Int. J. Appl. Math. Comput. Sci., 2003, Vol. 13, No. 4, 469–479

ON THE CONSTRAINED CONTROLLABILITY OF DYNAMICAL SYSTEMS WITH MULTIPLE DELAYS IN THE STATE

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Linear stationary dynamical systems with multiple constant delays in the state are studied. Their relative and approximate controllability properties with constrained controls are discussed. Definitions of various types of controllability with constrained controls for systems with delays in the state are introduced. Some theorems concerning the relative and the approximate relative controllability with constrained controls for dynamical systems with delays in the state are established. Various types of constraints are considered. Numerical examples illustrate the theoretical analysis. An example of a real technical dynamical system is given to indicate one of possible practical applications of the theoretical results.

Keywords: delay systems, constrained controllability, control, constraints, asymptotic stability

1. Introduction

The controllability of dynamical systems is one of the basic notions in control theory. The controllability problem of delay dynamical systems is especially extensive. It is directly connected with a variety of mathematical models for these systems. Delay dynamical systems can be encountered in many fields of science, and among other things, in industrial processes, medicine, biology and economy.

This article includes an analysis of an important class of dynamical systems with delays, i.e., dynamical systems with delays in the state. Linear systems with multiple constant delays in the state are discussed. Relative and approximate controllability properties with constrained controls are investigated. Various types of constraints are considered. Some criteria of relative and approximate controllability for linear stationary dynamical systems with a single constant delay in the state are known (Banks *et al.*, 1975; Salamon, 1982; Klamka, 1990). However, these criteria concern mainly unconstrained controls. Since, in practice, controls are almost always constrained, it is worth to analyse the controllability of dynamical systems with additional constraints on control. Controllability with some type of constraints for systems with delays in the state was studied in (Chukwu, 1979; Son, 1990).

Dynamical systems with delays in the state are met, among other things, in the case of systems with feedback containing a delay (e.g. regarding measurements) in the feedback loop. Examples of real technical systems with delays in the state can be found, e.g., in chemical reactors, in electric systems containing long lines and in the case of heat exchangers and acoustic systems (Campbell, 1962; Bieńkowska-Lipińska, 1974; Luyben, 1990).

In the example below, we present a chemical solution control system. This example of a technical dynamical system is given to indicate one of possible applications of the theoretical results presented in this article.

Example 1. (Model with delay in the state for a solution control system.) Consider the cascade connection of two fully filled mixers according to the scheme presented in Fig. 1, where $c_{\mathrm{in}1}(t)$ and $c_{\mathrm{in}2}(t)$ are the input concentrations of the product, Q_1^* and Q_2^* denote constant flow intensities for concentrations $c_{\mathrm{in}1}(t)$ and $c_{\mathrm{in}2}(t)$, V_1 and V_2 stand for the volumes of Mixers 1 and 2, $c_1(t)$ and $c_2(t)$ are the strength of solutions in Mixers 1 and 2, respectively, L denotes the length of the reactor, and h is a constant delay arising in the reactor.

Assume that $V_1 = V_2 = V$. The state equations describing the above chemical system have the form

$$\begin{cases} V\dfrac{\mathrm{d}c_1(t)}{\mathrm{d}t} = Q_1^* c_{\mathrm{in}1}(t) - Q_1^* c_1(t), \\ V\dfrac{\mathrm{d}c_2(t)}{\mathrm{d}t} = Q_1^* c_1(t-h) \\ \qquad\qquad + Q_2^* c_{\mathrm{in}2}(t) - (Q_1^* + Q_2^*) c_2(t). \end{cases}$$

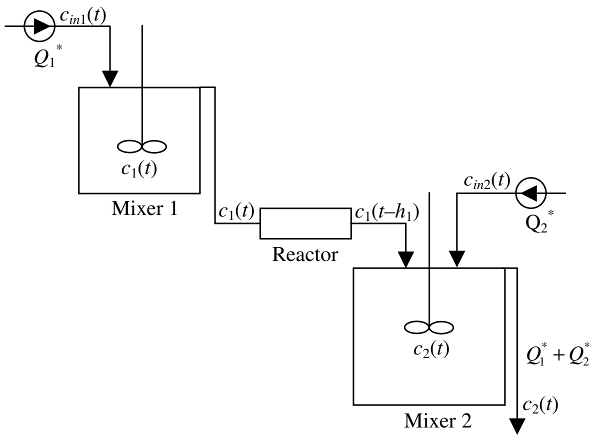

Fig. 1. Scheme of a cascade connection of two mixers.

After a transformation, we have

$$\begin{cases} \dfrac{\mathrm{d}c_1(t)}{\mathrm{d}t} &= -\dfrac{Q_1^*}{V}c_1(t) + \dfrac{Q_1^*}{V}c_{\mathrm{in}1}(t), \\ \dfrac{\mathrm{d}c_2(t)}{\mathrm{d}t} &= -\dfrac{(Q_1^*+Q_2^*)}{V}c_2(t) \\ & \quad +\dfrac{Q_1^*}{V}c_1(t-h) + \dfrac{Q_2^*}{V}c_{\mathrm{in}2}(t). \end{cases}$$

Taking $c_1(t) = x_1(t)$, $c_2(t) = x_2(t)$, $c_{\mathrm{in}1}(t) = u_1(t)$ and $c_{\mathrm{in}2}(t) = u_2(t)$, we get the mathematical model of the dynamical system with delay in the state, described by the following differential equation:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t),$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

and

$$A_0 = \begin{bmatrix} -\dfrac{Q_1^*}{V} & 0 \\ 0 & -\dfrac{Q_1^*+Q_2^*}{V} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ \dfrac{Q_1^*}{V} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \dfrac{Q_1^*}{V} & 0 \\ 0 & \dfrac{Q_2^*}{V} \end{bmatrix}.$$

This is an example of a dynamical system with constant delay in the state. Theoretical results presented in this article can be applied, among other things, to such technical systems. ♦

