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OPTIMALITY AND SENSITIVITY FOR SEMILINEAR BANG-BANG TYPE OPTIMAL CONTROL PROBLEMS

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In optimal control problems with quadratic terminal cost functionals and systems dynamics linear with respect to control, the solution often has a bang-bang character. Our aim is to investigate structural solution stability when the problem data are subject to perturbations. Throughout the paper, we assume that the problem has a (possibly local) optimum such that the control is piecewise constant and almost everywhere takes extremal values. The points of discontinuity are the switching points. In particular, we will exclude the so-called singular control arcs, see Assumptions 1 and 2, Section 2. It is known from the results by Agrachev *et al.* (2002) stating that regularity assumptions, together with a certain strict second-order condition for the optimization problem formulated in switching points, are sufficient for strong local optimality of a state-control solution pair. This finite-dimensional problem is analyzed in Section 3 and optimality conditions are formulated (Lemma 2). Using well-known results concerning solution sensitivity for mathematical programs in $\mathbb{R}^n$ (Fiacco, 1983) one may further conclude that, under parameter changes in the problem data, the switching points will change Lipschitz continuously. The last section completes these qualitative statements by calculating sensitivity differentials (Theorem 2, Lemma 6). The method requires a simultaneous solution of certain linearized multipoint boundary value problems.

Keywords: stability in optimal control, solution structure, bang-bang control, optimality conditions, strong local optima, sensitivity differentials

1. Introduction

From mathematical programming theory it is well known that the analysis of strong second-order optimality conditions and stability properties of the solution with respect to small data perturbations are closely related questions. Similar results have been obtained in the last decade for a wide range of nonlinear constrained optimal control problems (Malanowski, 2001; Dontchev and Malanowski, 2000; and the bibliographies therein) when control functions are continuous.

The investigation of discontinuous and, in particular, bang-bang optimal controls has recently found renewed interest in the control community. Sufficient optimality conditions for bang-bang controls in problems where the control enters the state equation linearly are considered, e.g., by Sarychev (1997), Agrachev *et al.* (2002), Maurer and Osmolovskii (2005) and by the author (Felgenhauer, 2003a). General optimality conditions admitting control discontinuities are derived, e.g., in the monograph by Milyutin and Osmolovskii (1998), see also (Osmolovskii, 2000; Osmolovskii and Lempio, 2002), and by Noble and Schaettler (2002). A particular result using a duality based Riccati approach was given in (Felgenhauer, 2003b).

Up to now, only few results have been known concerning stability properties of optimal solutions in case the control is of the bang-bang type (cf., e.g., Kim and Maurer, 2003; Maurer and Osmolovskii, 2005). For linear systems, in (Felgenhauer, 2003a) optimality conditions are formulated which ensure the switching structure stability and differentiability of switching times with respect to parameters (see also Felgenhauer, 2003c).

The investigations were accelerated when the attention was re-drawn to the properties of the finite-dimensional subproblem formulated in terms of switching times. This traditional heuristical idea was consequently used in optimality analysis first in (Agrachev *et al.*, 2002). It is due to H. Maurer to recognize this approach as a suitable tool for sensitivity investigation, too (Kim and Maurer, 2003). However, the method requires the assumption that, for the auxiliary problem, the Strong Second-Order Sufficiency Condition holds. Instead, in the present paper a method is used which is based on a shooting-type approach for solving canonical system equations (Felgenhauer, 2003a).

Section 2 summarizes the known regularity results with the emphasis on the so-called *strict bang-bang* condition. This condition characterizes the points of discontinuity of the control vector as regular zeros of the related switching function component (i.e., as zeros with a non-vanishing derivative value). We briefly discuss the linear case, where the given assumptions already ensure strict local optimality of the solution. Section 3 is devoted to optimization over the positions of switching points, for a fixed structure of bang-bang control. In the linear case, the Strong Second-Order Sufficiency Condition for this problem follows directly from the strict bang-bang property (Felgenhauer, 2003d). We find new formulas for the Hesse matrix of the objective functional with respect to switching points in the semilinear case, and conditions ensuring its positive definiteness, see Lemma 2.

Part 4 of the paper uses the optimality condition for analyzing the stability of the switching structure in a parametric version of the original semilinear control problem (with a special terminal functional). The main result consists in the calculation of *sensitivity differentials* of switching points (Theorem 2, Lemma 6). As auxiliary terms, one has to determine certain derivatives of the optimal state with respect to the parameter h and the shooting input z by solving multipoint boundary value problems for the linearized state equation. In principle, the procedure is suitable for a numerical application.

