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OUTPUT STABILIZATION FOR INFINITE–DIMENSIONAL BILINEAR SYSTEMS

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The purpose of this paper is to extend results on regional internal stabilization for infinite bilinear systems to the case where the subregion of interest is a part of the boundary of the system evolution domain. Then we characterize either stabilizing control on a boundary part, or the one minimizing a given cost of performance. The obtained results are illustrated with numerical examples.

Keywords: infinite bilinear systems, output stabilization, regional stabilization

1. Introduction

For distributed parameter system theory, the term 'regional analysis' has been used to refer to control problems in which the target of interest is not fully specified as a state, but refers only to a subregion ω of the spatial domain Ω on which the system is considered.

For a stabilization problem one normally considers a control system on a time interval $]0, +\infty[$ and searches for feedback control in such away that the state evolving on Ω close to its the steady state of the system when $t \to \infty$.

Recently, the question of regional stabilization for infinite-dimensional linear systems has been tackled and developed by Zerrik and Ouzahra (2003a). It consists in studying the asymptotic behaviour of a distributed system only within a subregion ω of its evolution domain Ω. This notion includes the classical one and enables us to analyse the behaviour of a distributed system in any subregion of its spatial domain. Also, it makes sense for the usual concept of stabilization taking account of the spatial variable and then becomes closer to real-world problems, where one wishes to stabilize a system in a critical subregion of its geometrical domain.

In real problems it is also plausible that the target region of interest be a portion of the boundary $\partial\Omega$ of Ω so that the stabilization is required only on $\Gamma \subset \partial\Omega$, rather than in an actual subregion.

In (Zerrik *et al.*, 2004) the question of regional internal stabilization for infinite bilinear systems was considered. The properties and characterizations of control ensuring regional stabilization in a subregion interior to the system domain with various illustrating examples were given.

A natural extension may be the case where the target part is located on the boundary of the evolution domain. Technically, the difficulty is that the relevant restriction map is now a trace map and cannot be expected to be continuous.

The principal reason for considering this case is that, firstly, there exist systems which are stable on some boundary subregion but are unstable in any neighbourhood $\omega \subset \Omega$ of Γ satisfying $\Gamma \subset \partial\omega$ (see the example in Section 2), and, secondly, it is closer to a real situation. (For example, the treatment of water by using a bioreactor where the objective is to regulate the concentration of the substrate at the boundary output of the bioreactor (see Fig. 1).)

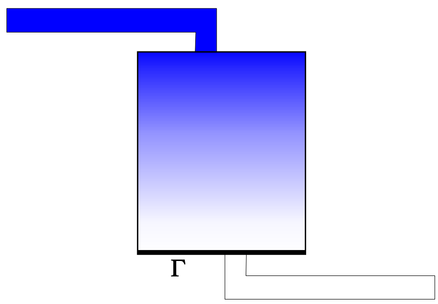

Fig. 1. Regulation of substrate concentration at the boundary output of the reactor.

This paper considers the question of regional boundary stabilization for an infinite bilinear system defined in a domain $\Omega \subset \mathbb{R}^n$, $(n \geq 2)$ with a regular boundary $\partial\Omega$:

$$\frac{\partial z}{\partial t} = Az + v(t)Bz, \text{on } Q = \Omega\times]0, \infty[,$$
$$z(\cdot, 0) = z_0 \quad \text{on} \quad \Omega, \quad (1)$$

where A is the infinitesimal generator of a linear strongly continuous semigroup $S(t)$, $t \geq 0$ on a Hilbert state space Z endowed with a complex inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$, B is a linear bounded operator from Z to Z. We suppose that for any initial state z_0, there exists a control function $v(t)$ such that (1) has a unique mild solution $z(t)$. The problem of regional boundary stabilization of (1) in a subregion Γ of $\partial\Omega$ consists in choosing the control $v(t)$ in such a way that the trace $\gamma_\Gamma z(t)$ of $z(t)$ on Γ converges to zero in some sense. This is the aim of this paper, which is organized as follows: In Section 2 we will define regional boundary stabilization for bilinear systems and give characterizations of stabilizing control. In the third section, we consider the problem of finding stabilizing control in a boundary subregion and minimizing a given cost of performance, and provide a characterization of such a control. Finally, the results are illustrated with a numerical example.

2. Regional Boundary Stabilization for Bilinear Systems

2.1. Notation and Definitions

For $\Gamma \subset \partial\Omega$ (or $\Gamma \subset \Omega$ such that meas$(\Gamma) = 0$), we consider the space $L^2(\Gamma)$ endowed with the complex inner product $(z, y) = \int_\Gamma z\overline{y}\,\mathrm{d}\sigma$, and the corresponding norm $|z| = (\int_\Gamma |z|^2\,\mathrm{d}\sigma)^{\frac{1}{2}}$ ($\mathrm{d}\sigma$ is the surface measure defined on Γ and induced by the Lebesgue measure).

The state space Z is such that for a subregion Γ with $\sigma(\Gamma) > 0$, the restriction map γ_Γ on Γ is bounded. Let $\gamma_\Gamma{}^*$, be its adjoint operator and consider the operator $i_\Gamma = \gamma_\Gamma^*\gamma_\Gamma$.

$\mathcal{L}(Z)$ will denote the space of bounded linear operators mapping Z into itself endowed with the uniform norm of operators $|||\cdot|||$.

Definition 1. The system (1) is said to be

1. Regionally weakly boundary stabilizable (r.w.b.s.) on Γ, if $\gamma_\Gamma z(t)$ tends to 0 weakly, as $t \to \infty$.

2. Regionally strongly boundary stabilizable (r.s.b.s.) on Γ, if $\gamma_\Gamma z(t)$ tends to 0 strongly, as $t \to \infty$.

3. Regionally exponentially boundary stabilizable (r.e.b.s.) on Γ, if $\gamma_\Gamma z(t)$ tends to 0 exponentially, as $t \to \infty$.

Remark 1.

