TWO-LEVEL STOCHASTIC CONTROL FOR A LINEAR SYSTEM WITH NONCLASSICAL INFORMATION

ZDZISŁAW DUDA*, WITOLD BRANDYS*

* Institute of Automatic Control
Silesian University of Technology
ul. Akademicka 16, 44–101 Gliwice, Poland
e-mail: zduda@ia.polsl.gliwice.pl

A problem of control law design for large scale stochastic systems is discussed. Nonclassical information pattern is considered. A two-level hierarchical control structure with a coordinator on the upper level and local controllers on the lower level is proposed. A suboptimal algorithm with a partial decomposition of calculations and decentralized local control is obtained. A simple example is presented to illustrate the proposed approach.

Keywords: stochastic control, nonclassical information, hierarchical structure

1. Introduction

This paper deals with control design for large-scale stochastic systems composed of interconnected linear subsystems. It is obvious that the quality of control depends on the assumed information and control structures. In the centralized structure (one-level structure) a central decision maker determines control values on the basis of the available information collected from all subsystems. However, in large-scale systems the process of transmission and transformation of information in a centralized way may be difficult to implement. This leads to the decentralization of information and control structures.

Control and optimization for large-scale systems are usually based on the decomposition of global system into subsystems in order to decrease computational requirements and the amount of information to be transmitted to and processed by decision makers.

Different control and coordination methods are described, e.g., in (Findeisen *et al.*, 1980; Mesarovic *et al.*, 1974; Aoki, 1973; Chong and Athans, 1971; Ho, 1980; Gessing, 1987). Decentralized control problems may be complicated in the case of a nonclassical information pattern (Witsenhausen, 1968). In this case decision makers have different information that is used for the determination of control.

In the present paper a hierarchical control problem with local decision makers (controllers) on the lower level and a coordinator on the upper level is considered. It is assumed that the local controllers have essential information of their subsystems while the coordinator has aggregated information on the whole system. The problem is to design control laws that minimize a quadratic performance index.

A primary problem statement was discussed in (Gessing and Duda, 1995), where the so-called elastic constraint (Gessing, 1987) was applied. A two-fold interpretation of a control variable was used in control law design. The i-th local control variable was treated as a decision variable for the i-th local controller and as a random variable for other decision makers. Consequently, the solution had a closed-form linear representation. It seemed that the obtained control laws were optimal.

Present paper differs in the synthesis of control laws that lead to a suboptimal solution. The control laws, however, have the same form as in (Gessing and Duda, 1995). This means that the two-fold interpretation of control variables does not lead to an optimal solution. The primary version of the problem was presented in (Duda and Brandys, 2002).

2. Problem Formulation

Let us consider a large-scale static system composed of M distributed subsystems and described by input-output equations

$$\begin{aligned} x_i &= B_{ii}^* u_i + \sum_{\substack{j=1 \\ j \neq i}}^{M} A_{ij} x_j + w_i^* \\ &= B_{ii}^* u_i + \sum_{j \neq i} A_{ij} x_j + w_i^*, \quad i = 1, 2, \dots, M, \end{aligned} \tag{1}$$

where x_i, u_i, w_i^* denote the output, control and random input vector variables of the i-th subsystem, respectively, B_{ii}^* and A_{ij} being given matrices with appropriate dimensions.

The system is observed via the following output

$$y_i = \phi_i(w_i^*, e_i), \quad i = 1, 2, \dots, M, \tag{2}$$

where y_i and e_i are the vectors of measurements and measurement errors of the i-th subsystem, respectively, ϕ_i being a given vector function. We assume that w_i^* and e_i are random variables with given probability distribution functions, independent of w_j^* and e_j, $i \neq j$. The form of the model (2) will be justified in the sequel.

For convenience, random variables will be denoted using bold type, while sample realizations of the random variables will be denoted by other types.

It will be clear from the context whether a variable should be treated as a random variable or as a realization of a random variable.