2. Problem and Regularity Conditions

Consider the following optimal control problem where the control vector enters the state equation linearly:

$$\min J(x,u) = \frac{1}{2}\,\|x(T)-b\|^2, \tag{1}$$

subject to

$$\dot{x}(t) = f\big(t,x(t)\big) + B(t)u(t), \quad x(0)=a\neq b, \tag{2}$$

$$|u_i(t)|\le 1, \quad i=1,\dots,m. \tag{3}$$

The state and control variables are denoted by x and u, respectively. They are considered in a generalized sense ($x\in W^1_\infty(0,T;\mathbb{R}^n)$, $u\in L_\infty(0,T;\ \mathbb{R}^m)$). All data functions in (2) are assumed to be sufficiently smooth. Introducing the Hamilton function,

$$H(t,x,u,p) = p^T f(t,x) + p^T B(t)u,$$

from Pontryagin's maximum principle we obtain

$$\dot{p}(t) = -A(t)^T p(t), \quad p(T) = x(T)-b \tag{4}$$

(where $A=\nabla_x f$), and the optimal control u^0 satisfies

$$u^0(t) = \arg\max_{|v_i|\le 1}\left\{-H\big(t,x(t),v,p(t)\big)\right\}.$$

In other words, using the so-called switching function σ, almost everywhere we have

$$\sigma = B^T p, \quad u^0 = -\operatorname{sign}(\sigma). \tag{5}$$

If $\sigma\equiv 0$ on a certain interval, then this part of the control trajectory is called a singular arc.

Assumption 1. (bang-bang regularity)
The pair (x^0,u^0) is a solution such that u^0 is piecewise constant and has no singular arcs. For every j, the set $\Sigma_j=\{\,t\in[0,T]:\ \sigma_j(t)=0\,\}$ is finite, and $0,T\notin\Sigma_j$.

The set Σ of points where one or more components of σ vanish consists of the so-called switching points. (Notice that we will speak of a *simple* switching point if only one σ-component is zero.) In general, we shall write

$$\Sigma_j = \big\{t_{js}:\ s=1,\dots,l(j)\big\},$$
$$\Sigma = \big\{t_{js}:\ s=1,\dots,l(j),\ j=1,\dots,m\big\}.$$

It will be assumed that the points of each Σ_j are monotonically ordered. Further, we set $t_{j0}=0,\ t_{j,l(j)+1}=T$ for all j.

Assumption 2. (strict bang-bang property)
For every j, $t_s\in\bigcup_j\Sigma_j$: $\sigma_j(t_s)=0\Rightarrow\dot{\sigma}_j(t_s)\neq 0$.

Under the given assumptions, the j-th control component switches in accordance with

$$\begin{aligned}\left[u_j^0\right]^s &= u_j^0(t_s+0)-u_j^0(t_s-0)\\ &= -2\,u_j^0(t_s-0) = -2\operatorname{sign}\big(\dot{\sigma}_j(t_s)\big).\end{aligned} \tag{6}$$

Sufficient optimality conditions for problems of the class (1)–(3) have been recently considered by several authors. For the *linear case* (i.e., $f(t,x)=A(t)x$), it was shown, e.g., in (Felgenhauer, 2003a) that Assumptions 1 and 2 are sufficient for strict local optimality of the solution pair (x^0,u^0) in an L_∞-neighborhood of x^0 (*strong* local optimality). Besides the assumptions about the extremal being regular and strict bang-bang, the proof is based on a primal-dual optimality condition which, in its main part, consists of

$$\min_{x,u,S}\int_0^T \left[H(t,x,u,\nabla_x S)+S_t\right]\mathrm{d}t = 0,$$

(see Felgenhauer, 2003a, Thm. 2.2; cf. also Maurer and Pickenhain, 1995, Thm. 3.2). Notice that this condition can be interpreted as an integrated form of the Hamilton-Jacobi inequality, i.e., a generalization of this well-known variational approach to constrained control problems.

Using the expansion

$$\begin{aligned}S &= S_0 + p^T(x-x^0) + 0.5(x-x^0)^T Q(x-x^0)\\ &\quad + \mathrm{o}\big(|x-x^0|^2\big),\end{aligned}$$

one has to show that, for a small positive γ, the matrix Riccati differential inequality

$$\dot{Q} + A^T Q + Q\,A \succeq \gamma\, I \quad \text{a.e.},$$
$$I - Q(T) \succeq 0$$

(a) has an absolutely continuous solution on $[0,T]$, and (b) this solution can be chosen such that $\|Q\|_\infty = \mathrm{O}(\gamma)$ (Felgenhauer, 2003a, Lem. 3.2). The result can be easily adapted to the case of other convex terminal functionals in (1).

For the semilinear situation, analogous optimality results are obtained, e.g., in (Agrachev *et al.*, 2002; Noble and Schaettler, 2002; Osmolovskii and Lempio, 2002), but under certain additional second-order type assumptions. The generalization of the duality based approach from (Felgenhauer, 2003a) under appropriate additional convexity type conditions is also possible and will be a subject of forthcoming research.

3. Optimization of Switching Points

The optimality and sensitivity properties of the control problem given in (1)–(3) are connected with solution properties of the following *auxiliary* mathematical program using switching points (i.e., the vector Σ) as decision variables.

