The space of traces of the elements of $\boldsymbol{V}$ on Γ is denoted by $(H^{1/2}(\Gamma))^n$, and $(H^{-1/2}(\Gamma))^n$ is its dual space. The duality pairing between them is denoted by $\langle \cdot, \cdot \rangle_\Gamma$. We further assume that

$$\begin{aligned} E_{ijkl}(\boldsymbol{x}) &\in L_\infty(\Omega), & \mu(\boldsymbol{x}) &\in L_\infty(\Gamma_c), \\ \boldsymbol{f}(\boldsymbol{x}) &\in \boldsymbol{H}, & \boldsymbol{F}(\boldsymbol{x}) &\in (L_2(\Gamma_\sigma))^n. \end{aligned} \tag{11}$$

Introduce the functionals

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sigma_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v}) \, \mathrm{d}x, \tag{12}$$

$$\langle k_N(u_N), v_N \rangle_{\Gamma_c} = \int_{\Gamma_c} k_N(u_N) v_N \, \mathrm{d}\Gamma, \tag{13}$$

$$j_T(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_c} \mu(\boldsymbol{x}) |\sigma_N(\boldsymbol{u})| |\boldsymbol{v}_T| \, \mathrm{d}\Gamma, \tag{14}$$

$$l(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_\sigma} \boldsymbol{F} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma. \tag{15}$$

Then we associate with Problem 1 the following problem:

Problem 2. Find $\boldsymbol{u} \in \boldsymbol{V}$, satisfying for all $\boldsymbol{v} \in \boldsymbol{V}$ the inequality

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) &+ \langle k_N(u_N), v_N - u_N \rangle_{\Gamma_c} \\ &+ j_T(\boldsymbol{u}, \boldsymbol{v}) - j_T(\boldsymbol{u}, \boldsymbol{u}) \ge l(\boldsymbol{v} - \boldsymbol{u}). \end{aligned} \tag{16}$$

Remark 1. The functional $l(\boldsymbol{v})$ is linear and continuous on $\boldsymbol{V}$. The bilinear form $a(\boldsymbol{u}, \boldsymbol{v})$ is symmetric, continuous and $\boldsymbol{V}$-elliptic, i.e., for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= a(\boldsymbol{v}, \boldsymbol{u}), \\ a(\boldsymbol{u}, \boldsymbol{v}) &\le M\|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_1, \\ a(\boldsymbol{u}, \boldsymbol{u}) &\ge m\|\boldsymbol{u}\|_1^2, \end{aligned} \tag{17}$$

where m and M are positive constants. We also have

$$\langle k_N(u_N), u_N \rangle_{\Gamma_c} \ge 0, \quad j_T(\boldsymbol{u}, \boldsymbol{v}) \ge 0, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \tag{18}$$

$$|\langle k_N(u_{N\,1}) - k_N(u_{N\,2}), v_N \rangle_{\Gamma_c}| \le \kappa \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1 \|\boldsymbol{v}\|_1, \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v} \in \boldsymbol{V}, \tag{19}$$

and for all $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V}$

$$|j_T(\boldsymbol{u}_1, \boldsymbol{v}_2) + j_T(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_T(\boldsymbol{u}_1, \boldsymbol{v}_1) - j_T(\boldsymbol{u}_2, \boldsymbol{v}_2)| \leq c\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_1 \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|_1, \tag{20}$$

where κ and c are positive constants, as c depends on the friction coefficient.

Remark 2. In the frictionless case, write

$$K_N(u_N) = \int_0^{u_N} k_N(\xi)\,\mathrm{d}\xi. \tag{21}$$

Introduce the functional

$$j_N(\boldsymbol{v}) = \begin{cases} \int_{\Gamma_c} K_N(v_N)\,\mathrm{d}\Gamma & \text{if} \quad K_N(v_N) \in L_1(\Gamma_c), \\ +\infty & \text{if} \quad K_N(v_N) \notin L_1(\Gamma_c). \end{cases} \tag{22}$$

Then the following result holds:

Lemma 1. *The functional $j_N(\boldsymbol{v}) : \boldsymbol{V} \to \mathbb{R}$ is convex, proper and lower semicontinuous.*