2. Mathematical Model

We consider linear stationary dynamical systems with lumped, multiple, constant delays in the state described by an ordinary differential equation with a delay argument of the following form:

$$\dot{x}(t) = \sum_{i=0}^{M} A_i x(t-h_i) + Bu(t), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ stands for the instantaneous n-dimensional state vector, $u \in L^2_{\mathrm{loc}}([0,\infty),\mathbb{R}^m)$ is the control, A_i, $i = 0,1,\dots,M$ are $(n\times n)$-dimensional matrices with real elements, B is an $(n\times m)$-dimensional matrix with real elements, and h_i, $i = 0,1,\dots,M$ denote constant delays satisfying the following inequalities:

$$0 = h_0 < h_1 < \dots < h_i < \dots < h_{M-1} < h_M, \tag{2}$$

with initial conditions $z_0 = (x(0), x_0) \in \mathbb{R}^n \times L^2([-h_M,0),\mathbb{R}^n)$, where $x(0) \in \mathbb{R}^n$ is the instantaneous state vector at time $t = 0$, and x_0 is a function given in the time interval $[-h_M,0)$, i.e., $x_0(t) = x(t)$ for $t \in [-h_M,0)$. The Hilbert space $\mathbb{R}^n \times L^2([-h_M,0),\mathbb{R}^n)$ endowed with the scalar product defined by

$$\begin{aligned} &\langle \{x(t), x_t\}, \{y(t), y_t\} \rangle \\ &\quad = \sum_{i=1}^{n} x_i(t) y_i(t) + \int_{-h_M}^{0} \langle x_t(\tau), y_t(\tau) \rangle_{\mathbb{R}^n} \,\mathrm{d}\tau, \end{aligned}$$

is denoted by $M_2([-h_M,0],\mathbb{R}^n)$.

Let $U \subset \mathbb{R}^m$ be a non-empty, convex and compact set such that $0 \in U$. Any control $u \in L^2_{\mathrm{loc}}([0,\infty)U)$ is called an admissible control for the dynamical system (1). The pair $z_t = (x(t), x_t) \in \mathbb{R}^n \times L^2([-h_M,0),\mathbb{R}^n) = M_2([-h_M,0],\mathbb{R}^n)$, where $x(t) \in \mathbb{R}^n$ is the vector of the current state and $x_t(\tau) = x(t+\tau)$ for $\tau \in [-h_M,0)$ is the segment of the trajectory of length h_M, which is defined in the time interval $[t-h_M, t)$, is called the complete state of the dynamical system (1) for $t \geq 0$.

For a given initial condition $z_0 = (x(0), x_0) \in M_2([-h_M,0],\mathbb{R}^n)$ and an admissible control $u \in L^2([0,t],U)$, for every $t \geq 0$ there exists a unique, absolutely continuous solution $x(t,z_0,u)$ of the differential equation (1). This solution has the form (Jacobs and Langenhop, 1976; Klamka, 1990):

$$x(t,z_0,u) = x(t,z_0,0) + \int_0^t F(t-\tau) Bu(\tau) \,\mathrm{d}\tau, \tag{3}$$

where the $(n\times n)$-dimensional transition matrix $F(t)$ is the solution of the following linear matrix integral equation:

$$F(t) = I + \sum_{i=0}^{M} \int_0^{t-h_i} F(\tau) A_i \,\mathrm{d}\tau \quad \text{for } t > 0, \tag{4}$$

with initial conditions $F(0) = I$ and $F(t) = 0$ for $t < 0$, and $x(t, z_0, 0)$ is the so-called free solution of (1) with zero control $u(t) = 0$ for $t \geq 0$, given by the formula

$$x(t, z_0, 0) = F(t)x(0) + \sum_{i=0}^{M} \int_{-h_i}^{0} F(t-\tau-h_i)A_i x_0(\tau)\,\mathrm{d}\tau. \quad (5)$$

The free component of the solution $x(t, z_0, 0)$ depends only on the initial complete state $z_0 = (x(0), x_0)$.

The set of all solutions of the differential equation with delay argument (1) at time $t_1 > 0$ with initial conditions $z_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbb{R}^n)$ and admissible control $u \in L^2([0, t_1], U)$ is called the attainable set in time $t_1 > 0$ of the dynamical system (1) from the initial complete state z_0 with constrained control. This set is denoted by $K_U([0, t_1], z_0)$. Therefore we can formulate the following definition of the attainable set:

Definition 1. The attainable set $K_U([0, t], z_0)$ of the dynamical system (1) from the initial complete state $z_0 = (x(0), x_0)$ in time $t > 0$ for $u(t) \in U$ is the set

$$K_U([0, t], z_0) = \Big\{ x(t) \in \mathbb{R}^n : \; x(t) = x(t, z_0, 0) + \int_0^t F(t-\tau)Bu(\tau)\,\mathrm{d}\tau, \; u \in L^2([0, t], U) \Big\}. \quad (6)$$

Remark 1. The attainable set $K_U([0, t], z_0)$ given by (6) is convex and closed, and $0 \in K_U([0, t], 0)$ for every $t \geq 0$ (Chukwu, 1979).

3. Basic Definitions

In this section we shall give the definitions of various types of controllability with constrained controls for continuous dynamical systems with delays in the state of the form (1). We also quote the definition and the necessary and sufficient condition for the asymptotic stability of the dynamical system (1).

Based on the definitions of systems with delay in the state, $\mathbb{R}^n$-controllability (Klamka, 1990) and systems without constrained control controllability (Klamka, 1990), we define various types of relative U-controllability for the dynamical system (1) in the time interval $[0, t_1]$, i.e. the relative controllability with constrained values of controls.

Let $S \subset \mathbb{R}^n$ be any non-empty set.