Let $\Sigma = (\Sigma_1, \dots, \Sigma_m)$ denote a vector of the size $L = l(1) + \cdots + l(m)$ composed of $\Sigma_j = (\tau_{js}\, :\ s = 1, \dots, l(j))$. We will require that all components of Σ be inner points of the time interval, i.e., $0 < \tau_{js} < T$. Assuming further that, for each Σ_j, the elements $\{\tau_{js}\}$ are strictly monotonically ordered, the feasible set may be described by

$$S = \big\{\, \Sigma = (\tau_{js}) \in \mathbb{R}^L : \ \tau_{js} < \tau_{j,s+1}, \\ s = 1, \dots, l(j)-1,\ j = 1, \dots, m \,\big\}.$$

Notice that S is an open subset of $\mathbb{R}^L$. Determine next $u = u(t,\Sigma),\ x = x(t,\Sigma)$ by

$$u_j(t,\Sigma) \equiv u_j^0(t_{js}+0) \quad \text{for } t \in (\tau_{js}, \tau_{j,s+1}), \tag{7}$$

$$\dot{x}(t) = f\big(t, x(t)\big) + B(t)\, u(t,\Sigma), \tag{8}$$

$$x(0) = a. \tag{9}$$

Then Σ^0 corresponding to (x^0, u^0) solves the finite-dimensional problem

$$\min \phi(\Sigma) = \frac{1}{2}\|x(T,\Sigma) - b\|^2 \quad \text{s.t. } \Sigma \in S. \tag{10}$$

This problem, where the number of switchings and the principal structure information are temporarily fixed, was considered, e.g., in (Agrachev *et al.*, 2002), where it was shown that a *Strong Second-Order Optimality Condition* for (10) together with the *strict bang-bang* behavior (Assumptions 1 and 2) are sufficient for strict strong local optimality of the solution given as $x = x(\cdot,\Sigma),\ u = u(\cdot,\Sigma)$ at $\Sigma = \Sigma^0$. Moreover, for the *linear* case with $f(t,x) = A(t)x$ the following result was obtained (Felgenhauer, 2003d):

Lemma 1. *Let (x^0, u^0) and Σ^0 be a solution and a switching set such that the strict bang-bang conditions given in Assumptions 1 and 2 are fulfilled. Then, at $\Sigma = \Sigma^0$, we have*

$$\nabla_\Sigma \phi(\Sigma^0) = 0, \quad \nabla^2_\Sigma \phi(\Sigma^0) \succ 0.$$

As we will see, generalization to the *semilinear* case requires certain additional convexity type assumptions about the data which arise from additional integral terms in the Hessian $\nabla^2_\Sigma \phi$.

To begin with, consider the first-order derivative information $\nabla_\Sigma \phi$, which can be expressed by means of $\eta_s = (\partial/\partial\tau_s) x(t,\Sigma)$. For simplicity, we will consider here only the so-called *simple* switches where at most one control component may jump at time τ_s. Generalization to multiple switching points remains to be true up to minor technical changes (for details, see Felgenhauer, 2003d).

For $t > \tau_s$, η_s solves

$$\dot{\eta}_s(t,\Sigma) = A(t,\Sigma)\eta_s(t,\Sigma) \ \text{ a.e.}, \ \ \eta_s(\tau_s) = -b_s(\Sigma), \tag{11}$$

where in the semilinear case we have

$$A(t,\Sigma) = \nabla_x f\big(t, x(t,\Sigma)\big),$$
$$b_s(\Sigma) = B(\tau_s)\, \big[u^0\big]^s,$$

with $\big[u^0\big]^s = u^0(t_s+0) - u^0(t_s-0)$, cf. (6).

In contrast to the linear case, the right-hand side matrix A implicitly depends on Σ. Nevertheless, for the solution representation we may use the so-called fundamental solutions $\Phi = \Phi(t,\Sigma)$, $\Psi = \Psi(t,\Sigma)$ determined by the matrix differential systems

$$\begin{aligned} \dot{\Phi} + A^T \Phi = 0, \quad \Phi(0) = I, \\ \dot{\Psi} - A \Psi = 0, \quad \Psi(0) = I. \end{aligned} \tag{12}$$

Thus, with the notation θ for the Heaviside function, we obtain

$$\eta_s(t,\Sigma) = -\theta(t,\tau_s)\Psi(t,\Sigma)\Phi(\tau_s,\Sigma)^T b_s(\Sigma). \tag{13}$$

As in the linear case, we use this formula to check the gradient of $\nabla_\Sigma \phi(\Sigma^0)$: by the chain rule,

$$\frac{\partial}{\partial \tau_s}\left(\frac{1}{2}\|x(T,\Sigma)-b\|^2\right) = \eta_s(T,\Sigma)^T\left(x(T,\Sigma)-b\right)$$
$$= \left(-B(\tau_s)\left[u^0\right]^s\right)^T \Phi(\tau_s,\Sigma)\Psi(T,\Sigma)^T\left(x(T,\Sigma)-b\right). \tag{14}$$