Proof. The first two properties follow from the properties of the function $K_N(\cdot) : \mathbb{R} \to \mathbb{R}$. Next we prove the lower semicontinuity. We consider the sequence $\{\boldsymbol{v}_m\} \to \boldsymbol{v} \in \boldsymbol{V}$ strongly. Since the trace operator $\gamma : \boldsymbol{V} \to (H^{1/2}(\Gamma_c))^n \subset (L^2(\Gamma_c))^n$ is linear, continuous and surjective, we have $\{v_{N\,m}\} \to v_N \in (L^2(\Gamma_c))^n$ strongly. If $j_N(\boldsymbol{v}) = \infty$, then the lower semicontinuity is proved. Suppose that $\liminf_{m\to\infty} j_N(\boldsymbol{v}_m) = C < \infty$. Hence we can extract the subsequence $\{v_{N\,m_k}\} \to v_N$ almost everywhere and such that $\lim_{k\to\infty} j_N(\boldsymbol{v}_{m_k}) = C$. Further we have $K_N(v_{N\,m_k}) \to K_N(v_N)$ also almost everywhere and $K_N(v_{N\,m_k})$ is bounded in $L_1(\Gamma_c)$. Then by Fatou's lemma it follows that $K_N(v_N) \in L_1(\Gamma_c)$ and $\liminf_{k\to\infty} j_N(\boldsymbol{v}_{m_k}) \geq j_N(\boldsymbol{v})$, which completes the proof. ∎

Since the functional $j_N(\boldsymbol{v})$ is convex and Gâteaux differentiable, we have

$$\langle j'_N(\boldsymbol{u}), v_N \rangle_{\Gamma_c} = \langle k_N(u_N), v_N \rangle_{\Gamma_c}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},$$

$$\langle k_N(u_{N\,1}) - k_N(u_{N\,2}), u_{N\,1} - u_{N\,2} \rangle_{\Gamma_c} \geq 0, \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{V}. \tag{23}$$

Then we obtain the following problem: Find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + \langle k_N(u_N), v_N \rangle_{\Gamma_c} = l(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{24}$$

or

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\boldsymbol{V}} J(\boldsymbol{v}),$$

$$J(\boldsymbol{v}) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) + j_N(\boldsymbol{v}) - l(\boldsymbol{v}). \tag{25}$$

A detailed study of frictionless contact problems can be found in (Rabier and Oden, 1987,1988).

The following result, common for such a kind of problems (Andersson and Klarbring, 2001; Căpătină and Cocu, 1991; Cocu, 1984; Demkowicz and Oden, 1982; Klarbring *et al.*, 1989; Lee and Oden, 1993a; 1993b; Nečas *et al.*, 1980; Oden and Carey, 1984), is valid.

Theorem 1. *For a sufficiently small coefficient of friction, there exists a unique solution of Problem 2.*

Remark 3. Since the constant c, cf. Remark 1, contains the coefficient of friction as a multiplier, the requirement that the coefficient of friction be sufficiently small means $c < m$, (Cocu, 1984; Demkowicz and Oden, 1982).

4. Two–Step Iterative Method

The proposed two-step iterative method consists in solving in each step problems of different natures: first, a problem with given friction stresses, and second, a problem with given normal stresses along the contact boundary (Panagiotopoulos, 1975). If this idea is not employed as in (Nečas *et al.*, 1980), or with additional restrictions on the magnitude of the applied forces, e.g., as in (Klarbring *et al.*, 1989), it may cause convergence problems, which is reported by many authors (Hlavaček *et al.*, 1988; Klarbring *et al.*, 1989). Here the convergence of the two-step iterative method is proved, as additional restrictions on the friction coefficient are imposed.

Let us consider the following two-step iterative process:

Problem 2$_k$. Given $\boldsymbol{u}_k \in \boldsymbol{V}$ find $\boldsymbol{u}_{k+1/2}, \boldsymbol{u}_{k+1} \in \boldsymbol{V}$, $k = 0, 1, 2, \ldots$ satisfying for all $\boldsymbol{v} \in \boldsymbol{V}$ the inequalities

$$\begin{aligned} &a(\boldsymbol{u}_{k+1/2}, \boldsymbol{v} - \boldsymbol{u}_{k+1/2}) \\ &\quad + \langle k_N(u_{N\,k+1/2}), v_N - u_{N\,k+1/2} \rangle_{\Gamma_c} \\ &\geq l(\boldsymbol{v} - \boldsymbol{u}_{k+1/2}) - j_T(\boldsymbol{u}_k, \boldsymbol{v}) \\ &\quad + j_T(\boldsymbol{u}_k, \boldsymbol{u}_{k+1/2}), \end{aligned} \tag{26}$$