Let the performance index which should be minimized have the form

$$I = E\Big[\sum_{i=1}^{M}(\mathbf{x_i^T} Q_i \mathbf{x_i} + \mathbf{u_i^T} H_i \mathbf{u_i})_{\mathbf{u_i}=a_i(\cdot)}\Big], \tag{3}$$

where E denotes the expectation operation and a_i is a control law. It is possible to design a control law a_i as a function of information $y = [y_1^T, y_2^T, \dots, y_M^T]^T$, i.e. $u_i = a_i(y)$. In this case the whole information from distributed subsystems is sent to a central controller. Next, the control value u_i determined from the designed control law a_i is forwarded to the i-th local subsystem.

Nevertheless, the proposed structure of information and control is not reasonable for large-scale distributed systems (large M) because of communication and computational complexities. Another way is to design a control law a_i as a function of the information measurement y_i, i.e. $u_i = a_i(y_i)$. This leads to a completely decentralized control system based on decentralized information. Unfortunately, an optimal solution cannot be designed whereas suboptimal algorithms are far from being optimal. Thus we propose a control strategy realized in a two-level hierarchical structure with a coordinator on the upper level and local controllers on the lower one. Let the available information for the decision makers be as follows: The i-th local controller receives a measurement y_i from the i-th subsystem. The coordinator receives an aggregated form of the measurement y_i given by

$$m_i = D_i y_i, \tag{4}$$

where m_i, $i = 1, 2, \dots, M$ is a vector of a dimension lower than y_i, D_i being a given matrix. Consequently, the amount of information transmitted and converted by the coordinator may be decreased. If no information is sent to the coordinator from the i-th subsystem, then dim $m_i = 0$. The coordinator determines the values of coordinating variables p_i, $i = 1, 2, \dots, M$ based on information $m = [m_1^T, m_2^T, \dots, m_M^T]^T$ and transmits them to local controllers.

The i-th local controller determines the value of the control u_i based on information y_i and the coordinating variable p_i. Therefore, by the admissible control laws of the coordinator and the i-th local controller we mean the functions $p_i = b_i(m)$ and $u_i = a_i(y_i, p_i)$, respectively.

The problem is to design optimal control laws b_i^o, $i = 1, 2, \dots, M$ for the coordinator and a_i^o for the i-th local decision maker that minimize the performance index (3) subject to the constraint (1).

3. Problem Solution

Denoting

$$\mathbf{v_i} = \sum_{i \neq j} A_{ij} \mathbf{x_j} \tag{5}$$

and inserting (5) into (1) and then the resulting relation into (3) gives

$$I = E\Big\{\sum_{i=1}^{M}\big[\mathbf{u_i^T} V_i \mathbf{u_i} + 2(\mathbf{v_i} + \mathbf{w_i^*})^T Q_i B_{ii}^* \mathbf{u_i} + \mathbf{v_i}^T Q_i \mathbf{v_i} + 2\mathbf{v_i^T} Q_i \mathbf{w_i^*} + \mathbf{w_i^{*T}} Q_i \mathbf{w_i^*}\big]_{\mathbf{u_i}=a_i[\mathbf{y_i}, b_i(\mathbf{m})]}\Big\}, \tag{6}$$

where $V_i = B_{ii}^{*T} Q_i B_{ii}^* + H_i$.

Control laws a_i^o and b_i^o, $i = 1, 2, \dots, M$ should minimize the performance index (6).

3.1. Synthesis of Local Control Laws

In order to control the i-th subsystem based on available information, the i-th decision maker requires some knowledge of interaction $(\mathbf{v_i})$.