- We are only interested in the behaviour of (1) on Γ without constraints on Ω, so the regularity of the solution $z(t)$ is needed only in a neighbourhood ω of Γ to obtain a trace operator γ_Γ on Γ. Moreover, if the system (1) is regionally strongly stabilizable on a subregion $\omega \subset \Omega$ satisfying $\Gamma \subset \partial\omega$, then (1) is regionally boundary stabilizable on Γ by the same control.

- If the system (1) is regionally boundary stabilizable on $\Gamma \subset \partial\Omega$, then it is regionally stabilizable on $\Gamma' \subset \Gamma$ using the same control.

- The regional stabilization problem can be seen as a special case of output stabilization for infinite dimensional systems with the partial observation $y = \gamma_\Gamma z$.

- This notion includes the case where the target part is an internal subregion $\omega \subset \Omega$, which is of null measure.

- Stabilizing a system on a boundary part Γ may be cheaper than stabilizing it in any neighbourhood $\omega \subset \Omega$ of Γ.

In the following, we shall give two examples illustrating the above remarks.

Example 1. A feedback may be a stabilizing control in part Γ of measure null, but not a stabilizing one in an internal part ω verifying $\partial\omega \supset \Gamma$.

Let us consider the system defined in $\Omega =]0, 1[^2$ by

$$\begin{cases} \dfrac{\partial z(t)}{\partial t} = \Delta z(t) + v(t) & \text{in } Q, \\ z(\cdot, 0) = z_0 & \text{in } \Omega. \end{cases} \quad (2)$$

Here we take $A = \Delta$ with Neumann boundary conditions.

The eigenpairs $(\lambda_{(k,l)}, \varphi_{(k,l)})$ of A are given by

$$\lambda_{(k,l)} = -(k^2 + l^2)\pi^2, \quad k, l \geq 0,$$

and

$$\varphi_{(k,l)}(x, y) = 2\cos(k\pi x)\cos(l\pi y), \quad \text{if } k\,l \neq 0,$$

and

$$\varphi_{(0,0)}(x, y) = 1, \text{ otherwise}$$

the feedback

$$\begin{aligned} v(t) &= Kz(t) \\ &= -\langle z(t), \varphi_{(0,0)}\rangle\varphi_{(0,0)} + 2\pi^2\langle z(t), \varphi_{(1,0)}\rangle\varphi_{(1,0)} \end{aligned}$$

does not stabilize (2) in any internal subregion. But for $\Gamma = \{\frac{1}{2}\} \times [0,1]$, we have $\chi_\Gamma \varphi_{(1,0)} = 0$. Then

$$\chi_\Gamma z(t) = \sum_{(k,l)\neq(1,0)} e^{t\lambda_{(k,l)}} \langle z_0, \varphi_{(k,l)}\rangle \chi_\Gamma \varphi_{(k,l)}$$

$$\to 0 \text{ exponentially as } t \to +\infty.$$

♦

Example 2. Let us consider the system defined on $\Omega =]0,2[^2$ by

$$\begin{cases} \dfrac{\partial z(t)}{\partial t} = Az(t) + v(t)|\langle z(t), \varepsilon(\cdot)\rangle|\varepsilon(\cdot) \quad \text{in } Q, \\ z_0 \in Z \end{cases} \tag{3}$$

evolving in the state space $Z := \{z \in H^1(\Omega) \mid z = 0 \text{ on } \Gamma_0 = \{0\} \times [0,1]\}$, which is a closed subspace of $H^1(\Omega)$ endowed with its natural inner product, so Z is a Hilbert space, where $Az = (\frac{1}{2} - y)z$, $\varepsilon(\cdot) = 1$ on Ω, and $\Gamma = \{0\} \times [0,2]$.

For $z_0 \in \mathcal{D}(A)$, we have

$$\|\chi_\Gamma S(t) z_0\|^2_{L^2(\Gamma)} = \int_1^2 e^{2(\frac{1}{2}-y)t} |z_0(0,y)|^2 \,\mathrm{d}y \le e^{-t} \int_1^2 |z_0(0,y)|^2 \,\mathrm{d}y.$$

This inequality holds, by density, in Z, which shows that (3) is exponentially stable on Γ.

Now for any subregion $\omega \subset \Omega$ verifying $\Gamma \subset \partial\omega$, $(\exists\ a > 0,\ 0 < b < \frac{1}{2})$, $\omega_0 =]0,a[\times]b,\frac{1}{2}[\subset \omega$, then $\|\chi_\omega z(t)\| \ge \|\chi_{\omega_0} z_0\|_{L^2(\omega_0)}$, so (3) is not regionally exponentially stable in ω, where χ_ω is the restriction map in ω.

The system (3) is exponentially stabilizable on Γ by $v(t) = 0$, but for a subregion ω such that $\Gamma \subset \partial\omega$, (3) is not exponentially regionally stable on ω. Then if we consider the functional cost $q(v) = \int_0^{+\infty} \|v(t)\|_V^2 \,\mathrm{d}t$, we obtain $\min_{\mathcal{V}_{ad}(\Gamma)} q(v) = 0 < \min_{\mathcal{V}_{ad}(\omega)} q(v)$. ♦

2.2. Stabilizing Control

In what follows we give sufficient conditions for the control $v(t)$ to be a stabilizing one for (1). For that purpose we have to ensure the existence and uniqueness of a global solution. It is known that if $v(\cdot) \in L^1(0,\infty;\mathbb{C})$, then (1) has a unique global mild solution (Ball *et al.*, 1982), and if $v(t)$ is a quadratic feedback control law $v(t) = -\langle Kz(t), z(t)\rangle$, where $K \in \mathcal{L}(Z)$, then (1) has a unique mild solution $z \in \mathcal{C}([0,t_{\max}[;Z)$ defined on a maximal interval $[0,t_{\max}[$. Moreover, if $z(t)$ is bounded on $[0,t_{\max}[$, the solution $z(t)$ is global: $t_{\max} = +\infty$, (Pazy, 1983). This is the case when $S(t)$ is a contraction and

$$\mathrm{Re}\left(\langle z, Kz\rangle\langle Bz, z\rangle\right) \ge 0, \quad \forall z \in Z. \tag{4}$$

In this case, the mapping $z_0 \to z(t)$ is continuous in $\mathcal{C}([0,t_{\max}[;\ Z)$. (Ball *et al.*, 1982, Zerrik *et al.*, 2004).