$$\begin{aligned} &a(\boldsymbol{u}_{k+1}, \boldsymbol{v} - \boldsymbol{u}_{k+1}) \\ &\quad + j_T(\boldsymbol{u}_{k+1/2}, \boldsymbol{v}) - j_T(\boldsymbol{u}_{k+1/2}, \boldsymbol{u}_{k+1}) \\ &\geq l(\boldsymbol{v} - \boldsymbol{u}_{k+1}) \\ &\quad - \langle k_N(u_{N\,k+1/2}), v_N - u_{N\,k+1} \rangle_{\Gamma_c}. \end{aligned} \tag{27}$$

The first inequality follows from

$$\begin{aligned}
&a(\boldsymbol{u}_{k+1/2}, \boldsymbol{v} - \boldsymbol{u}_{k+1/2}) \\
&\quad + \langle k_N(u_{N\,k+1/2}), v_N - u_{N\,k+1/2}\rangle_{\Gamma_c} \\
&= l(\boldsymbol{v} - \boldsymbol{u}_{k+1/2}) \\
&\quad + \Big\langle \sigma_T(\boldsymbol{u}_k), \boldsymbol{v}_T - \boldsymbol{u}_{T\,k+1/2}\Big\rangle_{\Gamma_c},
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\Big\langle \mu|\sigma_N(\boldsymbol{u}_k)|, |\boldsymbol{v}_T| - |\boldsymbol{u}_{T\,k+1/2}|\Big\rangle_{\Gamma_c} \\
&\quad + \Big\langle \sigma_T(\boldsymbol{u}_k), \boldsymbol{v}_T - \boldsymbol{u}_{T\,k+1/2}\Big\rangle_{\Gamma_c} \geq 0.
\end{aligned} \tag{29}$$

Theorem 2. *Under the additional restriction on the coefficient of friction, the iterative process defined by Problem 2_k is convergent.*

Proof. The existence and uniqueness of the solutions to (26) and (27) can be obtained by using the theory of elliptic variational inequalities (Glowinski, 1984; Kikuchi and Oden, 1988). Set $\boldsymbol{v} = \boldsymbol{u}_{k+3/2}$ and $\boldsymbol{v} = \boldsymbol{u}_{k+1/2}$ in the first couple of subproblems, and $\boldsymbol{v} = \boldsymbol{u}_{k+2}$ and $\boldsymbol{v} = \boldsymbol{u}_{k+1}$ in the second couple of subproblems of Problem 2_k and Problem 2_{k+1}, respectively. Adding the corresponding subproblems, we obtain

$$\begin{aligned}
&a(\boldsymbol{u}_{k+3/2} - \boldsymbol{u}_{k+1/2}, \boldsymbol{u}_{k+3/2} - \boldsymbol{u}_{k+1/2}) \\
&\quad + \langle k_N(u_{N\,k+3/2}) - k_N(u_{N\,k+1/2}), \\
&\qquad u_{N\,k+3/2} - u_{N\,k+1/2}\rangle_{\Gamma_c} \\
&\leq j_T(\boldsymbol{u}_{k+1}, \boldsymbol{u}_{k+1/2}) - j_T(\boldsymbol{u}_{k+1}, \boldsymbol{u}_{k+3/2}) \\
&\quad + j_T(\boldsymbol{u}_k, \boldsymbol{u}_{k+3/2}) - j_T(\boldsymbol{u}_k, \boldsymbol{u}_{k+1/2}),
\end{aligned} \tag{30}$$

$$\begin{aligned}
&a(\boldsymbol{u}_{k+2} - \boldsymbol{u}_{k+1}, \boldsymbol{u}_{k+2} - \boldsymbol{u}_{k+1}) \\
&\leq \langle k_N(u_{N\,k+3/2}) - k_N(u_{N\,k+1/2}), \\
&\qquad u_{N\,k+1} - u_{N\,k+2}\rangle_{\Gamma_c} \\
&\quad + j_T(\boldsymbol{u}_{k+3/2}, \boldsymbol{u}_{k+1}) - j_T(\boldsymbol{u}_{k+3/2}, \boldsymbol{u}_{k+2}) \\
&\quad + j_T(\boldsymbol{u}_{k+1/2}, \boldsymbol{u}_{k+2}) - j_T(\boldsymbol{u}_{k+1/2}, \boldsymbol{u}_{k+1}).
\end{aligned} \tag{31}$$