Definition 2. The dynamical system (1) is said to be *relatively U-controllable in the time interval $[0, t_1]$ from the complete state $z_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbb{R}^n)$ into the set S* if for every vector $\tilde{x} \in S$ there exists an admissible control $\tilde{u} \in L^2([0, t_1], U)$ such that the corresponding trajectory $x(t, z_0, \tilde{u})$ of the dynamical system (1) satisfies the condition

$$x(t_1, z_0, \tilde{u}) = \tilde{x}.$$

Definition 3. The dynamical system (1) is said to be *(globally) relatively U-controllable in the time interval $[0, t_1]$ into the set S* if it is relatively U-controllable in the interval $[0, t_1]$ into the set S for every initial complete state $z_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbb{R}^n)$.

Definition 4. The dynamical system (1) is said to be *(globally) relatively U-controllable from $t_0 = 0$ into the set S* if for every initial complete state $z_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbb{R}^n)$ there exists $t_1 \in [0, \infty)$ such that the dynamical system (1) is relatively U-controllable in the time interval $[0, t_1]$ into the set S.

If $S = \mathbb{R}^n$, then we deal with the (global) relative U-controllability in the time interval $[0, t_1]$. When $S = \{0\}$, we deal with the relative null U-controllability in $[0, t_1]$ from a complete state z_0, and the (global) relative null U-controllability in $[0, t_1]$.

Assume that S is a linear variety in $\mathbb{R}^n$ of the form

$$S = \{x \in \mathbb{R}^n : \quad Lx = c\}, \quad (7)$$

where L is a known $(p \times n)$-matrix of rank p and $c \in \mathbb{R}^p$ is a given vector. In the specific case, when $L = I_n$ (the $(n \times n)$-dimensional identity matrix) and $c = 0$, we get $S = \{0\}$.

Definition 5. The dynamical system (1) is said to be *approximately relatively U-controllable* from the complete state $z_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbb{R}^n)$ if there exists $t_1 > 0$ such that

$$\mathrm{cl}K_U([0, t_1], z_0) = \mathbb{R}^n.$$

The notation $\mathrm{cl}K_U([0, t_1], z_0)$ means the closure of the attainable set $K_U([0, t_1], z_0)$.

In the sequel, the asymptotic stability of the dynamical system (1) will be exploited.

Definition 6. (Kaczorek, 1977; Klamka and Ogonowski, 1999) The dynamical system (1) is said to be *asymptotically stable* if for any initial complete state $z_0 \in M_2([-h_M, 0], \mathbb{R}^n)$ and $u = 0$, the complete state at time $t > 0$, i.e., $z_t = (x(t), x_t)$ satisfies the condition

$$\lim_{t \to \infty} \|z_t\|_{M_2} = 0.$$

Theorem 1. (Kaczorek, 1977; Klamka and Ogonowski, 1999) *The dynamical system (1) is asymptotically stable if and only if all the roots s_i of the quasi-characteristic equation*

$$\varphi(s) = \det\Big(sI - \sum_{i=0}^{M} A_i e^{-sh_i}\Big) = 0$$

of the autonomous dynamical system (1) ($u(t) \equiv 0$) have negative real parts, i.e., $\Re[s_i] < 0$ for $i = 1, 2, \ldots$.

4. Controllability Results

In this section we shall formulate some theorems concerning the controllability with constrained controls for the dynamical system (1). We shall give criteria of the relative U-controllability into the set S of the form (7) in the time interval $[0, t_1]$ and the relative null U-controllability of dynamical systems with delays in the state (1).

Generalizing the results obtained in (Chukwu, 1979), we first give a sufficient condition for the relative null U-controllability of the dynamical system (1). To this end, we define the following matrix (Chukwu, 1979; Klamka, 1990):

$$Q_k(s) = \sum_{i=0}^{M} A_i Q_{k-1}(s - h_i), \quad s \in [0, \infty), \quad k = 1, 2, \ldots, \tag{8}$$

$$Q_0(s) = \begin{cases} B & \text{for } s = 0, \\ 0 & \text{for } s \neq 0, \end{cases} \tag{9}$$

and

$$\tilde{Q}_n(t_1) = \{Q_0(s), Q_1(s), \ldots, Q_{n-1}(s), \ \ s \in [0, t_1]\} \tag{10}$$

for $s = h_i, 2h_i, 3h_i, \ldots,\ i = 0, 1, 2, \ldots, M$.

We define the rank of $\tilde{Q}_n(t_1)$ as the rank of the block matrix composed of all matrices from the set $\tilde{Q}_n(t_1)$.

Lemma 1. (Chukwu, 1979; Klamka, 1990) *For every $t_1 \in (0, \infty)$ the following conditions are equivalent:*

(i) if $c^T F(t) B = 0$ for $t \in [0, t_1]$ and $c \in \mathbb{R}^n$, then $c = 0$,

(ii) $\operatorname{rank} \tilde{Q}_n(t_1) = n$,

(iii) the dynamical system (1) without constraints on controls is (globally) relatively controllable in the time interval $[0, t_1]$.

Remark 3. The equivalence of Conditions (i) and (ii) is shown in (Chukwu, 1979), whereas the equivalence of Conditions (ii) and (iii) is shown in the monograph (Klamka, 1990).

Based on Lemma 1 we shall formulate a sufficient condition for the (global) relative null U-controllability of the dynamical system (1).

Theorem 2. *If for the dynamical system (1), for all $t_1 > 0$, one of Conditions (i), (ii) or (iii) of Lemma 2 is satisfied and the system is asymptotically stable, then the dynamical system (1) is (globally) relatively null U-controllable.*

Proof. Fix a final time $t_1 > 0$. We assume that for the dynamical system (1) Condition (i) of Lemma 1 is satisfied and the system is asymptotically stable. Let $\Phi_{t_1} = \{z_0 \in M_2([-h_M, 0], \mathbb{R}^n)$: there exists $u \in U$ such that $x(t_1, z_0, u) = 0$, $t_1 \in (0, \infty)\}$ denote the domain of relative null U-controllability for the dynamical system (1) in time $t_1 > 0$. We notice that $0 \in \Phi_{t_1}$ (here, by 0 we mean the pair $(0, 0) \in \mathbb{R}^n \times L^2([-h_M, 0), \mathbb{R}^n)$), because with zero initial conditions, owing to the stability assumption, $x = 0$ is the solution of (1) for the admissible control $u = 0$.