Now we proceed to stabilization results for (1), and we begin with the following result, giving sufficient conditions for regional boundary weak stabilization:

Proposition 1. *Suppose that $S(t)$ is a contraction and B is compact. If*

$$\langle BS(t)\phi, S(t)\phi\rangle = 0,\ t \ge 0 \Longrightarrow \gamma_\Gamma \phi = 0, \tag{5}$$

then the system (1) is weakly regionally stabilizable on Γ by the feedback control

$$v(t) = -\langle Bz(t), z(t)\rangle.$$

Proof. From (Ball and Slemrod, 1979) there exists ϕ such that $\langle BS(t)\phi, S(t)\phi\rangle = 0$, $t \ge 0$, and that $z(t) \to \phi$ weakly as $t \to +\infty$, so by the continuity of $\chi_\omega z(t)$ we have $\langle i_\Gamma z(t), \varphi\rangle \to \langle i_\Gamma \phi, \varphi\rangle$, as $t \to +\infty, \forall \varphi \in Z$, and then the conclusion follows from (5). ■

For illustration, consider the system (1) governed by the dynamics

$$Az = -\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \quad \mathcal{D}(A) = H^2(A) \cap H_0^1(\Omega),$$

and

$$(Bz)(\cdot) = \Big(\int_0^1 z(0,y)\,\mathrm{d}y\Big) i_\Gamma\ \varepsilon(\cdot) = \langle z, i_\Gamma\ \varepsilon(\cdot)\rangle i_\Gamma\ \varepsilon(\cdot),$$

where in the state space $H^1(\Omega)$ with $\Omega =]0,+\infty[^2$, $\Gamma = \{0\} \times [0,1]$ and $\varepsilon(x) = 1$ a.e $x \in \Omega$.

The operator A generates in $L^2(\Omega)$ a semigroup of contractions defined by

$$\big(S(t)\phi\big)(x,y) = \begin{cases} \phi(x-t, y-t) & \text{if } x \ge t \text{ and } y \ge t, \\ 0 & \text{otherwise.} \end{cases}$$

For $z_0 \in H^1(\Omega)$ we have $\|S(t)z_0 - z_0\| = \|S(t)z_0 - z_0\|_{L^2(\Omega)} + \|(S(t)z_0)' - z_0'\|_{L^2(\Omega)}$. But $(S(t)z_0)' = S(t)z_0'$, and then, using the fact that $S(t)$ is a c_0-semigroup in $L^2(\Omega)$, we deduce that $\|S(t)z_0 - z_0\| \to 0$ as $t \to 0$. Then $S(t)$ induce a c_o-semigroup on $H^1(\Omega)$. Moreover $S(t)$, remains a semigroup of contraction in $H^1(\Omega)$. Indeed, for $z_0 \in H^1(\Omega)$ we have $\|S(t)z_0\| = \|S(t)z_0\|_{L^2(\Omega)} + \|S(t)z_0'\|_{L^2(\Omega)} = \|z_0\|_{L^2(\Omega)} + \|z_0'\|_{L^2(\Omega)} = \|z_0\|$.

Moreover, B is a compact operator and we have

$$\langle BS(t)\phi, S(t)\phi\rangle = \Big(\int_0^{1-t} \phi(0,y)\,\mathrm{d}y\Big)^2,\ 0 \le t \le 1.$$

Then

$$\langle BS(t)\phi, S(t)\phi\rangle = 0, \quad \forall t \geq 0 \Rightarrow \phi(0,y) = 0$$

a.e $y \in]0,1[$ i.e. $\gamma_\Gamma \phi = 0$. Then the control

$$v(t) = -\Big(\int_0^1 z(0,y,t)\,\mathrm{d}y\Big)^2$$

ensures weak stabilization of the analysed system on the boundary subregion Γ. This example shows in particular that (5) can be satisfied for $\Gamma \subset \partial\Omega$ but not for Ω.

2.3. Decomposition Method

In this part we shall give an approach based on the decomposition of a state space and a system. Let $\delta > 0$, and consider the subsets $\sigma_u(A)$ and $\sigma_s(A)$ of the spectrum $\sigma(A)$ of A, defined by

$$\sigma_u(A) = \big\{\lambda : \mathrm{Re}\,(\lambda) \geq -\delta\big\},$$

$$\sigma_s(A) = \big\{\lambda : \mathrm{Re}\,(\lambda) < -\delta\big\}.$$

Suppose that the set $\sigma_u(A)$ is bounded and is separated from the set $\sigma_s(A)$ in such a way that a rectifiable, simple, closed curve can be drawn so as to enclose an open set containing $\sigma_s(A)$ in its interior and $\sigma_u(A)$ in its exterior, which is the case if A is selfadjoint with a compact resolvent. In this case there are at most finitely many nonnegative eigenvalues of A, each with a finite dimensional eigenspace (Triggiani, 1975). Then the state space Z can be decomposed (Kato, 1980) according to $Z = Z_u + Z_s$ with $Z_u = PZ$ along $Z_s = (I-P)Z$, and $P \in \mathcal{L}(Z)$ is the projection given by

$$P = \frac{1}{2\pi i}\int_C (\lambda I - A)^{-1}\,\mathrm{d}\lambda,$$

where C is a curve surrounding $\sigma(A)$. Suppose that $BP = PB$, which is the case if B satisfies $\langle ABz, z\rangle - \langle BAz, z\rangle = 0, \;\; z \in \mathcal{D}(A)$.