Let the information provided by the coordinator to the i-th decision maker be the best estimate of the interaction

$$v_i^* = E_{|m} \mathbf{v_i} = E_{|m} \sum_{i \neq j} A_{ij} \mathbf{x_j}, \tag{7}$$

where $E_{|m}$ denotes the conditional mean given m. Therefore, a modified model of the i-th subsystem is described by

$$x_i = B_{ii}^* u_i + v_i^* + w_i^* \tag{8}$$

and the performance index (6) has the form

$$I^* = E\Big\{\sum_{i=1}^{M}[\mathbf{u_i^T}V_i\mathbf{u_i}+2(\mathbf{v_i^*}+\mathbf{w_i^*})^T Q_i B_{ii}^*\mathbf{u_i}+\mathbf{v_i^{*T}}Q_i\mathbf{v_i^*} + 2\mathbf{v_i^{*T}}Q_i\mathbf{w_i^*}+\mathbf{w_i^{*T}}Q_i\mathbf{w_i^*}]_{\mathbf{u_i}=a_i[\mathbf{y_i},b_i(\mathbf{m})]}\Big\}$$

$$= EE_{|\mathbf{m}}\Big\{\sum_{i=1}^{M}[\dots]_{\mathbf{u_i}=a_i(\mathbf{y_i},\mathbf{p_i})}\Big\}, \tag{9}$$

where $\mathbf{v_i^*} = E_{|\mathbf{m}}\mathbf{v_i}$.

We see from (9) that the optimal control laws $\mathbf{u_i^o} = a_i^o[\mathbf{y_i}, p_i]$, $i = 1, 2, \dots, M$ can be found by minimizing the expression

$$\bar{I}^* = E_{|m}\Big\{\sum_{i=1}^{M}[\mathbf{u_i^T}V_i\mathbf{u_i}+2(v_i^*+\mathbf{w_i^*})^T Q_i B_{ii}^*\mathbf{u_i} + v_i^{*T}Q_i v_i^* + 2v_i^{*T}Q_i\mathbf{w_i^*} + \mathbf{w_i^{*T}}Q_i\mathbf{w_i^*}]_{\mathbf{u_i}=a_i(\mathbf{y_i},p_i)}\Big\} \tag{10}$$

subject to (7). Let us notice that $E_{|\mathbf{m}}(.)$ is a random variable while $E_{|m}(\cdot)$ is a realization of the random variable. Therefore $\mathbf{p_i} = b_i(\mathbf{m})$ and $\mathbf{v_i^*} = E_{|\mathbf{m}}\mathbf{v_i}$ in (9) are random variables while $p_i = b_i(m)$ and $v_i^* = E_{|m}\mathbf{v_i}$ in (10) are deterministic variables treated as parameters.

In order to solve the minimization problem, we use the Lagrange multiplier method. The Lagrangian functional has the form

$$\bar{I}^{**} = E_{|m}\Big\{\sum_{i=1}^{M}[\mathbf{u_i^T}V_i\mathbf{u_i}+2(v_i^*+\mathbf{w_i^*})^T Q_i B_{ii}^*\mathbf{u_i} + v_i^{*T}Q_i v_i^* + 2v_i^{*T}Q_i\mathbf{w_i^*} + \mathbf{w_i^{*T}}Q_i\mathbf{w_i^*} + 2l_i^T(v_i^* - \sum_{j\neq i}A_{ij}\mathbf{x_j})]\Big\}$$

$$= E_{|m}\Big\{\sum_{i=1}^{M}[\mathbf{u_i^T}V_i\mathbf{u_i}+2(v_i^*+\mathbf{w_i^*})^T Q_i B_{ii}^*\mathbf{u_i} + v_i^{*T}Q_i v_i^* + 2v_i^{*T}Q_i\mathbf{w_i^*} + \mathbf{w_i^{*T}}Q_i\mathbf{w_i^*} + 2l_i^T v_i^* - 2\sum_{j\neq i}l_j^T A_{ji}\mathbf{x_i}]\Big\}, \tag{11}$$

where l_i is a Lagrange multiplier treated as a parameter.