Since the attainable set $K_U([0, t_1], 0)$ of the dynamical system (1), given by (6), is a convex and closed subset of the space $\mathbb{R}_n$ (cf. Remark 1), from Condition (i) of Lemma 1, $0 \in \mathbb{R}^n$ lies in the interior of the attainable set from the zero initial complete state of the dynamical system (1) with constraints, i.e., $0 \in \operatorname{int} K_U([0, t_1], 0)$ for every $t_1 > 0$ (Chukwu, 1979, Thm. 2.2).

We shall show that $0 \in \operatorname{int} \Phi_{t_1}$. Assume that $0 \notin \operatorname{int} \Phi_{t_1}$. Then there exists a sequence of initial complete states $\{z_{0n}\}_{n \in \mathbb{N}}$ convergent to zero, and for every z_{0n}, $z_{0n} \notin \Phi_{t_1}$ such that $z_{0n} \neq 0$ we have

$$0 \neq x(t_1, z_{0n}, u) = x(t_1, z_{0n}, 0) + \int_0^{t_1} F(t-\tau) B u(\tau)\, \mathrm{d}\tau$$

for every $t_1 > 0$ and $u \in U$.

We form a sequence of final states $\{x_n(t_1, z_{0n}, 0)\}_{n \in \mathbb{N}}$ corresponding to the sequence of complete initial states $\{z_{0n}\}_{n \in \mathbb{N}}$, with $u = 0$. The elements of this sequence are different from the zero elements of the attainable set $K_U([0, t_1], 0)$ and converge to zero as $n \to \infty$. Therefore, $0 \notin \operatorname{int} K_U([0, t_1], 0)$ for every $t_1 > 0$. But this contradicts the assumption that $0 \in \operatorname{int} K_U([0, t_1], 0)$ for every $t_1 > 0$ and any initial complete state z_0. Thus $0 \in \operatorname{int} \Phi$.

Since the set Φ_{t_1} contains 0 in the interior, it also contains a neighbourhood of $0 = (0, 0) \in \mathbb{R}^n \times L^2([-h_M, 0), \mathbb{R}^n)$. By a neighbourhood in the space $\mathbb{R}^n \times L^2([-h_M, 0), \mathbb{R}^n)$ we mean a pair of neighbourhoods consisting of a neighbourhood of $0 \in \mathbb{R}^n$ and a neighbourhood of $0 \in L^2([-h_M, 0), \mathbb{R}^n)$.

Let an initial complete state $z_0 = (x(0), x_0) \neq \mathbf{0}$ of the dynamical system (1) be given. Using the null control $u(t) = 0$, the solution $x(t, z_0, 0)$ satisfies the conditions

$$\lim_{t\to\infty} x(t, z_0, 0) \equiv 0, \quad x(t_1, z_0, 0) \in P$$

for some finite $t_1 \in (0, \infty)$, where P is a sufficiently small neighbourhood of $0 \in \mathbb{R}^n$. Then the instantaneous state $x(t_1, z_0, 0)$ can be steered to $0 \in \mathbb{R}^n$ in a finite time, so the dynamical system (1) is (globally) relatively null U-controllable. ■

In order to formulate the criteria of relative U-controllability for the dynamical system (1) on the assumption that the final set is of the form (7), introduce a scalar function $J : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ related to the attainable set $K_U([0, t_1], z_0)$ of the system (1). The function J has the following form:

$$\begin{aligned} J(z_0, t_1, v) &= v^T L x(t_1, z_0, 0) \\ &\quad + \int_0^{t_1} \sup\{v^T L F(t_1 - \tau) B u(\tau), \\ &\qquad u \in L^2([0, t], U)\}\, \mathrm{d}t - v^T c, \end{aligned} \tag{11}$$

where $v \in \mathbb{R}^p$ is any vector.

The scalar function J, called the supporting function of the attainable set, was used for dynamical systems without delays in (Schmitendorf and Barmish, 1981; Klamka, 1990).

The theorem below gives the necessary and sufficient condition of relative U-controllability from the complete state $z_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbb{R}^n)$ into the set S of the form (7) in the time interval $[0, t_1]$ for the dynamical system (1).

Theorem 3. *Let $E \subset \mathbb{R}^p$ be any set containing 0 as an internal point. Then the dynamical system with delays in the state (1) is relatively U-controllable from the complete state $z_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbb{R}^n)$ into the set S of the form (7) if and only if for some $t_1 \in [0, \infty)$ we have*

$$\min\{J(z_0, t_1, v) :\ v \in E\} = 0$$

or, equivalently, if and only if

$$J(z_0, t_1, v) \geq 0 \ \text{ for every } \ v \in E,$$

where the scalar function $J(z_0, t_1, v)$ is defined by (11).

Proof. The set $K_U([0, t_1], z_0)$ is compact. In order to prove this fact, we shall show that for any sequence of points $x_1(t_1), x_2(t_1), \ldots, x_k(t_1), \ldots$ belonging to the set $K_U([0, t_1], z_0)$ of the form (6) we can choose a subsequence convergent to a point $\bar{x}(t_1) \in K_U([0, t_1], z_0)$. Since the set of admissible controls $L^2([0, t_1], U)$ is weakly compact in the space $L^2([0, t_1], \mathbb{R}^m)$ (Lee and Marcus, 1972, Lem. 1A, p. 169), there exists a subsequence of controls $u_{k_i} \in L^2([0, t_1], U)$ which is weakly convergent to a control $\bar{u}$ such that

$$\lim_{k_i\to\infty} \int_0^{t_1} F(t_1 - \tau) B u_{k_i}(\tau)\, \mathrm{d}\tau = \int_0^{t_1} F(t_1 - \tau) B \bar{u}(\tau)\, \mathrm{d}\tau.$$