The system (1) may be decomposed into the following ones:

$$\frac{\partial z_u(t)}{\partial t} = A_u z_u(t) + v(t)B_u z_u(t), \qquad z_{0u} = Pz_0, \;\; z_u = Pz, \tag{6}$$

$$\frac{\partial z_s(t)}{\partial t} = A_s z_s(t) + v(t)B_s z_s(t), \qquad z_{0s} = (I-P)z_0, \;\; z_s = (I-P)z, \tag{7}$$

where $A_s = (I-P)A(I-P)$, $A_u = PAP$, $B_s = (I-P)B(I-P)$ and $B_u = PBP$.

In the internal case, if the operator A_s satisfies the spectrum growth assumption, namely,

$$\lim_{t\to+\infty}\frac{\ln |||S_s(t)|||}{t} = \sup \mathrm{Re}\,\big(\sigma(A_s)\big), \tag{8}$$

then stabilizing the system (1) boils down to stabilizing (6) (Zerrik *et al.*, 2004). In the boundary case we have a similar result.

Proposition 2. *Let A_s satisfy (8). If there exists $K_u \in \mathcal{L}(Z_u)$, such that the control*

$$v_u(t) = -\langle K_u z_u(t), z_u(t)\rangle \tag{9}$$

regionally weakly (strongly, exponentially) stabilizes the system (6) on Γ with a bounded state $z_u(t)$, then the system (1) is regionally weakly (strongly, exponentially) stabilizable on Γ using the same control (9), and the state $z(t)$ remains bounded.

Proof. Let $z_s(t)$ be the solution of (7) defined on a maximal interval $[0, t_{\max}[$. We shall show that $z_s(t)$ is bounded on $[0, t_{\max}[$ to conclude that $t_{\max} = +\infty$.

The solution of (7) is given by

$$z_s(t) = S_s(t)z_{0s} + \int_0^t v_u(\tau)S_s(t-\tau)B_s z_s(\tau)\,\mathrm{d}\tau, \tag{10}$$

where $S_u(t)$ and $S_s(t)$ denote the restrictions of $S(t)$ to Z_u and Z_s, which are strongly continuous semigroups generated respectively by A_u and A_s.

In view of the above decomposition, one has $\sup \mathrm{Re}\,(\sigma(A_s)) \leq -\delta$. Hence, if A_s satisfies (8), then for some $a > 0$, $0 < \delta' < \delta$, we obtain $\|S_s(t)\| \leq a\, e^{-\delta t}$, $\forall\; t \geq 0$ (Triggiani, 1975), and using (10) we have

$$\|z_s(t)\| \leq a\|z_{0s}\|\, e^{-\delta' t} + a\|B_s\|\int_0^t |v_u(\tau)|\, e^{-\delta'(t-\tau)}\|z_s(\tau)\|\,\mathrm{d}\tau.$$

By the Gronwall inequality we have

$$\|z_s(t)\| \leq a\|z_{0s}\|\, e^{-\eta t} + a^2\|z_{0s}\|\cdot\|B_s\| \times \int_0^t e^{-\eta\tau}|v_u(\tau)|\, e^{-\eta(t-\tau)}y(\tau)\,\mathrm{d}\tau. \tag{11}$$

Moreover, since $z_u(t)$ is bounded, from (9) we get that so is $v_u(t)$. Then there exists $M(z_0) > 0$, such that $y(\tau) \leq e^{M(z_0)te^{-\eta t}}, \forall \tau \leq t$, then

$$\|z_s(t)\| \leq \big[A(z_0) + B(z_0)te^{M(z_0)te^{-\eta t}}\big]e^{-\eta t} \tag{12}$$

for the positive functions $A(z_0)$ and $B(z_0)$, which shows that the state $z_s(t)$ is bounded on $[0, t_{\max}[$ and hence

$z_s(t)$ is defined for all $t \geq 0$. Consequently, the state $z(t) = z_u(t) + z_s(t)$ is a bounded global solution, and $\gamma_\Gamma z_s(t) \to 0$ exponentially, as $t \to +\infty$, which completes the proof. ■

Corollary 1. *Suppose that $S(t)$ is a semigroup of contractions and A_s satisfies (8). If B_u is compact and satisfies $\langle B_u S_u(t)\phi_u, S_u(t)\phi_u\rangle = 0,\ t \geq 0 \Rightarrow \gamma_\Gamma \phi_u = 0$, then the control $v(t) = -\langle B_u z_u(t), z_u(t)\rangle$ regionally weakly stabilizes (1) on Γ, and $z(t)$ remains bounded on Ω.*

Corollary 2. *Let A be self-adjoint with compact resolvent and suppose that $S(t)$ is a semigroup of contractions. If*

1.

$$\big(\forall\ (n,m) \in [1,N]^2\big)\big(\forall\ (j,k) \in [1,r_n] \times [1,r_m]\big),$$
$$\langle B\varphi_{n_j}, \varphi_{m_k}\rangle \neq 0 \ \text{ iff } \ n = m, \tag{13}$$

2. there exists $1 \leq n \leq N$ such that the matrix

$$B_n := \big(\langle B\varphi_{n_j}, \varphi_{n_k}\rangle\big)_{1\leq j,k\leq r_n}$$

satisfies

$$\langle B_n S_u(t)\phi_u, S_u(t)\phi_u\rangle = 0,\ t \geq 0 \Rightarrow \gamma_\Gamma \phi_u = 0,$$

where r_n is the multiplicity of λ_n, and φ_{n_j} are the eigenfunctions associated with λ_n, then (1) is regionally boundary weakly stabilizable on Γ.

