ADAPTIVE COMPENSATORS FOR PERTURBED POSITIVE REAL INFINITE-DIMENSIONAL SYSTEMS

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The aim of this investigation is to construct an adaptive observer and an adaptive compensator for a class of infinite-dimensional plants having a known exogenous input and a structured perturbation with an unknown constant parameter, such as the case of static output feedback with an unknown gain. The adaptive observer uses the nominal dynamics of the unperturbed plant and an adaptation law based on the Lyapunov redesign method. We obtain conditions on the system to ensure uniform boundedness of the estimator dynamics and the parameter estimates, and the convergence of the estimator error. For the case of a known periodic exogenous input we design an adaptive compensator which forces the system to converge to a unique periodic solution. We illustrate our approach with a delay example and a diffusion example for which we obtain convincing numerical results.

Keywords: infinite dimensional systems, positive real systems, adaptive controllers

1. Introduction

In this paper we construct adaptive observers for the infinite-dimensional linear system with structured perturbations on a complex Hilbert space X

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= (A_0 + B\Gamma C)\,x(t) + Bu(t) + f(t),\\ x(0) &= x_0 \in X,\\ y(t) &= Cx(t),\end{aligned} \tag{1}$$

where A_0 is an infinitesimal generator of an exponentially stable C_0 semigroup $T(t),\ t \geq 0$ on X (Pazy, 1983). The signals $u(t)$ and $y(t)$ are the vector-valued inputs and outputs, respectively, and $f(t)$ is an X-valued known exogenous input. The operators $B \in \mathcal{L}(\mathbb{R}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{R}^m)$ are known, but the gain matrix $\Gamma \in \mathbb{R}^{m\times m}$ is unknown. The structured perturbation term $B\Gamma C$ may represent a passive feedback loop. The gain Γ may depend on other factors such as temperature and age, and consequently it needs to be estimated in real time.

The proposed observer is of the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}(t) &= A_0\widehat{x}(t) + B\widehat{\Gamma}(t)y(t) + Bu(t) + f(t),\\ \widehat{x}(0) &= \widehat{x}_0,\\ \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Gamma}(t) &= G\big(y(t) - C\widehat{x}(t)\big)y^T(t),\\ \widehat{\Gamma}(0) &= \widehat{\Gamma}_0,\end{aligned} \tag{2}$$

where $G = G^T > 0$ is a pre-selected adaptation matrix gain. The objective of the paper is to analyze the stability and convergence properties of the proposed adaptive observer. Simplified versions of this adaptive observer scheme for special classes of systems were studied in earlier papers (Curtain *et al.*, 1997; Demetriou and Ito, 1996; Demetriou *et al.*, 1998). Our main result (Theorem 2) uses a Lyapunov equation of the form

$$\begin{aligned}A^*Qx + QAx &= -L^*Lx, \quad x \in D(A_0),\\ B^*Q &= C,\end{aligned} \tag{3}$$

with $A_0 + \mu I$ replacing A, where μ is a certain positive constant, $Q \in \mathcal{L}(X)$ and L is a bounded operator on $D(A_0)$. That is, if there exist a pair of operators (Q, L) satisfying (3), then the gain estimate $\widehat{\Gamma}(t)$ and the observer error $e(t) = x(t) - \widehat{x}(t)$ are bounded and $e^{\mu t}\|Q^{\frac{1}{2}} e(t)\|_X \to 0$ as $t \to \infty$ provided that $u,\ y \in L^{\infty}_{\ell oc}(0, \infty; \mathbb{R}^m)$. Moreover, under a persistence of excitation condition the parameter convergence $\widehat{\Gamma}(t) \to \Gamma$ as $t \to \infty$ is proved.

The earlier results on the existence of solutions to Lur'e equations in the literature are too restrictive for our application. Balakrishnan (1995) assumes that A is a Riesz spectral operator and the scalar inputs and outputs are very smooth; both Curtain (1996a; 1996b) and Pandolfi (1997) require the exact controllability, which is never satisfied by our class of systems. In the recent results by Curtain (2001) and Pandolfi (1998), the latter assumption is removed. The results in (Curtain, 2001) provide the type of the positive-real lemma suited to our applications.