From the transversality condition, $x(T,\Sigma^0)-b = p(T)$, so that from (4) and (12) we conclude that $\Phi(t_s)\Psi(T)^T p(T) = p(t_s)$. Using now the switching points definition, we are finally able to confirm the first-order stationarity condition for (10) at $\Sigma=\Sigma^0$,

$$\frac{\partial}{\partial \tau_s}\phi\left(x(T,\Sigma^0)\right) = -\left(B(t_s)\left[u^0\right]^s\right)^T p(t_s)$$
$$= -\sigma(t_s)^T\left[u^0\right]^s = 0. \tag{15}$$

Consider next the structure of the Hesse matrix $\nabla^2_\Sigma \phi$ at the reference solution. It was shown in (Felgenhauer, 2003d) that, for the linear case with the terminal cost functional $J(x,u) = k(x(T))$, the principal parts of $\nabla^2_\Sigma\phi(\Sigma^0)$ (where $\phi(\Sigma)=k(x(T,\Sigma))$) are given by

$$\nabla^2_\Sigma k\left(x(T,\Sigma^0)\right) = \eta^T\nabla^2_x k\left(x^0(T)\right)\eta + \mathrm{diag}_s\left\{D^s(H)\right\}, \tag{16}$$

with

$$D^s(H) = -\left[\frac{\mathrm{d}}{\mathrm{d}t}H\right]^s = -\dot\sigma(t_s)\left[u^0\right]^s > 0 \tag{17}$$

(see Assumption 2). Consequently, for every function $k=k(\xi)$ being convex near $\xi = x^0(T)$, the matrix $\nabla^2_\Sigma\phi$ is positive definite.

In the case of the *semilinear* system (2), the formula (16) does not apply for the Hessian of ϕ. The second-order derivatives have to be re-calculated from (14), where the matrices Φ and Ψ via A also depend on Σ, cf. (12). In order to find expressions for $\partial\Phi/\partial\tau_s$ and $\partial\Psi/\partial\tau_s$, we first consider

$$F_s(t,\Sigma) = \frac{\partial A}{\partial \tau_s}(t,\Sigma).$$

It is easy to see that F_s has the row-wise representation

$$F_{s,i}(t,\Sigma) = \eta_s(t,\Sigma)^T\nabla^2_x f_i\left(t,x(t,\Sigma)\right), \quad i=1,\dots,n.$$

Therefore, the matrix functions $M_s = \partial\Phi/\partial\tau_s$, $N_s = \partial\Psi/\partial\tau_s$ satisfy

$$\dot M_s + A^T M_s = -F_s^T\,\Phi, \quad M_s\equiv 0 \ \text{ for } \ t<\tau_s, \tag{18}$$

$$\dot N_s - A\,N_s = F_s\,\Psi, \qquad N_s\equiv 0 \ \text{ for } \ t<\tau_s. \tag{19}$$

In other words, for $t>\tau_s$ we can write

$$M_s(t,\Sigma) = -\Phi(t)\int_{\tau_s}^t \Psi(\tau)^T F_s(\tau)^T\Phi(\tau)\,\mathrm{d}\tau$$
$$N_s(t,\Sigma) = \Psi(t)\int_{\tau_s}^t \Phi(\tau)^T F_s(\tau)\Psi(\tau)\,\mathrm{d}\tau$$
$$= -\Psi(t,\Sigma)M_s(t,\Sigma)^T\Psi(t,\Sigma) \tag{20}$$

(and $M_s = -\Phi N_s^T\Phi$, resp.).

Now, using (13) and (14), we can find the partial derivatives $\partial^2\phi/\partial\tau_s\partial\tau_k$:

$$\frac{\partial^2}{\partial\tau_k\partial\tau_s}\phi(\Sigma) = \eta_s(T,\Sigma)^T\eta_k(T,\Sigma)$$
$$+\left(x(T,\Sigma)-b\right)^T\frac{\partial}{\partial\tau_k}\eta_s(T,\Sigma), \tag{21}$$

where for $k\neq s$,

$$\frac{\partial}{\partial\tau_k}\eta_s(T,\Sigma) = -\Big[N_k(T,\Sigma)\Phi(\tau_s,\Sigma)^T$$
$$+\Psi(T,\Sigma)M_k(\tau_s,\Sigma)^T\Big]\,b_s$$
$$=: P_{sk}(\Sigma), \tag{22}$$

and for $k=s$ we have

$$\frac{\partial}{\partial\tau_s}\eta_s(T,\Sigma) = P_{ss}(\Sigma) - \Psi(T,\Sigma)\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^T B\right)_{t=\tau_s}\left[u^0\right]^s$$
$$=: P_{ss}(\Sigma)+q_s(\Sigma). \tag{23}$$