Inserting (8) into (11) gives

$$\bar{I}^{**} = E_{|m}\Big\{\sum_{i=1}^{M}[\mathbf{u_i^T}V_i\mathbf{u_i}+2\big(v_i^{*T}Q_iB_{ii}^* + \mathbf{w_i^{*T}}Q_iB_{ii}^* - \sum_{j\neq i}l_j^TA_{ji}B_{ii}^*\big)\mathbf{u_i} + v_i^{*T}Q_iv_i^* + 2v_i^{*T}Q_i\mathbf{w_i^*} + \mathbf{w_i^{*T}}Q_i\mathbf{w_i^*} + 2l_i^Tv_i^* - 2\sum_{j\neq i}l_j^TA_{ji}(v_i^*+\mathbf{w_i^*})\Big\}. \tag{12}$$

From (12) we know that the local control laws can be found independently by the minimization of the local Lagrangian functionals:

$$\bar{I}^{i**} = E_{|m}\Big\{[\mathbf{u_i^T}V_i\mathbf{u_i}+2(v_i^{*T}Q_iB_{ii}^* + \mathbf{w_i^{*T}}Q_iB_{ii}^* - \sum_{j\neq i}l_j^TA_{ji}B_{ii}^*)\mathbf{u_i} + v_i^{*T}Q_iv_i^* + 2v_i^{*T}Q_i\mathbf{w_i^*} + \mathbf{w_i^{*T}}Q_i\mathbf{w_i^*} + 2l_i^Tv_i^* - 2\sum_{j\neq i}l_j^TA_{ji}(v_i^*+\mathbf{w_i^*})]_{\mathbf{u_i}=a_i(\mathbf{y_i},p_i)}\Big\}$$

$$= E_{|m}\big\{E_{|m,\mathbf{y_i}}[\dots]_{\mathbf{u_i}=a_i(\mathbf{y_i},p_i)}\big\}. \tag{13}$$

Therefore the optimal control u_i results from the minimization of the function

$$S^{i**} = E_{|m,y_i}\big[u_i^TV_iu_i + 2(v_i^{*T}Q_iB_{ii}^* + \mathbf{w_i^{*T}}Q_iB_{ii}^* - \sum_{j\neq i}l_j^TA_{ji}B_{ii}^*)u_i + v_i^{*T}Q_iv_i^* + 2v_i^{*T}Q_i\mathbf{w_i^*} + \mathbf{w_i^{*T}}Q_i\mathbf{w_i^*} + 2l_i^Tv_i^* - 2\sum_{j\neq i}l_j^TA_{ji}(v_i^*+\mathbf{w_i^*})\big]. \tag{14}$$

Observe that minimization with respect to the function $\mathbf{u_i} = a_i(\mathbf{y_i}, p_i)$ in (13) is replaced by the minimization with respect to the variable u_i in (14).

Performing the $E_{|m,y_i}$ operation in (14) gives

$$S^{i**} = \Big[u_i^TV_iu_i + 2\Big(v_i^{*T}Q_iB_{ii}^* + \hat{w}_i^{*T}Q_iB_{ii}^* - \sum_{j\neq i}l_j^TA_{ji}B_{ii}^*\Big)u_i + v_i^{*T}Q_iv_i^* + 2v_i^{*T}Q_i\hat{w}_i^* + 2l_i^Tv_i^* - 2\sum_{j\neq i}l_j^TA_{ji}(v_i^*+\hat{w}_i^*)\Big] + E_{|y_i}\mathbf{w_i^{*T}}Q_i\mathbf{w_i^*}, \tag{15}$$

where

$$\hat{w}_i^* = E_{|m,y_i}\mathbf{w_i^*} = E_{|y_i}\mathbf{w_i^*} \tag{16}$$

is the estimate of the random variable $\mathbf{w_i^*}$ given information y_i.

Making the derivative of (15) with respect to u_i equal to zero yields

$$u_i^o = V_i^{-1}\Big[\sum_{j\neq i} B_{ii}^{*T}A_{ji}^T l_j - B_{ii}^{*T}Q_i(\hat{w}_i^* + v_i^*)\Big]. \tag{17}$$

Denoting

$$p_i = E_{|m}\mathbf{u_i^o}$$
$$= E_{|m}\Big\{V_i^{-1}\Big[\sum_{j\neq i} B_{ii}^{*T}A_{ji}^T\mathbf{l_j} - B_{ii}^{*T}Q_i(\mathbf{\hat{w}_i^*} + \mathbf{v_i^*})\Big]\Big\} \tag{18}$$

and determining the expectation given m gives

$$p_i = V_i^{-1}\Big[\sum_{j\neq i} B_{ii}^{*T}A_{ji}^T l_j - B_{ii}^{*T}Q_i(\bar{w}_i^* + v_i^*)\Big], \tag{19}$$

where

$$\bar{w}_i^* = E_{|m}\mathbf{w_i^*} = E_{|m_i}\mathbf{w_i^*} \tag{20}$$

is the estimate of the random variable $\mathbf{w_i^*}$ given information m_i.