A key assumption to ensure the existence of μ, L, Q satisfying (3) is that there exists a positive number μ so that $(A_0 + \mu I, B, C)$ satisfies a positive-real condition (Curtain, 2001) as follows with $A = A_0 + \mu I$.

Definition 1. Let A be an infinitesimal generator of an exponentially stable semigroup on X. If the transfer function $G(s) = C(sI - A)^{-1}B : \mathbb{C}_0^+ \to \mathcal{L}(\mathbb{C}^m)$, where $\mathbb{C}_0^+ = \{s \in \mathbb{C} : \mathrm{Re}\ s > 0\}$ satisfies

$$\text{(i)} \qquad \overline{G(s)} = G(\bar{s}), \tag{4}$$

$$\text{(ii)} \qquad G(s) \text{ is holomorphic on } \mathbb{C}_0^+, \tag{5}$$

$$\text{(iii)} \qquad G(s) + G(s)^* \geq 0 \text{ for all } s = \imath\omega,\ \omega \in \mathbb{R}, \tag{6}$$

then G is *positive real*.

Although there are many results on the positive-real lemma in the infinite-dimensional literature (e.g., see (Staffans, 1995; 1997; 1998; 1999; Weiss, 1994; 1997; Weiss and Weiss, 1997), most are in terms of a certain Riccati equation. For our main result we need the singular equation (Lur'e) (3), for which no corresponding Riccati equation exists.

In Section 2, we state and discuss three distinct versions of a positive-real lemma that are in essence existence theorems for a Lur'e equation like (3). Moreover, we collect various sets of verifiable sufficient conditions. We discuss three examples which satisfy at least one version of the positive-real lemma. In Section 3 we prove the main theorem based on a Lyapunov method and a solution to the Lur'e equation, and state the persistence of the excitation condition we use for the gain estimate convergence. In Section 4, we propose an adaptive compensator design using a separation scheme with an LQR design on the resulting adaptive observer. To illustrate the above results we present some numerical results on our three examples in Section 5.

2. Positive-Real Lemmas

The adaptive observer scheme is only applicable to positive-real systems and the key is a positive real lemma. As is well known, it is possible to have different versions corresponding to spectral factors of different dimensions. We have found three useful versions. The first version is particularly useful for dissipative systems with collocated actuators and sensors, and it was utilized in (Demetriou and Ito, 1996). These systems are always positive-real, and the following lemma is trivial.

Lemma 1. *Suppose that A is the infinitesimal generator of a contraction semigroup on X and $B \in \mathcal{L}(\mathbb{R}^m, X)$. Then $Q = I$ is a solution to the constrained Lyapunov equation for $x \in D(A)$*

$$\langle Ax, Qx\rangle + \langle Qx, Ax\rangle \leq 0,$$

$$B^*Q = B^*.$$

In the adaptive observer application, one also needs to suppose that A generates an exponentially stable C_0-semigroup. An example satisfying Lemma 1 is the following.

Example 1. Consider the diffusion equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial \xi^2} + b(\xi)u(t), \quad z(0,t) = 0 = z(1,t),$$

$$z(\xi, 0) = z_0(\xi),$$

$$y(t) = \int_0^1 b(\xi) z(\xi, t)\, \mathrm{d}\xi,$$

where $b \in L^2(0,1) = X$.

We let

$$D(A) = \left\{ \begin{array}{l} h \in L^2(0,1) : h, \dfrac{\mathrm{d}h}{\mathrm{d}\xi} \text{ are absolutely continuous,} \\ \dfrac{\mathrm{d}^2 h}{\mathrm{d}\xi^2} \in L^2(0,1) \text{ and } h(0) = 0 = h(1) \end{array} \right\}$$

and define

$$Ah = \frac{\mathrm{d}^2 h}{\mathrm{d}\xi^2} \quad \text{for} \quad h \in D(A).$$

Then A has compact resolvent, eigenvalues $\lambda_n = -n^2\pi^2$, $n \in \mathbb{N}$ and eigenvectors $\phi_n = \sqrt{2}\sin(n\pi\xi)$,

$n \in \mathbb{N}$, which form an orthonormal basis for $L^2(0,1)$. A is exponentially stable, self-adjoint and for $x \in D(A)$

$$\langle x, Ax\rangle \leq -\|x\|^2,$$

and it generates a contraction semigroup.