Taking into account Remark 1 and (23), we obtain from (30) and (31):

$$\|\boldsymbol{u}_{k+3/2} - \boldsymbol{u}_{k+1/2}\|_1 \leq \frac{c}{m}\|\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\|_1, \tag{32}$$

$$\|\boldsymbol{u}_{k+2} - \boldsymbol{u}_{k+1}\|_1 \leq \frac{c+\kappa}{m}\|\boldsymbol{u}_{k+3/2} - \boldsymbol{u}_{k+1/2}\|_1. \tag{33}$$

Combining the above inequalities, we finally obtain

$$\begin{aligned}
&\|\boldsymbol{u}_{k+2} - \boldsymbol{u}_{k+1}\|_1 \leq q\|\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\|_1, \\
&q = \frac{c(c+\kappa)}{m^2}.
\end{aligned} \tag{34}$$

When the friction coefficient is such that

$$c \in \left(0, \frac{-\kappa + \sqrt{\kappa^2 + 4m^2}}{2}\right), \tag{35}$$

we have $0 < q < 1$, and since

$$\|\boldsymbol{u}_{k+2} - \boldsymbol{u}_{k+1}\|_1 \leq q^{k+1}\|\boldsymbol{u}_1 - \boldsymbol{u}_0\|_1, \tag{36}$$

it follows that

$$\lim_{k\to\infty} \|\boldsymbol{u}_{k+2} - \boldsymbol{u}_{k+1}\|_1 = 0, \tag{37}$$

i.e., $\{\boldsymbol{u}_s\}$, $s = 0, 1/2, 1, \ldots$ is a fundamental sequence. Therefore there exists an element $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\lim_{s\to\infty} \|\boldsymbol{u}_s - \boldsymbol{u}\|_1 = 0. \tag{38}$$

We shall show that it is the solution of Problem 2. From (26) we have

$$\begin{aligned}
&a(\boldsymbol{u}, \boldsymbol{v}) + \langle k_N(u_N), v_N\rangle_{\Gamma_c} + j_T(\boldsymbol{u}, \boldsymbol{v}) - l(\boldsymbol{v}) \\
&= \lim_{k\to\infty} \Big(a(\boldsymbol{u}_{k+1/2}, \boldsymbol{v}) + \langle k_N(u_{N\,k+1/2}), v_N\rangle_{\Gamma_c} \\
&\qquad + j_T(\boldsymbol{u}_k, \boldsymbol{v})\Big) - l(\boldsymbol{v}) \\
&\geq \liminf_{k\to\infty} \Big(a(\boldsymbol{u}_{k+1/2}, \boldsymbol{u}_{k+1/2}) \\
&\qquad + \langle k_N(u_{N\,k+1/2}), u_{N\,k+1/2}\rangle_{\Gamma_c} + j_T(\boldsymbol{u}_k, \boldsymbol{u}_{k+1/2})\Big) \\
&\qquad - \lim_{k\to\infty} l(\boldsymbol{u}_{k+1/2}) \\
&\geq a(\boldsymbol{u}, \boldsymbol{u}) + \langle k_N(u_N), u_N\rangle_{\Gamma_c} + j_T(\boldsymbol{u}, \boldsymbol{u}) - l(\boldsymbol{u}),
\end{aligned} \tag{39}$$

which is exactly the inequality (16). The same result follows from (27) and thus the proof is completed. ∎

Remark 4. If we rewrite (26) as follows:

Problem $2_k'$. Given $\boldsymbol{u}_k \in \boldsymbol{V}$ find $\boldsymbol{u}_{k+1} \in \boldsymbol{V}$, $k = 0, 1, 2, \ldots,$ satisfying for all $\boldsymbol{v} \in \boldsymbol{V}$ the inequality

$$\begin{aligned}
&a(\boldsymbol{u}_{k+1}, \boldsymbol{v} - \boldsymbol{u}_{k+1}) + \langle k_N(u_{N\,k+1}), v_N - u_{N\,k+1}\rangle_{\Gamma_c} \\
&\quad + j_T(\boldsymbol{u}_k, \boldsymbol{v}) - j_T(\boldsymbol{u}_k, \boldsymbol{u}_{k+1}) \geq l(\boldsymbol{v} - \boldsymbol{u}_{k+1}),
\end{aligned} \tag{40}$$

we obtain the iterative process defined in (Cocu, 1984; Demkowicz and Oden, 1982; Kikuchi and Oden, 1988; Klarbring *et al.*, 1989; Nečas *et al.*, 1980; Oden and Carey, 1984), which is convergent for $c < m$, i.e., without the additional restriction (35) on the magnitude of the friction coefficient. The same is valid (Căpătină and Cocu, 1991; Glowinski, 1984) if we consider the following:

Problem $2_k''$. Given $\boldsymbol{u}_k \in \boldsymbol{V}$, find $\boldsymbol{u}_{k+1} \in \boldsymbol{V}$, $k = 0, 1, 2, \dots,$ satisfying for all $\boldsymbol{v} \in \boldsymbol{V}$ the inequality

$$\begin{aligned}
&(\boldsymbol{u}_{k+1}, \boldsymbol{v} - \boldsymbol{u}_{k+1})_V + \beta \left(j_T(\boldsymbol{u}_k, \boldsymbol{v}) - j_T(\boldsymbol{u}_k, \boldsymbol{u}_{k+1}) \right) \\
&\quad \geq (\boldsymbol{u}_k, \boldsymbol{v} - \boldsymbol{u}_{k+1})_V + \beta l(\boldsymbol{v} - \boldsymbol{u}_{k+1}) \\
&\qquad - \beta \Big(a(\boldsymbol{u}_k, \boldsymbol{v} - \boldsymbol{u}_{k+1}) \\
&\qquad + \langle k_N(u_{N\,k}), v_N - u_{N\,k+1} \rangle_{\Gamma_c} \Big),
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
(\boldsymbol{u}, \boldsymbol{v})_V &= \int_\Omega (u_i v_i + \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v}))\, \mathrm{d}x, \\
\|\boldsymbol{u}\|_V &= (\boldsymbol{u}, \boldsymbol{u})_V^{1/2}, \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},
\end{aligned}$$

or

$$\begin{aligned}
(\boldsymbol{u}, \boldsymbol{v})_V &= \int_\Omega \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v})\, \mathrm{d}x, \\
\|\boldsymbol{u}\|_V &= (\boldsymbol{u}, \boldsymbol{u})_V^{1/2}, \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},
\end{aligned}$$

are inner product and norm, equivalent to $\|\cdot\|_1$,

$$0 < \beta < \frac{2(m_1 - c)}{(M_1^2 - c^2)}, \tag{42}$$

m_1 and M_1 are positive constants, such that for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ and

$$\langle A(\boldsymbol{u}), \boldsymbol{v} \rangle_\Omega = a(\boldsymbol{u}, \boldsymbol{v}) + \langle k_N(u_N), v_N \rangle_{\Gamma_c}, \tag{43}$$

the following conditions hold:

$$\langle A(\boldsymbol{v}) - A(\boldsymbol{u}), \boldsymbol{v} - \boldsymbol{u} \rangle_\Omega \geq m_1 \|\boldsymbol{v} - \boldsymbol{u}\|_V^2, \tag{44}$$

$$|\langle A(\boldsymbol{v}) - A(\boldsymbol{u}), \boldsymbol{v} \rangle_\Omega| \leq M_1 \|\boldsymbol{v} - \boldsymbol{u}\|_V \|\boldsymbol{v}\|_V. \tag{45}$$

5. Algorithms and Numerical Results

Using conforming finite elements for partition Ω and after an appropriate regularization of the nondifferentiable friction functional in Problem 2 (Angelov and Liolios, 2004; Kikuchi and Oden, 1988; Lee and Oden, 1993b; Oden and Carey, 1984), we obtain the following finite-dimensional system of mildly nonlinear equations:

$$\boldsymbol{K}\boldsymbol{u} + \boldsymbol{K}_N(\boldsymbol{u}) + \boldsymbol{K}_T(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{R}, \tag{46}$$

where $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{K}_N(\boldsymbol{u})$ and $\boldsymbol{K}_T(\boldsymbol{u}, \boldsymbol{u})$ are the vectors of the contact forces on Γ_c, $\boldsymbol{R}$ is the vector of the prescribed nodal forces and $\boldsymbol{u}$ is the vector of the nodal displacements. Applying the iterative schemes defined by Problems 2_k, $2_k'$ and $2_k''$ to the system of equations (46), the following algorithms are obtained:

Algorithm 1. For given $\boldsymbol{u}^0$, find $\boldsymbol{u}^{k+1/2}, \boldsymbol{u}^{k+1}, k = 0, 1, \dots,$ satisfying

$$\boldsymbol{K}\boldsymbol{u}^{k+1/2} + \boldsymbol{K}_N(\boldsymbol{u}^{k+1/2}) = \boldsymbol{R} - \boldsymbol{K}_T(\boldsymbol{u}^k, \boldsymbol{u}^k), \tag{47}$$

$$\boldsymbol{K}\boldsymbol{u}^{k+1} + \boldsymbol{K}_T(\boldsymbol{u}^{k+1/2}, \boldsymbol{u}^{k+1}) = \boldsymbol{R} - \boldsymbol{K}_N(\boldsymbol{u}^{k+1/2}), \tag{48}$$

until $\|\boldsymbol{u}^{k+1} - \boldsymbol{u}^k\|_h / \|\boldsymbol{u}^{k+1}\|_h < \delta$.

Algorithm 2. For given $\boldsymbol{u}^0$, find $\boldsymbol{u}^{k+1}, k = 0, 1, \dots,$ satisfying

$$\boldsymbol{K}\boldsymbol{u}^{k+1} + \boldsymbol{K}_N(\boldsymbol{u}^{k+1}) + \boldsymbol{K}_T(\boldsymbol{u}^k, \boldsymbol{u}^{k+1}) = \boldsymbol{R} \tag{49}$$

until $\|\boldsymbol{u}^{k+1} - \boldsymbol{u}^k\|_h / \|\boldsymbol{u}^{k+1}\|_h < \delta$.

Algorithm 3. For given $\boldsymbol{u}^0$, find $\boldsymbol{u}^{k+1}, k = 0, 1, \dots,$ satisfying

$$\begin{aligned}
&\boldsymbol{L}\boldsymbol{u}^{k+1} + \beta \boldsymbol{K}_T(\boldsymbol{u}^k, \boldsymbol{u}^{k+1}) \\
&\quad = \boldsymbol{L}\boldsymbol{u}^k - \beta \left(\boldsymbol{K}\boldsymbol{u}^k + \boldsymbol{K}_N(\boldsymbol{u}^k) \right) + \beta \boldsymbol{R},
\end{aligned} \tag{50}$$

until $\|\boldsymbol{u}^{k+1} - \boldsymbol{u}^k\|_h / \|\boldsymbol{u}^{k+1}\|_h < \delta$.

Here $\|\cdot\|_h$ is a vector norm, δ is the error tolerance, $\boldsymbol{L} = \boldsymbol{K}_0 + \boldsymbol{M}$, or $\boldsymbol{L} = \boldsymbol{K}_0$ is a matrix such that $\boldsymbol{K}_0$ is a stiffness matrix with elasticity constants $E = 1$, $\nu = 0$ and $\boldsymbol{M}$ is a mass matrix with density $\rho = 1$. Supposing further that $\boldsymbol{K}_N(\boldsymbol{u})$ and $\boldsymbol{K}_T(\cdot, \boldsymbol{u})$ are continuously differentiable with respect to $\boldsymbol{u}$, we can apply Newton's method.

Computational experiments show that Algorithm 1 is slower than Algorithm 2 and is not convergent in the frictionless case when $\kappa \geq m$. In the frictional case, Algorithm 1 is stable and convergent for any κ. Since the larger κ, the smaller c, the Signorini problem is approximated by a frictional problem with friction coefficients approaching 0^+. It can be finally concluded that Algorithm 1 behaves well if the theoretical restictions are satisfied. Algorithm 3 is stable, but slower than the two other algorithms, and as β close to its upper bound was found to produce a good speed of convergence. Using the three algorithms, the following example problem, considered in (Kikuchi and Oden, 1988), was solved and the computed results are given below:

Example 1. Find the displacements and stresses of a homogeneous, isotropic, linear elastic slab resting on a nonlinear elastic Winkler foundation along Γ_c, cf. Fig 1. The slab is subjected to distributed tractions $\boldsymbol{p} = (0, -20)$ on a part of Γ_σ, and no body forces are acting. We suppose that $u_N = 0$, $\boldsymbol{u}_T(8, 0) = 0$ on Γ_u and

$$k_N(u_N) = \begin{cases} 0 & \text{for } u_N \le 0, \\ 0.5 \times 10^5 u_N^2 & \text{for } 0 < u_N \le 10^{-3}, \\ 10^2 u_N - 0.05 & \text{for } u_N > 10^{-3}, \end{cases} \tag{51}$$