Proof. Here the space Z_u is finite dimensional, so the operator B_u is of a finite rank and hence it is compact. Now if $\langle B_u S_u(t) z_{0u}, S_u(t) z_{0u}\rangle = 0$, then under the condition (13) we have

$$\sum_{n=1}^{N} e^{2\lambda_n t} \sum_{j,k=1}^{r_n} \langle z_{0u}, \varphi_{n_j}\rangle \langle z_{0u}, \varphi_{n_k}\rangle \langle B\varphi_{n_j}, \varphi_{n_k}\rangle = 0, \quad \forall t \geq 0,$$

which implies

$$\sum_{j,k=1}^{r_n} \langle z_{0u}, \varphi_{n_j}\rangle \langle z_{0u}, \varphi_{n_k}\rangle \langle B\varphi_{n_j}, \varphi_{n_k}\rangle = 0, \quad \forall 1 \leq n \leq N.$$

In other words, $\langle B_n z_{0u}, z_{0u}\rangle = 0,\ \forall 1 \leq n \leq N$, and hence $\gamma_\Gamma z_{0u} = 0$. Then from the above corollary the system (1) is weakly stabilizable on Γ by the control $v(t) = -\langle B_u z_u(t), z_u(t)\rangle$. ■

3. Regional Stabilization Problem

The aim of this section is to determine the minimum energy control that yields regional boundary stabilization of the system (1) on Γ.

A natural approach to the regional boundary stabilization problem is to formally differentiate $|\gamma_\Gamma z(t)|^2$ along the trajectories of (1), which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}|\gamma_\Gamma z(t)|^2 = 2\mathrm{Re}\,\langle i_\Gamma A z(t), z(t)\rangle + 2\mathrm{Re}\, v(t)\langle i_\Gamma B z(t),\ z(t)\rangle.$$

So if the operator $i_\Gamma A$ is dissipative, then an obvious choice of the feedback control is $v(t) = -\langle z(t), i_\Gamma B z(t)\rangle$, since it yields the "dissipating energy inequality"

$$\frac{\mathrm{d}}{\mathrm{d}t}|\gamma_\Gamma z(t)|^2 \leq -2\langle z(t), i_\Gamma B z(t)\rangle^2.$$

Then let A satisfy

$$\langle i_\Gamma A z, z\rangle + \langle z, i_\Gamma A z\rangle + \langle i_\Gamma R z, z\rangle = 0, \quad z \in \mathcal{D}(A) \tag{14}$$

for a linear self-adjoint and positive operator R.

Our problem can be formulated as follows:

$$\begin{cases} \min q_\Gamma(v) = \displaystyle\int_0^{+\infty} \langle i_\Gamma R z(t), z(t)\rangle\,\mathrm{d}t + \int_0^{+\infty} |\langle i_\Gamma B z(t), z(t)\rangle|^2\,\mathrm{d}t + \int_0^{+\infty} |v(t)|^2\,\mathrm{d}t, \\ v \in \mathcal{U}_{ad}(\Gamma) = \big\{v \mid z(t) \text{ is a global solution and } q_\Gamma(v) < \infty\big\}. \end{cases} \tag{15}$$

Suppose that for some non-negative constants α, β, and δ, we have

$$|\gamma_\Gamma S(t) z| \leq \alpha |\gamma_\Gamma z|, \quad t \geq 0$$

and

$$|\gamma_\Gamma B z| \leq \beta |\gamma_\Gamma z|, \quad z \in Z, \tag{16}$$

and

$$\int_0^1 |\langle i_\Gamma B S(t) z, S(t) z\rangle|\,\mathrm{d}t \geq \delta |\gamma_\Gamma z|^2, \quad z \in Z. \tag{17}$$

We note that (16) means that the operators $\gamma_\Gamma S(t)$ and $\gamma_\Gamma B$ are continuous with respect to $\gamma_\Gamma z$.

3.1. Direct Approach

We shall characterize the solution of the problem (15) without taking into account the internal behaviour of (1). For this let us establish the following result, which gives a bound on the initial state on Γ:

Lemma 1. *Let*

$$\lambda := \int_0^1 |v(t)|^2 \,\mathrm{d}t, \quad \mu := \int_0^1 |\langle i_\Gamma Bz(t), z(t)\rangle|^2 \,\mathrm{d}t.$$

There exist $0 < \eta < 1$ *and* $l > 0$ *independent of* z_0*, such that*

$$\lambda < \eta \Rightarrow |\gamma_\Gamma z_0|^2 < l\sqrt{\mu}.$$

Proof. Let $\psi(t) = \gamma_\Gamma(z(t) - S(t)z_0)$, $0 \le t \le 1$. We have

$$\psi(t) = \int_0^t v(s)\gamma_\Gamma S(t-s)BS(s)z_0 \,\mathrm{d}s + \int_0^t v(s)\gamma_\Gamma S(t-s)B(z(s) - S(s)z_0)\,\mathrm{d}s.$$

Using (16), we obtain $|\psi(t)| \le \alpha\beta|\gamma_\Gamma z_0| \int_0^t |v(s)|\,\mathrm{d}s + \alpha\beta \int_0^t |v(s)| \cdot |\psi(s)|\,\mathrm{d}s$. Then using the triangle inequality, from (16) we obtain that there exist two non-negative constants a and b independent of z_0 such that (Quinn, 1980):

$$\int_0^1 |\langle i_\Gamma BS(t)z_0, S(t)z_0\rangle|\,\mathrm{d}t \le \sqrt{\mu} + (a\sqrt{\lambda} + b\lambda)|\gamma_\Gamma z_0|^2.$$

Taking $\lambda \le 1$ and using (17), we obtain

$$\big(\delta - (a+b)\sqrt{\lambda}\big)|\gamma_\Gamma z_0|^2 \le \sqrt{\mu}.$$

Then

$$0 < \eta < \frac{1}{2}\inf\Big(1, \frac{\delta^2}{(a+b)^2}\Big)$$

realizes the desired estimate. ∎

Theorem 1. *Let* $v^*(t) = -\langle z^*(t), i_\Gamma Bz^*(t)\rangle$*, and suppose that the corresponding solution* $z^*(t)$ *of (1) is global. Then* $v^*(t)$ *is the unique feedback control solution of (15), which strongly stabilizes (1) on* Γ*. Moreover, if there exists* $\eta > 0$ *such that*

$$\frac{1}{2}\langle i_\Gamma Rz, z\rangle + |\langle z, i_\Gamma Bz\rangle|^2 \ge \eta \,\mathrm{Re}\big(\langle Bz, z\rangle\langle z, i_\Gamma Bz\rangle - \langle Az, z\rangle\big), \quad \forall z \in \mathcal{D}(A), \tag{18}$$

then the state remains bounded on Ω*.*

Proof. For $z_0 \in \mathcal{D}(A)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\gamma_\Gamma z^*(t)|^2 = -\langle i_\Gamma Rz^*(t), z^*(t)\rangle - 2|\langle i_\Gamma Bz^*(t), z^*(t)\rangle|^2,$$

which implies

$$\int_0^t |\langle i_\Gamma Bz^*(s), z^*(s)\rangle|^2 \,\mathrm{d}s \le \langle i_\Gamma z_0, z_0\rangle, \quad t \ge 0. \tag{19}$$

Since $z^*(t)$ is supposed to be continuous with respect to the initial conditions, (19) holds for all $z_0 \in Z$, so $q_\Gamma(v)$ is finite for all $z_0 \in Z$.