Finally, note that $y(t) = \langle b, z(\cdot,t)\rangle = Cz(\cdot,t)$ and $B = C^*$. ♦

To treat systems for which the actuators and sensors were not collocated, the following version was proven in (Curtain *et al.*, 1997).

Lemma 2. *Suppose that A is self-adjoint, has compact resolvent, it generates an exponentially stable semigroup, its eigenvalues $\{\lambda_n, n \in \mathbb{N}\}$ are simple and its eigenvectors $\{\phi_n, n \in \mathbb{N}\}$ form an orthonormal basis for X. Suppose that $b, c \in X$ satisfy*

$$\langle c,\phi_n\rangle\langle b,\phi_n\rangle > 0, \quad n \in \mathbb{N},$$
$$\sup_{n\in\mathbb{N}} \left| \frac{\langle c,\phi_n\rangle}{\langle b,\phi_n\rangle} \right| < \infty.$$

Then there exist operators $0 \leq Q = Q^* \in \mathcal{L}(X)$, $L \in \mathcal{L}(D(A), X)$ and $\mu > 0$ such that for $x, y \in D(A)$

$$\langle (A+\mu I)x, Qy\rangle + \langle Qx, (A+\mu I)y\rangle = -\langle Lx, Ly\rangle, \quad \langle x, c\rangle = \langle x, Qb\rangle. \tag{7}$$

Proof. Show by direct substitution that the following operators satisfy the constrained Lyapunov equation:

$$Qx = \sum_{n=1}^{\infty} \frac{\langle c,\phi_n\rangle}{\langle b,\phi_n\rangle}\langle x,\phi_n\rangle\phi_n, \quad Lx = \sum_{n=1}^{\infty} \left(\frac{-2(\mu+\lambda_n)\langle c,\phi_n\rangle}{\langle b,\phi_n\rangle} \right)^{\frac{1}{2}} \langle x,\phi_n\rangle\phi_n. \tag{8}$$

■

This lemma applies to Example 1, where we can take $\mu = \pi^2 - \epsilon$ for any $\epsilon > 0$.

The following example from (Curtain *et al.*, 1997) does not satisfy the conditions of Lemma 1, but Lemma 2 does apply.

Example 2. Consider the diffusion equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial \xi^2} - \alpha\frac{\partial z}{\partial \xi} + b(\xi)u(t) + f, \quad \alpha > 0,$$
$$z(0,t) = 0, \quad z(1,t) = 0, \quad z(\xi,0) = z_0(\xi),$$

with the output given by

$$y(t) = \int_0^1 e^{-\alpha\xi} z(\xi,t)\,\mathrm{d}\xi,$$

where

$$b(\xi) = \begin{cases} 1 & \text{on } [0,1/2), \\ 0 & \text{elsewhere.} \end{cases}$$

Take $X = L^2(0,1)$ to be the Hilbert space with the weighted inner product

$$\langle f, g\rangle = \int_0^1 e^{-\alpha\xi} f(\xi)g(\xi)\,\mathrm{d}\xi.$$

Then, defining

$$D(A_0) = \left\{ \begin{array}{c} h : h,\ \dfrac{\mathrm{d}h}{\mathrm{d}\xi} \text{ are absolutely continuous} \\ \text{and } \dfrac{\mathrm{d}^2 h}{\mathrm{d}\xi^2} \in X,\ h(0) = 0 = h(1) \end{array} \right\}$$

and

$$A_0 h = \frac{\mathrm{d}^2 h}{\mathrm{d}\xi^2} - \alpha\frac{\mathrm{d}h}{\mathrm{d}\xi} \quad \text{for} \quad h \in D(A_0),$$