Using (19) in (17) gives

$$u_i^o = p_i - V_i^{-1}B_{ii}^{*T}Q_i(\hat{w}_i^* - \bar{w}_i^*). \tag{21}$$

The i-th local control depends on the coordinating variable p_i and the local estimates $\hat{w}_i^*$ and $\bar{w}_i^*$.

In order to determine the local estimates defined by (16) and (20), a model of measurements is required. This model is described by (2).

3.2. Synthesis of Optimal Control Laws for the Coordinator

Write

$$\mathbf{x} = [\mathbf{x_1^T}\ \ \mathbf{x_2^T}\ \ \dots\ \ \mathbf{x_M^T}]^T,$$
$$\mathbf{u^o} = [\mathbf{u_1^{oT}}\ \ \mathbf{u_2^{oT}}\ \ \dots\ \ \mathbf{u_M^{oT}}]^T,$$
$$\mathbf{p} = [\mathbf{p_1^T}\ \ \mathbf{p_2^T}\ \ \dots\ \ \mathbf{p_M^T}]^T,$$
$$\mathbf{w}^* = [\mathbf{w_1^{*T}}\ \ \mathbf{w_2^{*T}}\ \ \dots\ \ \mathbf{w_M^{*T}}]^T,$$
$$Q_d = \mathrm{diag}\,[Q_1\ \ Q_2\ \ \dots\ \ Q_M],$$
$$H_d = \mathrm{diag}\,[H_1\ \ H_2\ \ \dots\ \ H_M],$$
$$V_d^{-1} = \mathrm{diag}\,[V_1^{-1}\ \ V_2^{-1}\ \ \dots\ \ V_M^{-1}],$$
$$B_d = \mathrm{diag}\,[B_{11}^*\ \ B_{22}^*\ \ \dots\ \ B_{MM}^*],$$

and

$$B^* = \mathbf{1} - \begin{bmatrix} \mathbf{0}_1 & A_{12} & \dots & A_{1M} \\ A_{21} & \mathbf{0}_2 & \dots & A_{2M} \\ \dots & \dots & \dots & \dots \\ A_{M1} & \dots & \dots & \mathbf{0}_M \end{bmatrix}, \tag{22}$$

where $\mathbf{1}$ is a unit matrix and $\mathbf{0}_i$, $i = 1, 2, \dots, M$ are zero-element matrices of appropriate dimensions.

Therefore, (3) and (1) can be written in the form

$$I = E\big[(\mathbf{x^T}Q_d\mathbf{x} + \mathbf{u^{oT}}H_d\mathbf{u^o})\big], \tag{23}$$

$$\mathbf{x} = B\mathbf{u^o} + \mathbf{w}, \tag{24}$$

where

$$\mathbf{u^o} = \mathbf{p} - V_d^{-1}B_d^TQ_d(\mathbf{\hat{w}}^* - \mathbf{\bar{w}}^*), \tag{25}$$

$$B = (B^*)^{-1}B_d, \quad \mathbf{w} = (B^*)^{-1}\mathbf{w}^*. \tag{26}$$

Inserting (24) and (25) into (23) yields

$$I = E\big[(\mathbf{p^T}V\mathbf{p} + 2\mathbf{p^T}B^TQ_d\mathbf{\bar{w}})_{\mathbf{p}=b(\mathbf{m})}\big] + s$$
$$= E\big[E_{|\mathbf{m}}(\cdot)_{\mathbf{p}=b(\mathbf{m})}\big] + s, \tag{27}$$