Let $\bar{x}(t)$ be the solution corresponding to the control $\bar{u}(t)$. Then, in the time interval $[0, t_1]$ we have

$$\begin{aligned} \bar{x}(t) &= F(t)x(0) + \sum_{i=1}^{M} \int_{-h_i}^{0} F(t - \tau - h_i) A_i x_0(\tau) \mathrm{d}\tau \\ &\quad + \int_0^t F(t - \tau) B(\tau) \bar{u}(\tau) \mathrm{d}\tau = \lim_{k_i\to\infty} x_{k_i}(t) \end{aligned}$$

because

$$x(t, z_0, 0) = F(t)x(0) + \sum_{i=1}^{M} \int_{-h_i}^{0} F(t - \tau - h_i) A_i x_0(\tau) \mathrm{d}\tau.$$

Therefore

$$\lim_{k_i\to\infty} x_{k_i}(t_1) = \bar{x}(t_1) \in K_U([0, t_1], z_0),$$

which implies that the attainable set $K_U([0, t_1], z_0)$ is compact in the space $\mathbb{R}^n$.

Moreover, the attainable set $K_U([0, t_1], z_0)$ of the dynamical system (1) is convex (see Remark 1).

Owing to the convexity and compactness of the set $K_U([0, t_1], z_0)$, the set $\tilde{K}_U([0, t_1], z_0)$ of the form

$$\tilde{K}_U([0, t_1], z_0) = \{y \in \mathbb{R}^p :\ y = Lx,\ x \in K_U([0, t_1], z_0)\}$$

is also convex and compact. An initial complete state z_0 can be steered to the set S in time t_1 if and only if the vector c and the set $\tilde{K}_U([0, t_1], z_0)$ are not exactly separable by a hyperlane, i.e., if for all vectors $v \in \mathbb{R}^p$ we have

$$v^T c \leq \sup\{v^T \tilde{x} :\ \tilde{x} \in \tilde{K}_U([0, t_1], z_0)\}.$$

This fact follows from the theorem about separating convex sets.

Taking into account the form of the set $\tilde{K}_U([0, t_1], z_0)$, we can equivalently write the above inequality as follows:

$$\sup\left\{ \int_0^{t_1} v^T L F(t_1 - \tau) B u(\tau)\, \mathrm{d}t,\ u \in L^2([0, t], U) \right\} - v^T c \geq 0.$$

Interchanging integration and the supremum operation, we conclude that $c \in \tilde{K}_U([0,t_1],z_0)$ if and only if $J(z_0,t_1,v) \geq 0$ for all vectors $v \in \mathbb{R}^p$.

Moreover, we can show that

$$kJ(z_0,t_1,v) = J(z_0,t_1,kv) \quad \text{for every} \quad k \geq 0.$$

Therefore, limiting attention to vectors $v \in E$, we obtain the assertion of the theorem. ∎

Corollary 1. *Let $E \subset \mathbb{R}^n$ be any set containing 0 as an internal point. Then the dynamical system with delays in the state (1) is relatively null U-controllable from the complete state $z_0 = (x(0),x_0) \in M_2([-h_M,0],\mathbb{R}^n)$ if and only if for some $t_1 \in [0,\infty)$ the equality*

$$\min\{J(z_0,t_1,v) : \ v \in E\} = 0$$

is satisfied or, equivalently, if and only if

$$J(z_0,t_1,v) \geq 0 \ \textit{for every} \ v \in E,$$

where the scalar function $J(z_0,t_1,v)$ is defined by (11).

Proof. This corollary follows directly from Theorem 3 for $S=\{0\}$, i.e., for $L = I_n$ and $c = 0$. Then E is a subset of the space $\mathbb{R}^n$. ∎

Now, assume that $U = \mathbb{R}^m_+$. We shall give a sufficient condition of the (global) relative controllability with positive controls for the dynamical system (1).

Theorem 4. *The dynamical system (1) is (globally) relatively $\mathbb{R}^m_+$-controllable in the time interval $[0,t_1]$ if the $(n \times n)$-dimensional controllability matrix $W(t_1)$ of this system, given by*

$$W(t_1) = \int_0^{t_1} F(t_1-\tau)BB^TF^T(t_1-\tau)\mathrm{d}\tau, \tag{12}$$

satisfies

$$\operatorname{rank} W(t_1) = n \tag{13}$$

and $W^{-1}(t_1) \in \mathbb{R}^{n\times n}_+$, $B^TF^T(t_1-t) \in \mathbb{R}^{m\times n}_+$ for every $t \in [0,t_1]$ and $(\tilde{x} - x(t_1,z_0,0)) \in \mathbb{R}^n_+$, where the transition matrix $F(t)$ is determined by (4).

Proof. Let the assumptions of Theorem 4 be satisfied. Let $z_0 = (x(0),x_0) \in \mathbb{R}^n \times L^2([-h_M,0),\mathbb{R}^n)$ be any initial complete state of the dynamical system (1) and $\tilde{x} \in \mathbb{R}^n$ be any vector. We shall prove that the control $u \in L^2([0,t],\mathbb{R}^m_+)$ of the form

$$u(t) = B^TF^T(t_1-t)W^{-1}(t_1)(\tilde{x} - x(t_1,z_0,0)), \tag{14}$$

for $t \in [0,t_1]$ steers the system (1) from the initial state z_0 to the state $x(t_1,z_0,u) = \tilde{x}$. Substituting (14) into (3) for $t = t_1$, we get

$$\begin{aligned} &x(t_1,z_0,u) \\ &\quad = x(t_1,z_0,0) + \int_0^{t_1} \Big(F(t_1-\tau)BB^TF^T(t_1-\tau) \\ &\qquad \times W^{-1}(t_1)\big(\tilde{x} - x(t_1,z_0,0)\big)\Big)\,\mathrm{d}\tau \\ &\quad = x(t_1,z_0,0) \\ &\qquad + W(t_1)W^{-1}(t_1)(\tilde{x} - x(t_1,z_0,0)) = \tilde{x}. \end{aligned}$$

Since z_0 and $\tilde{x}$ were arbitrary, the dynamical system (1) is relatively $\mathbb{R}^m_+$-controllable in the time interval $[0,t_1]$. ∎

We can also formulate an analogous theorem with the aid of a generalized permutation matrix. A matrix with non-negative elements is called the generalized permutation matrix or, briefly, the GPM, if in each row and in each column only one element is positive and the remaining entries are zero (Kaczorek, 2000; 2002). Recall that in a permutation matrix all these elements are equal to one.