it is straightforward to show that A_0 is self-adjoint with eigenvalues $\lambda_n = -\frac{\alpha^2}{4} - n^2\pi^2$ and normalized eigenvectors $\phi_n(\xi) = \sqrt{2}e^{\alpha\xi/2}\sin(n\pi\xi)$, $n \in \mathbb{N}$. The set $\{\phi_n,\ n \in \mathbb{N}\}$ forms an orthonormal basis for X. Let

$$c_n := \langle c,\phi_n\rangle = \frac{4n\pi\sqrt{2}\left(1 - e^{-\frac{\alpha}{2}}(-1)^n\right)}{4n^2\pi^2 + \alpha^2}, \quad n \in \mathbb{N},$$

$$b_n := \langle b,\phi_n\rangle = \frac{4n\pi\sqrt{2}\left(1 + e^{-\frac{\alpha}{4}}\cos(\frac{n\pi}{2}) - \frac{\alpha}{2n\pi}e^{-\frac{\alpha}{4}}\sin(\frac{n\pi}{2})\right)}{4n^2\pi^2 + \alpha^2}.$$

So $b_n c_n > 0$ for all n and for certain constants m and M

$$m \leq \sup_{n\geq 1} \left| \frac{c_n}{b_n} \right| \leq M.$$

So the assumptions of Lemma 2 are satisfied, Q given by (8) satisfies the constrained Lyapunov equation (7) and it is boundedly invertible; L is unbounded.

♦

Note that in both Lemmas 1 and 2 the L term will be unbounded in general, even though B and C are bounded, and that L maps into the state-space X. This is in contrast to the usual finite-dimensional version for which L maps into the output space $\mathbb{R}^m$. The latter version is much harder to prove for infinite-dimensional systems, and earlier versions in (Balakrishnan, 1995; Curtain, 1996a; 1996b; Pandolfi, 1997) assumed very strong conditions on the system, such as exact controllability. Here we extract some useful results from (Curtain, 2001), where only mild conditions are assumed on the system operators (A, B, C).

First we need some extra notation:

$$H_\infty(\mathbb{Z}) = \left\{ \begin{array}{l} f : \mathbb{C}_0^+ \to \mathbb{Z} \text{ and } f \text{ is holomorphic and} \\ \|f\|_\infty = \sup_{\omega\in\mathbb{R}} \|f(\imath\,\omega)\|_X < \infty \end{array} \right\}$$

$$H^2(\mathbb{Z}) = \left\{ \begin{array}{l} f : \mathbb{C}_0^+ \to \mathbb{Z} \text{ and } f \text{ is holomorphic and} \\ \|f\|_2^2 = \sup_{x>0} \int_{-\infty}^{\infty} \|f(x+\imath\,\omega)\|_X^2 \,\mathrm{d}\omega < \infty \end{array} \right\}$$

$$L^2\big((-\imath\,\infty,\imath\,\infty);\mathbb{Z}\big) = \left\{ \begin{array}{l} f : (-\imath\,\infty,\imath\,\infty) \to \mathbb{Z} \text{ and } f \text{ is measurable and} \\ \|f\|_2 = \left(\int_{-\infty}^{\infty} \|f(\imath\,\omega)\|_X^2 \,\mathrm{d}\omega\right)^{\frac{1}{2}} < \infty \end{array} \right\}.$$

Let $\mathbb{Z}$ be a complex Hilbert space. $H^\infty(\mathbb{Z})$ is a Banach space under the sup norm, and $H^2(\mathbb{Z})$ and $L^2((-\imath\,\infty,\imath\,\infty);\mathbb{Z})$ are Hilbert spaces under their $\|\cdot\|_2$ norms. Furthermore, $f \in H^2(\mathbb{Z})$ uniquely defines a function $\widetilde{f} \in L^2((-\imath\,\infty,\imath\,\infty);\mathbb{Z})$ and $\widetilde{f}$ is isomorphic to f with $\|f\|_2 = \|\widetilde{f}\|_2$. f and $\widetilde{f}$ are usually identified with each other and with this identification $H^2(\mathbb{Z})$ is a subspace of $L^2\left((-\imath\,\infty,\imath\,\infty);\mathbb{Z}\right)$. We denote by P_{H^2} the orthogonal projection of $H^2(\mathbb{Z})$ onto $L^2((-\imath\,\infty,\imath\,\infty);\mathbb{Z})$.