Let us show that each control $v \in \mathcal{U}_{ad}(\Gamma)$ strongly stabilizes (1) on Γ. To this end, let η be the constant given by the above lemma and let $0 < \epsilon < \eta$.

Since $q_\Gamma(v)$ is finite, there exists $T > 0$ such that for $t > T$ we have

$$\int_t^{t+1} |\langle i_\Gamma Bz(s), z(s)\rangle|^2 \,\mathrm{d}s < \epsilon$$

and

$$\int_t^{t+1} |v(t)|^2 \,\mathrm{d}t < \epsilon.$$

Taking $z_0 = z(t)$, we get $|\gamma_\Gamma z(t)|^2 < l\sqrt{\epsilon}$, $\forall t > T$ so $|\gamma_\Gamma z(t)| \to 0$, as $t \to +\infty$. But

$$\frac{\mathrm{d}}{\mathrm{d}t}|\gamma_\Gamma z^*(t)|^2 = |\langle z(t), i_\Gamma Bz(t)\rangle + v(t)|^2 - |\langle i_\Gamma Bz(t), z(t)\rangle|^2 - |v(t)|^2 - \langle i_\Gamma Rz(t), z(t)\rangle,$$

and then we have

$$q_\Gamma(v) = |\gamma_\Gamma z_0|^2 + \int_0^\infty |\langle z(t), i_\Gamma Bz(t)\rangle + v(t)|^2 \,\mathrm{d}t, \quad \forall z_0 \in \mathcal{D}(A).$$

Setting $v = v^*$, we obtain $q_\Gamma(v^*) = |\gamma_\Gamma z_0|^2$, and then $q_\Gamma(v) \ge q_\Gamma(v^*)$, $\forall v \in \mathcal{U}_{ad}(\Gamma)$.

Let $z_0 \in Z$ and $z_{0n} \subset \mathcal{D}(A)$ such that $z_{0n} \to z_0$, as $n \to +\infty$. For $v \in \mathcal{U}_{ad}(\Gamma)$, we have

$$V\big(z_n(t)\big) - V\big(z_{on}\big) = \int_0^t |\langle z_n(s), P_\Gamma Bz_n(s)\rangle + v(s)|^2 \,\mathrm{d}s - \int_0^t \big(|\langle z_n(s), P_\Gamma Bz_n(s)\rangle|^2 + |v(s)|^2\big)\,\mathrm{d}s.$$

Since V is continuous, we have $V(z_{0n}) \to V(z_0)$, as $n \to +\infty$, so

$$q(v) + \big(V(z(t)) - V(z_0)\big) = \int_0^t |\langle z(s), P_\Gamma Bz(s)\rangle + v(s)|^2 \,\mathrm{d}s \ge 0, \quad \forall z_0 \in H.$$

We deduce that $q_\Gamma(v) \geq q_\Gamma(v^*)$. Moreover, it is clear that $v^*(t)$ is unique. Now, if (18) holds, then for $z_0 \in \mathcal{D}(A)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\gamma_\Gamma z(t)|^2 \geq \eta \frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2,$$

so

$$\langle i_\Gamma z(t), z(t)\rangle \geq \eta \|z(t)\|^2 - \eta \|z_0\|^2.$$

This inequality holds for all initial states, since its terms are continuous in z_0, and the stability of (1) on Γ completes the proof. ■

Remark 3.

1. Note that if A is dissipative, B is a monotone operator and commutes with i_Γ, then the operator $K = i_\Gamma B$ satisfies (4), which implies that the solution $z^*(t)$ is bounded and global.

2. $v^*(t)$ is a feedback of $i_\Gamma z^*(t)$, which can be seen as a feedback of the trace $\gamma_\Gamma z^*(t)$.

In the following result we give an estimate of $|\gamma_\Gamma z^*(t)|$ for "conservative systems":

Proposition 3. *Let $i_\Gamma R = 0$. If the solution $z^*(t)$ is global, then for any initial state z_0 such that $\gamma_\Gamma z_0 \neq 0$, we have*

$$|\gamma_\Gamma z^*(t)| = O(t^{-\frac{1}{2}}) \ \textit{as}\ t \to +\infty.$$

Proof. Let us consider the sequence

$$V_k = \frac{1}{2}|\gamma_\Gamma z^*(k)|^2, \quad k \in \mathbb{N}.$$

For $z_0 \in \mathcal{D}(A)$ we have

$$\begin{aligned} V_k - V_{k+1} &= \int_k^{k+1} |\langle i_\Gamma B z^*(t), z^*(t)\rangle|^2 \,\mathrm{d}t \\ &= \int_k^{k+1} |v^*(t)|^2 \,\mathrm{d}t. \end{aligned}$$

Taking $z_0 = z(k)$ in the above lemma, we obtain

$$V_k - V_{k+1} \geq \eta \ \text{ or } \ V_{k+1} - V_k < \frac{-1}{l^2}|\gamma_\Gamma z_0|^4. \qquad (20)$$