Theorem 5. *The dynamical system (1) is (globally) relatively $\mathbb{R}^m_+$-controllable in the time interval $[0,t_1]$ if the $(n \times n)$-dimensional controllability matrix $W(t_1)$ of this system*

$$W(t_1) = \int_0^{t_1} F(t_1-\tau)BB^TF^T(t_1-\tau)\,\mathrm{d}\tau$$

is a GPM, $B^TF^T(t_1-t) \in \mathbb{R}^{m\times n}_+$ for every $t \in [0,t_1]$ and $(\tilde{x} - x(t_1,z_0,0)) \in \mathbb{R}^n_+$, where the transition matrix $F(t)$ is determined by (4).

Proof. We only have to notice that if $W(t_1)$ is a GPM, then $W^{-1}(t_1) \in \mathbb{R}^{n\times n}_+$. Further, the proof is analogous to that of Theorem 4. ∎

In general, the calculation of the transition matrix from (4) is complicated. A more practical way to calculate the transition matrix is the so-called method of steps, as known from the literature (Bieńkowska-Lipińska, 1974; Muszyński and Myszkis, 1984). The method of steps is based on searching for the solution of the differential equation with a delay argument in succeeding intervals whose lengths depend on the delays occurring in the dynamical system.

Consider the following system with a single delay:

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + Bu(t), \quad t \geq 0. \tag{15}$$

In the first step, we get a solution $x_h(t)$ of (15) in the interval $[0,h]$. Then $(t-h) \in [-h,0]$ and $x(t-h) =$

$x_0(s)$ for $s \in [-h, 0]$, where x_0 is a function known from the initial conditions. Therefore, we get an ordinary differential equation without delays of the following form:

$$\dot{x}(t) = A_0 x(t) + A_1 x_0(t-h) + Bu(t), \quad t \in [0, h], \tag{16}$$

which has a unique solution $x(t) = x_h(t)$ in the interval $t \in [0, h]$,

$$\begin{aligned} x_h(t) &= x(t) \\ &= e^{A_0 t} x(0) + \int_0^t e^{A_0(t-\tau)} [A_1 x_0(\tau - h) + Bu(t)] \, \mathrm{d}\tau. \end{aligned}$$

In the next step, we look analogously for the solution of the equation

$$\dot{x}(t) = A_0 x(t) + A_1 x_h(t-h) + Bu(t) \tag{17}$$

for $t \in [h, 2h]$, with the initial condition $x(h) = x_h(h)$. Thus

$$\begin{aligned} x_{2h}(t) &= x(t) \\ &= e^{A_0 t} x(h) + \int_h^t e^{A_0(t-\tau)} [A_1 x_h(\tau - h) + Bu(t)] \, \mathrm{d}\tau. \end{aligned}$$

Generally, in the n-th step we look for the solution of the equation

$$\dot{x}(t) = A_0 x(t) + A_1 x_{nh}(t-h) + Bu(t) \tag{18}$$

for $t \in [nh, (n+1)h]$ with the initial condition $x(nh) = x_{nh}(nh)$.

Consequently, we get the solution of the differential equation with the delay argument (15) and the initial condition $z_0 = (x(0), x_0)$ at time $t \in [nh, (n+1)h]$ by solving (18) in the interval $[nh, (n+1)h]$ with the initial condition $x(nh) = x_{nh}(nh)$, where x_{nh} is obtained by the method of steps, i.e.,

$$x(t, z_0, u) = x_{nh}(t_1) \text{ for } t_1 \in [nh, (n+1)h].$$

Finally, the solution of (15) at time $t_1 > 0$ has the form

$$\begin{aligned} x(t_1, x_0, u) &= e^{A_0 t_1} x(nh) \\ &\quad + \int_{nh}^{t_1} e^{A_0(t_1-\tau)} [A_1 x_{nh}(\tau - h) + Bu(\tau)] \, \mathrm{d}\tau. \end{aligned}$$

We can also formulate the criteria for the controllability of the dynamical system (1) without the assumption on the convexity of the set of admissible control values $U \subset \mathbb{R}^m$. The theorem which characterizes the approximate controllability in the space $\mathbb{R}^n \times L^2([-h_M, 0], \mathbb{R}^n)$ for dynamical systems of the form (1) assuming that U is a cone in the space $\mathbb{R}^m$, whose convex hull has a nonempty interior was given in the article (Son, 1990). A convex hull is denoted by $\operatorname{co} U$. Similarly, we can formulate a criterion of approximate relative controllability for the dynamical system (1) with the same constraints on the control values.

Introduce the following notation:

$$\Phi(s) = sI - \sum_{i=0}^{\mathrm{M}} A_i e^{-sh_i}$$

and

$$\varphi(s) = \det \Phi(s),$$

where $\Phi(s)$ is an $(n \times n)$-dimensional matrix.

Theorem 6. *Assume that U is a cone in the space $\mathbb{R}^m$ such that $\operatorname{int} \operatorname{co} U \neq \emptyset$. Let $\det A_M \neq 0$ and $m = 1$. The dynamical system (1) is approximately relatively U-controllable if and only if*

$$\operatorname{rank} [\Phi(s), B] = n \text{ for every } s \in \mathbb{C}$$

and the quasi-characteristic equation $\varphi(s) = 0$ has no real roots.