The results depend on the *Popov function* Π defined by

$$\Pi(\imath\,\omega) = G(\imath\,\omega) + G(\imath\,\omega)^*. \tag{9}$$

Theorem 1. *Suppose that A is the infinitesimal generator of an exponentially stable C_0-semigroup on X, $B \in \mathcal{L}(U,X)$ and $C \in \mathcal{L}(X,U)$, where U is a separable Hilbert space. Assume that*

(i) there exists an outer function[1] $\Xi \in H^\infty(\mathcal{L}(U))$ *such that*

$$\Pi(\imath\,\omega) = \Xi(\imath\,\omega)^*\Xi(\imath\,\omega) \text{ for almost all } \omega \in \mathbb{R}, \tag{10}$$

(ii) (A,B) is approximately controllable,

(iii) there exists a $C_\Xi \in \mathcal{L}(D(A),U)$ such that for all $z \in D(A)$ and some $\gamma > 0$

$$\int_0^\infty \left\|C_\Xi T(t) z\right\| \mathrm{d}t \le \gamma \|z\|^2 \tag{11}$$

and

$$\Xi(s) = C_\Xi (sI - A)^{-1} B, \tag{12}$$

for all s in some right half-plane.

[1] $\Xi \in H^\infty(\mathcal{L}(U))$ is outer if its range as a multiplication operator on $H^2(U)$ is dense in $H^2(U)$.

Then there exists a $Q = Q^* \in \mathcal{L}(X)$ which satisfies, for all $x \in D(A)$, the following Lur'e equations:

$$A^*Qx + QAx = -C_\Xi^* C_\Xi x \tag{13}$$

$$B^*Qx = Cx. \tag{14}$$

Proof. This follows from Theorem 3.2 in (Curtain, 2001). The existence of the map $\Psi_\Xi \in \mathcal{L}(X, L^2(0,\infty;U))$ referred to in (i) of Theorem 3.2 is shown in Theorem 5.1 of (Curtain, 2001), where we note that for our Popov function formula (5.1) simplifies to the expression (12). We also use the fact that B is bounded, which also ensures that the spectral factor will be regular as required in (iii) of Theorem 3.2. In fact, Ψ_Ξ is defined for $x \in D(A)$ by

$$(\Psi_\Xi x)(t) = C_\Xi T(t) x.$$

The infinite-time admissibility assumption (11) ensures that Ψ_Ξ extends to a bounded map from X to $L^2(0,\infty;U)$. Moreover, $Q = \Psi_\Xi^*\Psi_\Xi$. ■

Sufficient conditions for (10) to hold are

$$\begin{gathered} \Pi(\imath\,\omega) \ge 0 \quad \text{and} \\ \Pi(\imath\,\omega) \text{ is invertible for almost all } \omega \in \mathbb{R}, \end{gathered} \tag{15}$$

and

$$\int_{-\infty}^{\infty} \frac{\log^+ \left\|\Pi^{-1}(\imath\,\omega)\right\|_{\mathcal{L}(U)}}{1+\omega^2} \,\mathrm{d}\omega < \infty, \tag{16}$$

where $\log^+ \alpha = \max(\log\alpha, 0)$.

(11) and (12) are often difficult to verify, so the following result from Proposition 5.5 of (Curtain, 2001) will be useful.