If $\gamma_\Gamma z_0 \neq 0$, then (20) gives $V_{k+1} - V_k \leq -a(z_0)V_k^2$, where $a(z_0) = \min(2/l^2, \eta/V_0^2)$. Then since V_k is a positive non-increasing sequence, we have

$$V_k \leq \frac{V_0}{(a(z_0)V_0)k + 1}, \quad k \geq 0,$$

which implies the estimate (Quinn, 1980):

$$|\gamma_\Gamma z^*(t)|^2 \leq \frac{2V_0}{(a(z_0)V_0)t + 1}, \quad t \geq 0.$$

Using the same techniques as in the above proposition, we show that this inequality holds for all $z_0 \in Z$. Then we obtain the desired estimate. ■

3.2. Internal Approach

In this part we give a link between the boundary stabilization problem (15) and the internal one. Then we show that one may consider the internal regional stabilization problem to stabilize the system on a boundary part. On each point x of Γ we consider the ingoing normal vector $\overrightarrow{N}_x$ to Γ on which we take a segment of length $1/n$ as illustrated in Fig. 2 Then we obtain a non-increasing

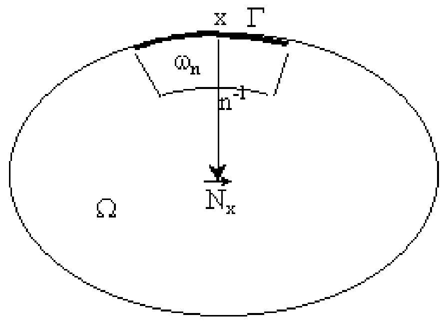

Fig. 2. Target boundary part Γ and the neighbourhood ω_n.

sequence (ω_n) of the subsets of Ω, which converges to $\Gamma = \bigcap_{n\geq 1} \overline{\omega_n}$. Let set $Z = H^1(\Omega)$. The bilinear form $a : (z, y) \to \langle \chi_{\omega_n} z, \chi_{\omega_n} y\rangle_{L^2(\omega)}$ is an hermitian positive and continuous map in $H^1(\Omega)\times H^1(\Omega)$. Then there exists a self-adjoint and positive what denoted by $i_{\omega_n} \in \mathcal{L}(H)$ such that

$$\langle i_{\omega_n} z, z\rangle = n\|\chi_{\omega_n} z\|^2_{L^2(\Omega)}, \quad \forall z \in H.$$

Assume that

$$\langle i_{\omega_n} Az, z\rangle + \langle z, i_{\omega_n} Az\rangle + \langle i_{\omega_n} Rz, z\rangle = 0, \qquad z \in \mathcal{D}(A) \quad (21)$$

and $\exists\ \alpha, \beta$ and δ non-negative numbers such that for all $n \geq 1$ we have $\forall\, t \geq 0$, $\forall z \in L^2(\Omega)$,

$$\begin{cases} \|\chi_{\omega_n} S(t)z\| \leq \alpha\|\chi_{\omega_n} z\|, \\ \|\chi_{\omega_n} Bz\| \leq \beta\|\chi_{\omega_n} z\|, \end{cases} \qquad (22)$$

and

$$\int_0^1 |\langle i_{\omega_n} BS(t)z, S(t)z\rangle| \,\mathrm{d}t \geq \delta\|\chi_{\omega_n} z\|^2, \qquad z \in L^2(\Omega). \quad (23)$$

Consider the cost

$$\begin{aligned} q_n(v) = n &\int_0^{+\infty} \langle i_{\omega_n} Rz(t), z(t)\rangle \,\mathrm{d}t \\ &+ n \int_0^{+\infty} |\langle i_{\omega_n} Bz(t), z(t)\rangle|^2 \,\mathrm{d}t \\ &+ \int_0^{+\infty} |v(t)|^2 \,\mathrm{d}t, \end{aligned}$$

and the problem

$$\begin{cases} \min q_n(v), \\ v \in \mathcal{U}_{ad}(\omega_n) = \{v;\ z(t) \text{ is a global solution and} \\ \qquad\qquad\qquad\qquad q_n(v) < \infty\}. \end{cases} \tag{24}$$

This problem has only one solution given by $v_n^*(t) = -n\langle z(t), i_{\omega_n} Bz(t)\rangle$ (Zerrik *et al.*, 2004), and we have the following result:

Proposition 4. *The sequence* $v_n^*(t)$ *converges to* $v^*(t) = -(\gamma_\Gamma z^*(t), \gamma_\Gamma Bz^*(t))$ *which is the solution of (15).*

Proof. For $z, y \in Z$ the average $n\int_{\omega_n} zy\,\mathrm{d}x$ converges to $\int_\Gamma zy\,\mathrm{d}\sigma$ as $n \to +\infty$ (Chilov, 1970). Multiplying (21) by n and setting $n \to +\infty$, we obtain (14). In much the same way, we obtain conditions (17) and (16) from (23) and (22). Then by the unicity of $v^*(t)$, $v_n^*(t)$ converges to $v^*(t)$. ■

3.3. Numerical Example

Here we discuss a numerical example illustrating the above results. They concern a system evolving in circular domains. We shall examine the convergence of the sequence of optimal controls $v_n^*(t)$ on ω_n to the optimal control $v^*(t)$ on Γ.

Consider the system defined on $\Omega = \{(x,y) \mid x^2 + y^2 < 1\}$ by

$$\begin{cases} \dfrac{\partial z(t)}{\partial t} = f(x,y)z(t) + v(t)\chi_D z(t) \quad \text{in } Q, \\ z(0) = z_0, \end{cases} \tag{25}$$

with $D = \{(x,y) \mid 0.9 \le x^2 + y^2 \le 1\}$,

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \in D, \\ \dfrac{1}{10}(x^2+y^2-0.9)^2 & \text{otherwise.} \end{cases}$$

Consider the problem of stabilizing (25) on $\Gamma = \{(x,y) \mid x^2 + y^2 = 1\}$, and minimizing $q_\Gamma(v)$ in the case when $R = 0$. We shall use direct internal approaches.