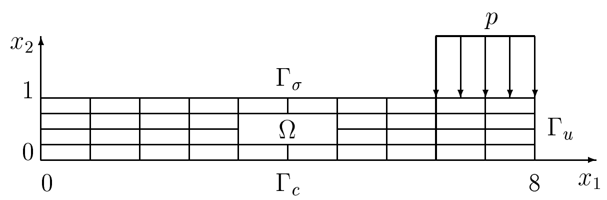

Fig. 1. Slab on a nonlinear Winkler foundation.

on Γ_c. The problem is in a plane-stress state. The elasticity constants are: Young's modulus $E = 10^3$, Poisson's ratio $\nu = 0.3$. Four-noded bilinear and two-noded linear, isoparametric finite elements are used for the discretization of the domain and the contact boundary. The finite element matrices are formed by applying the corresponding Gauss integration rules. The parameter $\beta = 0.01$ is taken in the computations. The results are obtained within an accuracy of $\delta = 10^{-6}$, for about 80–140, 15–30 and 170–240 iterations, depending on the algorithm and the friction coefficient used. It should be mentioned that for this example, Algorithm 1 is not convergent for $\mu = 0$ and $\mu \geq 1$. The computed normal and tangential displacements for various coefficients of friction are shown in Figs. 2 and 3.

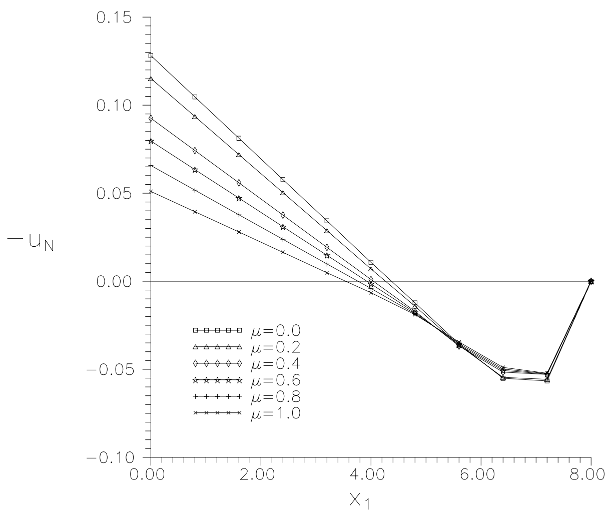

Fig. 2. Normal displacements along the contact interface.

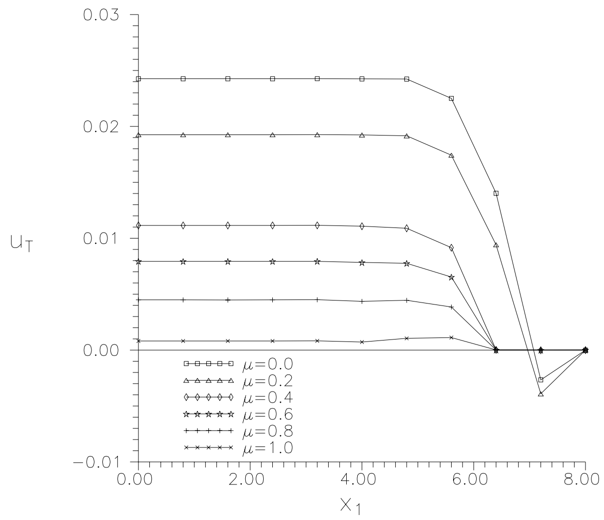

Fig. 3. Tangential displacements along the contact interface.

6. Concluding Remarks

In this work, we have considered a class of contact problems in elastostatics, with nonlinear normal and friction boundary conditions. For the corresponding variational problems, we applied and analysed the two-step iterative method proposed by Panagiotopoulos (1975). Its convergence is proved under an additional restriction on the magnitude of the friction coefficient. The method is compared with two other iterative methods and its applicability is demonstrated. Computational experiments show that the method behaves well if the theoretical restictions on the friction coefficient are satisfied.

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