Lemma 3. *Suppose that A, B, C are as in Theorem 1 and*

(i) conditions (15) and (16) hold,

(ii) A has eigenvalues $\{\lambda_n \,|\, n \in \mathbb{N}\}$ and the corresponding eigenvectors, ϕ_n, are such that the span of $\{\phi_n, n \in \mathbb{N}\}$ is dense in X,

(iii) $\Xi(-\overline{\lambda}_n)$ is invertible in $\mathcal{L}(U)$ for all $n \in \mathbb{N}$,

(iv) C_Ξ satisfies (11), where C_Ξ is given by

$$C_\Xi \phi_n = \left(\Xi(-\overline{\lambda}_n)^*\right)^{-1} C\phi_n. \tag{17}$$

Then, there exists a $Q = Q^* \in \mathcal{L}(X)$ satisfying (13) and (14).

We note that some easily verifiable conditions for C_Ξ to satisfy (11) are given in (Hansen and Weiss, 1997) and cited in Lemma 5.6 in (Curtain, 2001).

Lemma 4. (Curtain, 2001, Lem. 5.6) *Suppose that A, the generator of $T(t)$, is a diagonal matrix on $X = \ell^2$ with eigenvalue eigenvector pairs $\{\lambda_n, \phi_n \,|\, n \in \mathbb{N}\}$ satisfying the following conditions:*

(i) $\operatorname{Re} \lambda_n < 0$ *for* $n \in \mathbb{N}$,

(ii) *either* $T(t)$ *is analytic or there exist numbers* $\alpha \geq 0$ *and* $0 < a \leq b$ *such that*

$$a|\operatorname{Im}\,\lambda_n|^\alpha \leq -\operatorname{Re}\,\lambda_n \leq b|\operatorname{Im}\,\lambda_n|^\alpha. \tag{18}$$

Then $T(t)$ is exponentially stable and $C \in \mathcal{L}(D(A), \ell^2)$ satisfies (11) if and only if there exists $m \geq 0$ such that

$$\Big\| \sum_{-\lambda_n \in R(h,\omega)} (C\phi_n)(C\phi_n)^* \Big\|_{\mathcal{L}(\ell^2)} \leq Mh,$$

where $R(h,\omega) = \{z \in \mathbb{C} \,|\, 0 < \operatorname{Re}\, z \leq h,\ \omega - h \leq \operatorname{Im}\, z < \omega + h\}$.

Finally, in Lemma 5.7 of (Curtain, 2001), various bounds on the spectral factor $\Xi(s)$ are derived which help to verify that $\Xi(-\overline{\lambda_n})$ is invertible for the case of single-input single-output systems. Several examples of SISO parabolic systems are analyzed in Section 6 of that paper, including some with boundary control and point sensing. Applying the approach of Section 6 to our Examples 1 and 2 we see that the transfer functions both have the form

$$g(s) = \sum_n \frac{\langle c, \phi_n\rangle \langle b, \phi_n\rangle}{s + \lambda_n},$$

where $b_n = \langle b, \phi_n\rangle$ and $c_n = \langle c, \phi_n\rangle$ are both of order $1/n$ for large n and $\lambda_n \sim -n^2$ in both examples. Consequently, the analysis and the conclusions are the same as in Example 6.1 of (Curtain, 2001): $\Pi(\imath\omega) \sim m/|\omega|^{3/2}$ for some $m > 0$ and sufficiently large ω, and Π has an outer spectral factor Ξ as in (10) which satisfies

$$\left|\Xi(n^2)\right| \geq \frac{\gamma_1}{n^{3/2}}$$

for some $\gamma_1 > 0$ and sufficiently large n. So C_Ξ is well defined by (17) and it satisfies (11). It is interesting to note that C_Ξ is unbounded, $|C_\Xi \phi_n| \sim \sqrt{n}$. Here we have satisfied all the conditions of Lemma 3 and there exist Q and $C_\Xi \in \mathcal{L}(D(A), \mathbb{C})$ satisfying the constrained Lyapunov equations (13) and (14).

As has already been noted, Lemmas 1 and 2 only apply to special classes of SISO systems. Lemma 3 applies to a much wider class of partial differential equations, but it is not applicable to delay systems. In the following example we show how Theorem 1 can be applied to a delay system.