Proof. This theorem results directly from Theorem 4.3 and Corollary 4.4 in (Son, 1990). The condition $\operatorname{rank} [\Phi(s), B] = n$ for every $s \in \mathbb{C}$ with the assumptions $\det A_M \neq 0$ and $m = 1$ is a necessary and sufficient condition for the approximate relative controllability of the dynamical system (1) without constraints (Klamka, 1990; Son, 1990). If, moreover, the quasi-characteristic equation $\varphi(s) = 0$ has no real roots, the condition (18) of Theorem 4.3 in (Son, 1990) is satisfied and the dynamical system (1) with constrained controls is approximately relatively U-controllable. ■

Notice that, in a particular case, we can use this theorem for dynamical systems with positive controls.

Corollary 2. *Assume that $U = \mathbb{R}^m_+$. Let $\det A_M \neq 0$ and $m = 1$. The dynamical system (1) is approximately relatively $\mathbb{R}^m_+$-controllable if and only if*

$$\operatorname{rank} [\Phi(s), B] = n \text{ for every } s \in \mathbb{C}$$

and the quasi-characteristic equation $\varphi(s) = 0$ has no real roots.

Proof. We only have to notice that $U = \mathbb{R}^m_+$ is a cone in the space $\mathbb{R}^m$ such that $\operatorname{int} \operatorname{co} U \neq \emptyset$. ■

Approximate relative controllability is a weaker notion than relative controllability, but it appears sufficient for many important controllability tasks.

5. Examples

This section contains numerical examples illustrating the theoretical analysis. Moreover, in Examples 2 and 3 we shall formulate stability conditions for some type of delays and the elements of matrices A_i, $i = 0, 1, \dots, M$ and B. In Example 4, the way of constructing a transition matrix by the method of steps will be introduced.

Example 2. We consider a dynamical system with two delays in the state given by the equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_2 x(t-2) + Bu(t), \tag{19}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{20}$$

with control values in the m-dimensional unit hypercube

$$C^m = \{u(t) \in \mathbb{R}^m : \ |u_j(t)| \le 1, \ \ j = 1, 2, \dots, m\}, \ t > 0.$$

It is a convex and compact set containing $0 \in \operatorname{int} C^m$.

Then we have $h_0 = 0$, $h_1 = 1$, $h_2 = 2$, $n = 2$, $m = 1$, $M = 2$ and $U = C^m$.

We shall study the stability of the dynamical system (19). The quasi-characteristic polynomial of the autonomous equation of this system has the following form:

$$\begin{aligned}
\varphi(s) &= \det(sI - A_0 - e^{-sh_1} A_1 - e^{-sh_2} A_2) \\
&= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \right. \\
&\qquad \left. - e^{-s} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} - e^{-2s} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \\
&= \det \begin{bmatrix} s & -1 \\ 1 - e^{-2s} & s + 4 + 2e^{-s} \end{bmatrix} \\
&= s^2 + 4s + 2se^{-s} - e^{-2s} + 1.
\end{aligned}$$

We get the quasi-characteristic equation

$$s^2 + 4s + 2se^{-s} - e^{-2s} + 1 = 0. \tag{21}$$

We shall show that all roots of the above quasi-characteristic equation have negative real parts. To obtain a contradition, assume that $s = \alpha + \beta i$ is a root of (21) with a non-negative real part $\alpha \ge 0$. Notice that $\beta = 0$ is impossible, because in that case the left-hand side of (21) is positive. So $\beta \neq 0$ and we have

$$\begin{aligned}
&\Im(s^2 + 4s + 2se^{-s} - e^{-2s} + 1)/\beta \\
&\quad = 2\alpha + 4 + 2e^{-\alpha} \left(\cos\beta - \alpha \frac{\sin\beta}{\beta} \right) \\
&\qquad + 2e^{-2\alpha} \frac{\sin 2\beta}{2\beta} > 4 - 2 - 1 > 0.
\end{aligned}$$

This contradicts the assumption that the complex number $s = \alpha + \beta i$ is a root of (21). Therefore, all roots of the characteristic equation (21) have negative real parts. Based on Theorem 1, we conclude that the dynamical system (19) is stable.

Moreover, we shall show that Condition (ii) of Lemma 1 is satisfied, i.e., $\tilde{Q}_n(t_1) = n$, $n = 2$ for every $t_1 > 0$. To this end, we find all matrices belonging to the set $\tilde{Q}_2(t_1)$:

$$Q_0(0) = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ \text{and} \ Q_0(s) = 0 \ \text{for} \ s \neq 0.$$

Since

$$\begin{aligned}
Q_1(s) &= \sum_{i=0}^{2} A_i Q_0(s - h_i) \\
&= A_0 Q_0(s) + A_1 Q_0(s - h_1) + A_2 Q_0(s - h_2),
\end{aligned}$$

for $s = h_i, 2h_i, 3h_i, \dots$, $i = 0, 1, 2$, we get

$$Q_1(0) = A_0 B = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix},$$

$$Q_1(h_1) = A_1 Q_0(0) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix},$$

$$Q_1(h_2) = A_2 Q_0(0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$Q_1(2h_1) = A_2 Q_0(0) = Q_1(h_2), \ \text{as} \ 2h_1 = h_2.$$

Hence

$$\tilde{Q}_2(t_1) = \{Q_0(0), Q_1(0), Q_1(h_1), Q_1(h_2)\}, \quad n = 2,$$

and

$$\operatorname{rank} \tilde{Q}_2(t_1) = \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -4 & 8 & 1 \end{bmatrix} = 2.$$

We proved that Condition (ii) of Lemma 1 is satisfied in any time interval $[0, t_1]$ and the dynamical system (19) is stable, so based on Theorem 2, the dynamical system (19) is (globally) relatively null U-controllable. ♦

Example 3. Now, consider a more general case of a dynamical system with two delays in the state of the form

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t-h_1)\\ &\quad + A_2x(t-h_2) + Bu(t),\end{aligned} \tag{22}$$

where

$$\begin{aligned}A_0 &= \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -q \end{bmatrix},\\ A_2 &= \begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\end{aligned} \tag{23}$$

and k, b, q, d, h_1 and h_2 are positive constants.