where $V = H_d + B^TQ_dB$, $\mathbf{\bar{w}} = E_{|\mathbf{m}}\mathbf{w}$ and

$$s = E\Big[(\mathbf{\hat{w}}^* - \mathbf{\bar{w}}^*)^TQ_dB_dV_d^{-1}VV_d^{-1}B_d^TQ_d(\mathbf{\hat{w}}^* - \mathbf{\bar{w}}^*)$$
$$+\mathbf{w^T}Q_d\mathbf{w} - 2(\mathbf{\hat{w}}^* - \mathbf{\bar{w}}^*)^TQ_dB_dV_d^{-1}B^TQ_d\mathbf{w}\Big]. \tag{28}$$

We see that s is independent of the designed control laws.

From (27) we know that coordinating variables $p = [p_1^T, \dots, p_M^T]^T$ can be found by the minimization of the function

$$S = p^TVp + 2p^TB^TQ_d\bar{w}. \tag{29}$$

Differentiating (29) with respect to p and equating the result to zero gives

$$p^o = -V^{-1}B^TQ_d\bar{w} = -V^{-1}B^TQ_d(B^*)^{-1}\bar{w}^*. \tag{30}$$

The value of p_i^o is forwarded to the i-th local controller.

Inserting (30) into (27) gives

$$I^o = s - E(\mathbf{\bar{w}^T}Q_dBV^{-1}B^TQ_d\mathbf{\bar{w}}). \tag{31}$$

Using (31), we can compare the quality of control for different kinds of information sent from local subsystems to the coordinator.

4. Example

Consider a simple system composed of two subsystems for which

$$B_{11}^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

$$B_{22}^* = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \tag{32}$$

$$Q_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad H_1 = [\ 1\ ],$$

$$Q_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad H_2 = [\ 2\ ]. \tag{33}$$

Let the model of measurements for the i-th subsystem have the form

$$y_i = C_i w_i^* + e_i \tag{34}$$

for which

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{35}$$

Assume that Gaussian random variables $\mathbf{w_1^*}$, $\mathbf{w_2^*}$, $\mathbf{e_1}$ and $\mathbf{e_2}$ are characterized by

$$E\mathbf{w_1^*} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad E\mathbf{w_2^*} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$P_{\mathbf{w_1^*}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad P_{\mathbf{w_2^*}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \tag{36}$$

$$E\mathbf{e_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad E\mathbf{e_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$P_{\mathbf{e_1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_{\mathbf{e_2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{37}$$

Also, assume that $D_1 = [\ 1 \quad 1\ ]$ and $\dim m_2 = 0$ (no information is sent from the second subsystem to the coordinator).

The control laws of the local controllers have the form

$$u_1^o = p_1 + \begin{bmatrix} -0.5 & 0.5 \end{bmatrix} (\hat{w}_1^* - \bar{w}_1^*),$$

$$u_2^o = p_2 + \begin{bmatrix} -0.26 & -0.15 \end{bmatrix} (\hat{w}_2^* - \bar{w}_2^*). \tag{38}$$

The optimal decisions of the coordinator have the form

$$p^o = \begin{bmatrix} -0.39 & -0.03 & 0.02 & -0.24 \\ -0.10 & 0.10 & -0.11 & 0.17 \end{bmatrix} \bar{w}^*. \tag{39}$$

The estimate $\hat{w}_i^*$ can be determined from the conventional formulae

$$\hat{w}_i^* = E\mathbf{w_i^*} + P_{\mathbf{w_i^*}\mathbf{y_i}} P_{\mathbf{y_i}\mathbf{y_i}}^{-1} (y_i - E\mathbf{y_i}), \tag{40}$$

where

$$P_{\mathbf{w_i^*}\mathbf{y_i}} = E(\mathbf{w_i^*} - E\mathbf{w_i^*})(\mathbf{y_i} - E\mathbf{y_i})^T,$$

$$P_{\mathbf{y_i}\mathbf{y_i}} = E(\mathbf{y_i} - E\mathbf{y_i})(\mathbf{y_i} - E\mathbf{y_i})^T.$$