<u>*Direct approach*</u>

The semigroup $S(t)$ satisfies (16) and (17). Then $v^*(t) = -|\gamma_\Gamma z^*(t)|^2$, the solution of (15), regionally strongly stabilizes (25) on Γ, and $z^*(t)$ is fulfiled as $|\gamma_\Gamma z^*(t)| = O(1/\sqrt{t})$ as $t \to +\infty$.

For $z_0 = 1$ we have

$$v^*(t) = -\frac{2\pi}{4\pi t + 1},$$

and $q_\Gamma(v^*) = |\gamma_\Gamma z_0|^2 = \sigma(\Gamma) = 2\pi$.

Figures 3 and 4 describe the evolution of (25).

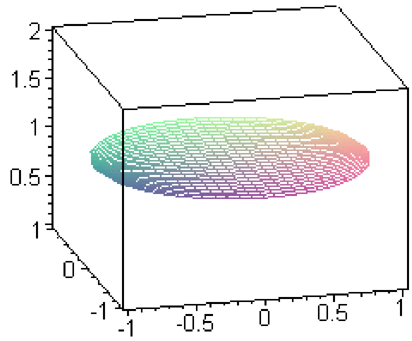

Fig. 3. Initial state $(t = 0)$.

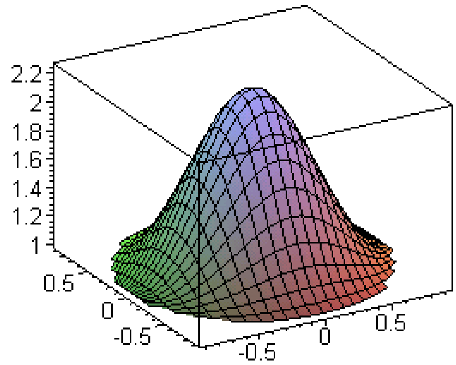

Fig. 4. Stabilized state on Γ.

<u>*Internal approach*</u>

Consider the subsequence

$$\omega_n = \left\{(x,y) \in \Omega \mid x^2 + y^2 \ge \left(1 - \frac{1}{n}\right)^2\right\}$$

of the neighbourhoods of Γ,

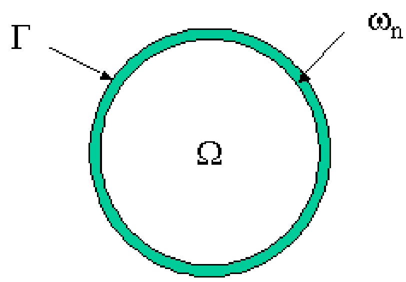

Fig. 5. Sequence of neighbourhoods ω_n of Γ.

and consider the problem of stabilizing the system (25) on ω_n and minimizing the cost

$$q_n(v) = n \int_0^{+\infty} \|\chi_{\omega_n} z(t)\|^2 \, dt + \int_0^{+\infty} |v(t)|^2 \, dt.$$

For $n \geq 10$, the semigroup $S(t)$ and the operator B satisfy (22). Then the control $v_n^*(t) = -n\|\chi_{\omega_n} z^*(t)\|^2$ minimizes $q_n(v)$ and regionally strongly stabilizes (25) on ω_n. The solution corresponding to $z_0 = 1$ is given by

$$z^*(t) = \begin{cases} \dfrac{1}{\sqrt{4\pi(1-\frac{1}{2n})t+1}} & \text{on } D, \\ e^{\frac{t}{10}(x^2+y^2-0.9)^2} & \text{otherwise.} \end{cases}$$

The system (25) is then stabilized on ω_n by

$$v_n^*(t) = -\frac{2\pi(1-\frac{1}{2n})}{4\pi(1-\frac{1}{2n})t+1},$$

(cf. Figs. 6 and 7), which converges to $v^*(t)$, and $q_n(v_n^*) = \pi(2 - \dfrac{1}{n}) \to q_\Gamma(v^*)$ as $n \to +\infty$.

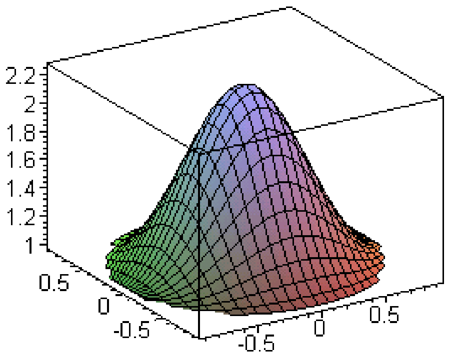

Fig. 6. Stabilized state on ω_{10}.

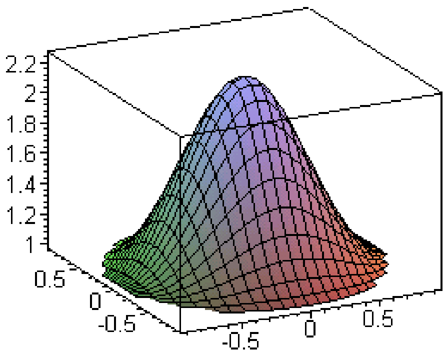

Fig. 7. Stabilized state on ω_{15}.

Figure 8 illustrates the convergence of $v_n^*(t)$ to $v^*(t)$.

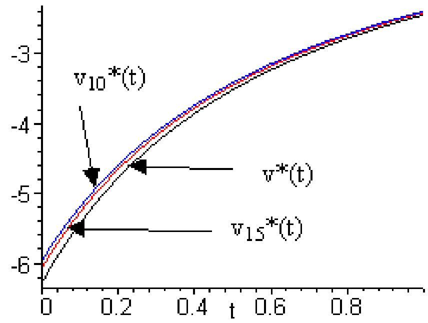

Fig. 8. Evolution of controls $v^*(t)$ and $v_n^*(t)$.

4. Conclusion

In this paper we have extended the results established in (Zerrik *et al.*, 2004) for internal regional stabilization for infinite-dimensional bilinear systems to the case where the target part is located on the boundary of the geometric domain where the system is considered. The obtained results characterize either stabilizing control or the one minimizing a cost of performance. Also they give rise to other questions. This is the case of semi-linear systems.

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