Taking into account (20), (18) and (13), the terms P_{ks} can be rewritten as

$$P_{ks} = \Psi(T)\left(M_k(T)^T\Psi(T) - M_k(\tau_s)^T\Psi(\tau_s)\right)\Phi(\tau_s)^T b_s$$
$$= \Psi(T)\int_{\tau_{sk}}^T \Phi(\tau)^T F_k(\tau)\Psi(\tau)\,\mathrm{d}\tau\;\Phi(T)^T\eta_s(T),$$

where $\tau_{sk} = \max\{\tau_s,\tau_k\}$. For the corresponding parts in (21) at $\Sigma=\Sigma^0$ we obtain

$$\left(x(T,\Sigma^0)-b\right)^T P_{ks}(\Sigma^0)$$
$$= \int_{\tau_{sk}}^T\Big[p(T)^T\Psi(T)\Phi(\tau)^T F_k(\tau)\Psi(\tau)\Phi(T)^T\eta_s(T)\Big]\mathrm{d}\tau$$
$$= \int_{\tau_{sk}}^T p(\tau)^T F_k(\tau)\eta_s(\tau)\,\mathrm{d}\tau.$$

The integrand in the last expression is symmetric since

$$p^T F_k\eta_s = \sum_{i,j,l} p_i\left(\nabla^2_x f_i\right)_{jl}(\eta_k)_l(\eta_s)_j$$
$$= \eta_k^T\left(\sum_i p_i\nabla^2_x f_i\right)\eta_s = \eta_k^T\,\nabla^2_x H\,\eta_s.$$

Thus,

$$\big(x(T,\Sigma^0)-b\big)^T P_{ks}(\Sigma^0) = \int_{\tau_{sk}}^{T} \eta_k(\tau)^T\, \nabla_x^2 H[\tau]\, \eta_s(\tau)\, \mathrm{d}\tau. \tag{24}$$

In case $k = s$, we further need the terms $(x(T,\Sigma) - b)^T q_s(\Sigma)$ which, at $\Sigma = \Sigma^0$, give

$$\begin{aligned}\big(x(T,\Sigma^0) - b\big)^T q_s(\Sigma^0) &\\ &= -\, p(T)^T \Psi(T) \frac{\mathrm{d}}{\mathrm{d}t} \left(\Phi^T B\right)_{t=t_s} \left[u^0\right]^s \\ &= -\, \frac{\mathrm{d}}{\mathrm{d}t} \left(p(T)^T \Psi(T) \Phi^T B\right)_{t=t_s} \left[u^0\right]^s \\ &= -\, \dot{\sigma}(t_s)^T \left[u^0\right]^s = D^s(H) > 0 \end{aligned} \tag{25}$$

due to Assumption 2.

Inserting the information from (24) and (25) into (21), the final representation for $\nabla_\Sigma^2 \phi$ consists of the following parts:

$$\begin{aligned}\nabla_\Sigma^2 \phi(\Sigma^0) &= \eta(T,\Sigma^0)^T \eta(T,\Sigma^0) + \mathrm{diag}_s \left\{D^s(H)\right\} \\ &\quad + \int_0^T \eta(\tau,\Sigma^0)^T\, \nabla_x^2 H[\tau]\, \eta(\tau,\Sigma^0)\, \mathrm{d}\tau. \end{aligned} \tag{26}$$

The last result allows us to formulate a generalization of Lemma 1 to the semilinear problem case (1)–(3).

Lemma 2. *Let (x^0, u^0) and Σ^0 be a solution and a switching set, respectively, such that the* strict *bang-bang conditions given in Assumptions 1 and 2 are fulfilled. Suppose further that almost everywhere in $[0,T]$ the Hessian $\nabla_x^2 H$ evaluated along the solution trajectories is positive semi-definite. Then, at $\Sigma = \Sigma^0$,*

$$\nabla_\Sigma \phi(\Sigma^0) = 0, \quad \nabla_\Sigma^2 \phi(\Sigma^0) \succ 0. \tag{27}$$

4. Sensitivity Result

In this section we consider a parametric version of the problem (1) with data functions depending on $h \in \mathcal{H} \subset \mathbb{R}$:

$$\min J(x,u;h) = \frac{1}{2} \left\| x(T) - b(h) \right\|^2 \tag{28}$$

subject to

$$\dot{x}(t) = f\big(t, x(t), h\big) + B(t,h) u(t), \quad x(0) = a(h), \tag{29}$$

$$|\, u_i(t)| \;\le\; 1, \quad i = 1, \dots, m.$$

The set $\mathcal{H}$ stands for a neighborhood of the reference parameter $h^0 = 0$, which is assumed to correspond to the reference data in (1)–(3). The above functions as well as the derivatives $\nabla_x f$ and $\dot{B}$ are assumed to be sufficiently smooth functions with respect to h on $\mathcal{H}$.