Therefore, we have

$$\hat{w}_1^* = \begin{bmatrix} -0.8 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.6 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} y_1, \tag{41}$$

$$\hat{w}_2^* = \begin{bmatrix} -0.4 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 0.6 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} y_2. \tag{42}$$

The estimate $\bar{w}_1^*$ can be determined from the formulae

$$\bar{w}_1^* = E\mathbf{w_1^*} + P_{\mathbf{w_1^*}\mathbf{m_1}} P_{\mathbf{m_1}\mathbf{m_1}}^{-1} (m_1 - E\mathbf{m_1}). \tag{43}$$

For given data we have

$$\bar{w}_1^* = \begin{bmatrix} -1.14 \\ 0.57 \end{bmatrix} + \begin{bmatrix} 0.43 \\ 0.29 \end{bmatrix} m_1. \tag{44}$$

We get the estimate $\bar{w}_2^* = E\mathbf{w_2^*}$ since no information is sent to the coordinator.

The estimate $\bar{w}$ results from (26) and has the form

$$\bar{w} = (B^*)^{-1} \begin{bmatrix} \bar{w}_1^* \\ \bar{w}_2^* \end{bmatrix}, \tag{45}$$

where $\bar{w}_1^*$ results from (44).

Therefore,

$$\bar{w} = \begin{bmatrix} -1.36 \\ 0.17 \\ -0.02 \\ -0.19 \end{bmatrix} + \begin{bmatrix} 0.07 \\ -0.17 \\ -0.26 \\ -0.09 \end{bmatrix} m_1. \tag{46}$$

The effect of the aggregated information m_i on the control quality was investigated. The results are presented in Tab. 1.

Table 1. Quality of control in the hierarchical control structure.

$m_i = D_i y_i$	I^o
$D_1 = \mathbf{1},\ D_2 = \mathbf{1}$	5.1764
$D_1 = [1\ \ 1]$, dim $m_2 = 0$	5.6162
$D_1 = \mathbf{1},\ D_2 = [1\ \ 1]$	5.1894
$D_1 = [1\ \ 1],\ D_2 = \mathbf{1}$	5.2651
$D_1 = [1\ \ 1],\ D_2 = [1\ \ 1]$	5.2803
dim $m_1 = 0$, dim $m_2 = 0$	6.2816

If $D_1 = \mathbf{1}$ and $D_2 = \mathbf{1}$, then the measurements $y_1 = [y_1^1\ \ y_1^2]^T$ and $y_2 = [y_2^1\ \ y_2^2]^T$ are sent to the coordinator. In this case, $u_i^o = p_i^o$ and the algorithm is optimal. The value of the performance index is equal to 5.1764. If $D_1 = \mathbf{1}$ and $D_2 = [1\ \ 1]$, then the measurements $m_1 = y_1 = [y_1^1\ \ y_1^2]^T$ and $m_2 = y_2^1 + y_2^2$ are sent from the local subsystems to the coordinator. The algorithm is suboptimal. The value of the performance index is equal to 5.1894. The loss of optimality is about 0.2%. In this case it is interesting to realize control in a two-level hierarchical control structure instead of sending all information to the central decision maker.

If dim $m_1 = 0$ and dim $m_2 = 0$, then no information is sent to the coordinator. The value of the performance index is equal to 6.2816. The loss of optimality is about 21%.

5. Conclusions

In this paper a suboptimal control algorithm realized by decision makers having different information has been proposed. In the synthesis of local control laws it is assumed that the variable representing an interaction between subsystems is replaced by its best estimate calculated by the coordinator. Consequently, it is possible to partially decompose calculations and decentralize local controls.

It is found that the suboptimal local control laws are linear functions of local random input (disturbance) estimates and coordinating variables. An interaction is taken into account by the coordinator. It takes an optimal decision that is a linear function of an estimate of global disturbances.

It is possible to compare the qualities of control realized in one and two-level hierarchical control structures. Sometimes it is reasonable to consider suboptimal control realized in a two-level hierarchical control structure instead of optimal control realized by one central controller.

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