The quasi-characteristic equation of the autonomous equation (22) has the form

$$s^2 + bs + qse^{-sh_1} - de^{-sh_2} + k = 0.$$

Making an analysis analogous to that in Example 2, we conclude that the dynamical system (22) with system matrices A_0, A_1, A_2 and B of the form (23) is stable if

$$b > q + d.$$

If

$$\begin{aligned}A_0 &= \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -q \end{bmatrix},\\ A_2 &= \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\end{aligned} \tag{24}$$

then the quasi-characteristic equation has the form

$$s^2 + bs + qse^{-sh_1} + sde^{-sh_2} + k = 0.$$

Again, (22) with system matrices A_0, A_1, A_2 and B of the form (24) is stable if

$$b > q + d.$$

Next, we can easily generalize this method to the case of a finite number of delays, i.e., to dynamical systems (1) with system matrices A_i, $i = 2, \dots, M$ of the form

$$\begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix}.$$

Then the dynamical system (1) is stable if

$$b > q + \sum_{i=2}^{M} d_i.$$

The stability of dynamical systems with delays in the state can also be investigated by other methods, e.g. by the method of Lapunov functionals or the D-section method (Bieńkowska-Lipińska, 1974; Muszyński and Myszkis, 1984). ♦

Example 4. Let a dynamical system of the form (1) be given in the time interval $[0, 3]$, where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with a single delay $h_1 = 1$ and the initial conditions $z_0 = (1, 1)$.

We compute the transition matrix $F(t)$ of the dynamical system with delay using the method of steps.

With the aid of the Laplace transform, we calculate

$$e^{A_0 t} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

In the first step, taking into consideration the initial conditions, in the interval $[0, 1]$ we get

$$F(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} + \int_0^1 \begin{bmatrix} 1 & 0 \\ t-\tau & 1 \end{bmatrix} \mathrm{d}\tau = \begin{bmatrix} 2 & 0 \\ 2t - \dfrac{1}{2} & 2 \end{bmatrix}.$$

In the next step, with the new initial conditions

$$x_1(1) = \begin{bmatrix} 2 & 0 \\ \dfrac{3}{2} & 2 \end{bmatrix},$$

in $[1, 2]$ we get

$$\begin{aligned}F(t) &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ \dfrac{3}{2} & 2 \end{bmatrix} + \int_1^2 \begin{bmatrix} 1 & 0 \\ t-\tau & 1 \end{bmatrix}\\ &\quad \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2(t-\tau) - \dfrac{1}{2} & 2 \end{bmatrix} \mathrm{d}\tau\\ &= \begin{bmatrix} 2t - 3 & 2 \\ 2t^2 - 4t + \dfrac{49}{6} & 2t - 1 \end{bmatrix}.\end{aligned}$$

In the last step, with the initial conditions

$$x_2(2) = \begin{bmatrix} 1 & 2 \\ \dfrac{49}{6} & 1 \end{bmatrix},$$

in the interval $[2,3]$ we have

$$F(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ \frac{49}{6} & 1 \end{bmatrix} + \int_2^3 \begin{bmatrix} 1 & 0 \\ t-\tau & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 2(t-\tau)-3 & 2 \\ 4(t-\tau)^2 - 4(t-\tau) + \frac{49}{6} & 2(t-\tau)-3 \end{bmatrix} \mathrm{d}\tau$$

$$= \begin{bmatrix} 4t^2 - 24t + \frac{89}{2} & 2t-6 \\ 4t^3 - 34t^2 + 103t - 110\frac{7}{12} & 2t^2 - 11t + \frac{71}{6} \end{bmatrix}.$$

Finally, the transition matrix in the time interval $[0,3]$ of the analysed dynamical system with delay has the following form:

$$F(t) = \begin{cases} F_1 & \text{for} \quad t \in [0,1], \\ F_2 & \text{for} \quad t \in [1,2], \\ F_3 & \text{for} \quad t \in [2,3], \end{cases}$$

where

$$F_1 = \begin{bmatrix} 2 & 0 \\ 2t - \frac{1}{2} & 2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 2t-3 & 2 \\ 2t^2 - 4t + \frac{49}{6} & 2t-1 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 4t^2 - 24t + \frac{89}{2} & 2t-6 \\ 4t^3 - 34t^2 + 103t - 110\frac{7}{12} & 2t^2 - 11t + \frac{71}{6} \end{bmatrix}.$$

♦

Example 5. Consider the dynamical system (1) with $n = 2,\ m = 1,\ M = 2,\ h_1 = 1,\ h_2 = 2$ and

$$A_0 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -a & 0 \\ 0 & d \end{bmatrix}, \quad B = \begin{bmatrix} p \\ q \end{bmatrix},$$

where a, b, c, d, p and q are positive numbers. We assume that U is a cone in $\mathbb{R}^m$, having a convex hull with non-empty interior.

The quasi-characteristic equation of the dynamical system has the form

$$s^2 - (a+b)s - (d-a)se^{-2s} \\ - (b-d)ae^{-2s} - ade^{-4s} + ab = 0.$$

Notice that 0 is a real root of this equation. Then, based on Theorem 6, this system is not approximately relatively controllable. ♦

6. Concluding Remarks

In this paper, linear stationary dynamical systems with multiple constant delays in the state of the form (1) have been considered. Their relative and approximate controllability properties with constrained controls were discussed. Definitions of various types of controllability with constrained controls for systems with delays in the state (1) were introduced. Some theorems concerning the relative and the approximate relative controllability with constrained controls for the dynamical system (1) were established. Various types of constraints were considered. The notions of the supporting function (Schmitendorf and Barmish, 1981; Klamka, 1990) and the general permutation matrix (Kaczorek, 2000; 2002) were exploited. The results obtained in this paper constitute an extension of those published in (Chukwu, 1979; Klamka, 1990) to systems with delays in the state and constrained controls. As one of possible practical applications of the theoretical results, an example of a real technical dynamical system was presented.

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Received: 16 February 2003
Revised: 26 June 2003