Suppose that, for $h = h^0$, the reference problem (1) has a solution (x^0, u^0) with the switching set Σ^0 and the related adjoint p and the switching function σ such that Assumptions 1 and 2 are fulfilled. Further assume that $\nabla_x^2 H$ is positive semi-definite so that Lemma 2 applies. Then, from a well-known sensitivity result of mathematical programming theory it is known that, for h sufficiently close to h^0, the parametric problem (28) has a locally unique solution $\Sigma = \Sigma(h)$ smoothly depending on the parameter:

Theorem 1. *Let for the problem (1) corresponding to $h = h^0$ the strict bang-bang conditions of Assumptions 1 and 2 hold true. Suppose further that the related vector Σ^0 of switching times satisfies the necessary and second-order sufficient optimality conditions (27). Then, for the parametric problem (28) with data smoothly depending on h, with each h sufficiently close to h^0, we can associate a switching vector $\Sigma(h)$ such that the following holds:*

(a) the mapping $h \to \Sigma(h)$ is continuously differentiable,

(b) the control defined by (7) and the corresponding trajectory from (8) provide a strict strong minimum to the perturbed control problem (28) at h.

This sensitivity result was formulated for a wider problem class including general boundary constraints, as well as possibly free final time in (Kim and Maurer, 2003), cf. Theorem 4.3 therein. In the case of the problem (1) where the final time is fixed, and for the state trajectory for which an IVP is given, the idea of the proof consists in the following:

Locally, the constraints defining S are inactive so that no constraint qualification is needed for sensitivity analysis. Every stationary point of the parametric version of the problem (10) related to (28) solves

$$\nabla_\Sigma \phi(\Sigma, h) = 0. \tag{30}$$

The left-hand side mapping is differentiable in both arguments. Moreover, by (27), the matrix $\nabla_\Sigma^2 \phi$ is regular at $\Sigma = \Sigma^0$, so that the Implicit Function Theorem yields the existence of a stationary solution $\Sigma = \Sigma(h)$ for h sufficiently close to h^0. From the positive definiteness of $\nabla_\Sigma^2 \phi$ at Σ^0, we conclude by the standard continuity arguments that the second-order optimality condition (27) is fulfilled for $\Sigma = \Sigma(h)$ in a neighborhood of Σ^0. Thus, the stationary solution $\Sigma(h)$ provides a strict local minimum for (10).

For the corresponding state-control pair $x^h = x(t, \Sigma(h))$, $u^h = u(t, \Sigma(h))$ from (7), (8), an adjoint function $p = p(t, \Sigma(h))$ can be constructed by analogy to (4). According to (30) and the parametric version of (15), the solution satisfies the maximum principle. Further, by continuity, the strict bang-bang property remains to be valid for a sufficiently small h. Consequently, (x^h, u^h) is a strict strong local minimizer of (28), see (Agrachev *et al.*, 2002).

The above theorem consists mainly in a qualitative statement since, for many practical problems, the exact Hesse matrix (26) will not be available. In (Kim and Maurer, 2003), some general remarks are made about a possible numerical calculation of the so-called *sensitivity differentials* $\mathrm{d}t_{js}/\mathrm{d}h$. In the sequel, we will use an alternative approach which has been first proposed for the *linear case* (Felgenhauer, 2003a). The main idea consists in the analysis of the following *shooting type* procedure:

For a given guess of the adjoint initial value $p(0) = z$, construct the functions $p = p(t, z, h)$ and $x = x(t, z, h)$ by

$$\begin{aligned} \dot p(t) &= -A(t,h)^T p(t), \quad p(0) = z, \\ \sigma(t,z,h) &= B(t,h)^T p(t,z,h), \\ u(t,z,h) &= -\operatorname{sign}\big(\sigma(t,z,h)\big) \\ \dot x(t) &= f(t,x,h) + B(t,h)u(t,z,h), \\ \text{with } x(0) &= a(h). \end{aligned}$$

Notice that, in general, $A = A(t, x, h)$, where $x = x(t, z, h)$, so that in the nonlinear situation the above system cannot be decoupled.

The above process yields an extremal to the parametric control problem at h if the following transversality condition is fulfilled:

$$F(z,h) = 0, \tag{31}$$

where

$$F(z,h) = b(h) + p(T,z,h) - x(T,z,h). \tag{32}$$

In the following, we will take the derivatives with respect to h or z of the functions $x = x(t, z, h)$, $p = p(t, z, h)$ and $F = F(z, h)$. The partial derivatives will be indicated by the related subscript, e.g., x_h for $\partial x/\partial h$, etc. Further, we use the fundamental solutions Φ and Ψ defined as in (12).

Lemma 3. *Under Assumptions 1 and 2, at* $h = h^0$, $z^0 = p(0)$, *the derivatives* $x_h = x_h(t, z^0, h^0)$ *and* $p_h = p_h(t, z^0, h^0)$ *solve the following linear ODE system with coupled multiple boundary conditions:*

$$\dot p_h = -A^T p_h - C x_h - w, \tag{33}$$

$$p_h(0) = 0,$$

$$\dot x_h = A\, x_h + y, \quad \textit{piecewise}, \tag{34}$$

$$x_h(0) = a_h, \;\; [x_h]^s = -\Psi(t_s) d^s,$$

$$\sigma_h = B_h^T p + B^T p_h, \tag{35}$$

$$d^s = \sum_{j \in I(s)} r_{js} \Gamma_j(t_s)^T \sigma_{j,h}(t_s) \tag{36}$$

(for $r_{js} = 2|\dot\sigma_j(t_s)|^{-1}$, $\Gamma = B^T\Phi$, $C = p^T \nabla_x^2 f$, $w = A_h^T p$, $y = f_h + B_h u$*).*

It should be noticed that the above system can be partially decoupled by solving the ODE piecewise on every $[t_s, t_{s+1}]$: Indeed, starting with $t_0 = 0$, we find x_h, p_h for $t < t_1$ from (34), (33) and calculate σ_h by (35). After utilizing the jump condition, we find $x_h(t_1 + 0)$ and can repeat the procedure for the next time intervals until we end up with $x_h(T)$ and $p_h(T)$.

Sketch of the proof: Formally, the differentiation of the state and adjoint equations with respect to h leads to

$$\begin{aligned} \dot p_h &= -A^T p_h - C x_h - A_h^T p, \\ \dot x_h &= A\, x_h + B\, u_h + f_h + B_h u. \end{aligned}$$

The term u_h herein is a sum of Dirac measures, see (Felgenhauer, 2003a, Section 4) for details. The solutions can be written as

$$\begin{aligned} x_h(t) = \Psi(t) a_h &+ \Psi(t) \int_0^t \Phi(s)^T y(s)\, \mathrm{d}s \\ &+ \Psi(t) \int_0^t \Phi(s)^T B(s)\, u_h(s)\, \mathrm{d}s, \end{aligned}$$

where, by analogy to (Felgenhauer, 2003a; or 2003c, Lem. 2), we have

$$\begin{aligned} &\int_0^t \Phi(s)^T B(s)\, u_h(s)\, \mathrm{d}s \\ &\quad = -\sum_{(j,s):\, t_s < t} r_{js} \Phi(t_s)^T B_j(t_s) \sigma_{j,h}(t_s) \end{aligned}$$

or, consequently, (36). ■

In essentially the same way, the derivative matrix functions with respect to z are determined as solutions of the following system:

Lemma 4. *Under Assumptions 1 and 2, at $h = h^0, z^0 = p(0)$ the derivatives $x_z = x_z(t, z^0, h^0)$ and $p_z = p_z(t, z^0, h^0)$ solve*

$$\dot{p}_z = -A^T p_z - C x_z, \tag{37}$$

$$p_z(0) = I,$$

$$\dot{x}_z = A x_z, \quad \textit{piecewise}, \tag{38}$$

$$x_z(0) = 0, \ [x_z]^s = -\Psi(t_s) e^s,$$

$$\sigma_z = B^T p_z, \tag{39}$$

$$e^s = \sum_{j \in I(s)} r_{js} \Gamma_j(t_s)^T B_j^T(t_s) p_z(t_s) \tag{40}$$

(for $r_{js} = 2|\dot{\sigma}_j(t_s)|^{-1}$, $\Gamma = B^T \Phi$, $C = p^T \nabla_x^2 f$).

The last lemmas show that, in principle, all partial derivatives of the terms in (32) are available after solving some coupled linear systems. Now, we can formulate the following result:

Theorem 2. *Let (x^0, u^0) be a solution of the problem (1) related to $h = h^0$, and suppose that the adjoint and switching functions p, σ are such that Assumptions 1 and 2 hold. Then, the vector $z^0 = p(0)$ solves (31) at $h = h^0$. If, in addition, the Jacobi matrix*

$$\nabla_z F = p_z(T) - x_z(T) \tag{41}$$

with p_z, x_z from (37)–(40) is regular, then, for all $h \in \mathcal{H}$ sufficiently close to h^0, Eqn. (31) has a locally unique solution $z = z(h)$ near z^0. As a function of h, $z = z(h)$ is differentiable at h^0, and

$$\frac{\partial z}{\partial h} = -\left(\nabla_z F(z^0, h^0)\right)^{-1} \frac{\partial F}{\partial h}(z^0, h^0)$$

can be calculated by (41) together with

$$\frac{\partial F}{\partial h} = b_h + p_h(T) - x_h(T).$$

The theorem is a consequence of the Implicit Function Theorem. For the *linear* case, the requirement about the Jacobi matrix $\nabla_z F$ is always fulfilled, cf. (Felgenhauer, 2003a, Theorem 4.1). Although Theorem 1 gives a reason to expect this result to be valid in general under the Second-Order Optimality Conditions (27), too, the direct proof of this property is an open question. To give an impression of the (technical) difficulties, consider for simplicity the case of a single switching point (simple or not simple), i.e., the case $|\cup \Sigma_j| = 1$:

Lemma 5. *Let (x^0, u^0) be a solution of (28) for $h = h_0$ such that u^0 has exactly one switching point, t_s, such that $I = \{j : \sigma_j(t_s) = 0\} \neq \emptyset$. Further, let Assumptions 1 and 2 hold true together with the condition $C(t) = \nabla_x^2 H[t] \succeq 0$ (a.e. on $[0, T]$). Then the Jacobi matrix (41) from Theorem 2 is regular.*

Proof. Consider the system (37)–(40). Starting with t in the time interval $[0, t_s)$, we obtain

$$x_z(t) \equiv 0, \quad p_z(t) = \Phi(t),$$

$$[x_z]^s = -\Psi(t_s) G^s, \quad \sigma_z(t) = \Gamma(t),$$

$$G^s = \sum_{j \in I} r_{js} \Gamma_j(t_s)^T \Gamma_j(t_s) \succeq 0.$$

Continuing the solution process for $t \in [t_s, T)$, we arrive at

$$x_z(t) = -\Psi(t) G^s,$$

$$p_z(t) = \Phi(t) + \Phi(t) \int_{t_s}^t \Psi^T C \Psi \, \mathrm{d}s \, G^s.$$

Therefore,

$$\begin{aligned} \nabla_z F &= p_z(T) - x_z(T) \\ &= \Phi(T) \left[I + M_s G^s\right] + \Psi(T) G^s \\ &= \Phi(T) \left[I + \left(M_s + \Psi(T)^T \Psi(T)\right) G^s\right], \end{aligned}$$

where M_s abbreviates to $\int \Psi^T C \Psi \, \mathrm{d}s$. Under the assumption about $\nabla_x^2 H$, the matrix $M = M_s + \Psi^T \Psi$ is positive definite. Thus, we can write

$$\nabla_z F = \Phi(T) \, M \, \left[M^{-1} + G^s\right].$$

Since M^{-1} is positive definite and G^s is positive semidefinite, the term in the brackets is a positive definite matrix. Then the Jacobian $\nabla_z F$ as a product of regular matrices is regular. ■

We will conclude the sensitivity analysis with the following result on $\mathrm{d}t_{js}/\mathrm{d}h$:

Lemma 6. *Let for the solution of the problem (28) corresponding to $h = h^0$ the strict bang-bang conditions of Assumptions 1 and 2 hold together with the Second-Order Sufficiency Condition in terms of the switching points (27). Further assume that, for h sufficiently close to h^0, the system (31) has a unique solution $z = z(h)$, which is a differentiable function of h, and $z_h = (\mathrm{d}z/\mathrm{d}h)$ at h^0. Then, for $\Gamma = \Gamma(t, h) = B(t, h)^T \Phi(t, h)$, we have*

$$\frac{\mathrm{d}\, t_{js}}{\mathrm{d}h} = -\left(\dot{\sigma}_j(t_{js})\right)^{-1} \left(\Gamma_{j,h}(t_{js}) z + \Gamma_j(t_{js}) z_h\right). \tag{42}$$

Proof. Remembering the general smoothness assumptions about f from (29) we can see that B and $\dot{B}$, as well as Φ and $\dot{\Phi}$, are differentiable functions with respect to the parameter h. This remains to be valid also for

$$\sigma(t,h) = \Gamma(t,h)z(h),$$

$$\dot{\sigma}(t,h) = \dot{\Gamma}(t,h)z(h).$$

Consider the equations

$$\sigma_j(t,h) = 0, \quad j = 1,\dots,m. \tag{43}$$

In view of Assumptions 1 and 2, for $h = h^0$, the j-th equation has $l(j)$ isolated zeros t_{js} on $(0,T)$. The strict bang-bang property together with the differentiability of $\dot{\sigma}$ yields

$$\dot{\sigma}_j(t,h) \neq 0$$

for (t,h) sufficiently close to (h^0, t_{js}).

Thus, for fixed (j,s) we may apply the Implicit Function Theorem to (43) and conclude that, in a neighborhood of $t = t_{js}$, the j-th equation has a unique solution $t_{js}(h)$, which is a differentiable function of h. The derivative can be obtained from

$$\dot{\sigma}_j(t_{js})\frac{\mathrm{d}\, t_{js}}{\mathrm{d}h}(h^0) + \frac{\partial \sigma_j}{\partial h}(t_{js}, h^0) = 0,$$

where $\sigma_{j,h} = \Gamma_{j,h}z + \Gamma_j z_h$. Since by our assumptions all derivative terms are well-defined and, moreover, $\dot{\sigma}_j(t_{js}) \neq 0$ for all switching points, the last equation yields (42). ■

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