Example 3. Consider the delay system

$$\dot{x}(t) = -ax(t) - bx(t-1) + u(t), \quad a, b > 0, \tag{19}$$

$$y(t) = x(t) \tag{20}$$

with the transfer function

$$g(s) = \frac{1}{s + a + be^{-s}} \in H^\infty.$$

Now

$$\begin{aligned}\Pi(\imath\omega) &= g(\imath\omega) + g(\imath\omega)^* \\ &= \frac{2(a + b\cos(\omega))}{(a + b\cos(\omega))^2 + (\omega - b\sin(\omega))^2\}} \\ &\geq 0 \quad \text{if } a \geq |b|\,.\end{aligned}$$

In this case, it is easy to find the spectral factor $\Xi \in H^\infty$ given by

$$\Xi(s) = \frac{\alpha + \beta e^{-s}}{s + a + be^{-s}}, \quad \alpha^2 + \beta^2 = 2a, \quad \alpha\beta = b. \tag{21}$$

The delay system (19), (20) can be formulated on the state-space $X = \mathbb{C} \oplus L^2(-1, 0)$ with generating operators defined by

$$Bu = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad C\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = r, \tag{22}$$

$$D(A) = \left\{ \begin{aligned} &\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in X \ \mid f \text{ is absolutely continuous,} \\ &\frac{\mathrm{d}f}{\mathrm{d}\theta}(\cdot) \in L^2(-1,0) \text{ and } f(0) = r \end{aligned} \right\},$$

$$A\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} -ar - bf(-1) \\ \dfrac{\mathrm{d}f}{\mathrm{d}\theta} \end{pmatrix} \tag{23}$$

(see Curtain and Zwart, 1995, Ch. 2.4). Clearly, B and C are bounded operators and we recall from (Infante and Walker, 1977) that A generates an exponentially stable semigroup if $a - |b| \geq \mu > 0$ for some positive constant μ. (A, B) is approximately controllable (see Curtain and Zwart, 1995, Thm. 4.2.10).

The candidate for C_Ξ is

$$(C_\Xi x)(t) = \alpha x(t) + \beta x(t-1). \tag{24}$$

C_Ξ is not bounded, but it does satisfy (11) (see (Salamon, 1984), and note that $T(t)$ is exponentially stable). We verify that (12) holds:

$$\Xi(s) = C_\Xi(sI - A)^{-1}B.$$

The resolvent is now given by (Curtain and Zwart, 1995, Lem. 2.4.5)

$$(sI - A)^{-1}\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} q(0) \\ q(\cdot) \end{pmatrix}, \tag{25}$$

where

$$q(\theta) = e^{s\theta}q(0) - \int_0^\theta e^{s(\theta-\mu)} f(\mu)\,\mathrm{d}\mu$$

$$q(0) = \frac{1}{\Delta(s)}\left(r - b\int_{-1}^0 e^{-s(\mu+1)} f(\mu)\,\mathrm{d}\mu\right),$$

$$\Delta(s) = s + a + be^{-s}.$$

So

$$C_\Xi(sI - A)^{-1}\begin{pmatrix} r \\ f(\cdot)\end{pmatrix} = C_\Xi\begin{pmatrix} q(0) \\ q(\cdot)\end{pmatrix}$$

$$= \Big(\alpha q(0) + \beta q(-1)\Big) = \frac{\alpha+\beta e^{-s}}{\Delta(s)} r$$

$$+ \left(1 - \frac{b(\alpha+\beta e^{-s})}{\Delta(s)}\right) e^{-s}\int_{-1}^0 e^{-s\theta} f(\theta)\,\mathrm{d}\theta,$$

and

$$C_\Xi(sI - A)^{-1}B = \frac{\alpha+\beta e^{-s}}{s+a+be^{-s}} = \Xi(s),$$

as required. In fact, it is readily verified that the solution to the Lur'e equation (13), (14) is

$$Q = \begin{pmatrix} I & 0 \\ 0 & \beta^2 I\end{pmatrix}.$$

For the general retarded system with vector-valued inputs and outputs, see Section 7 in (Curtain, 2001).

3. An Adaptive Observer: Main Results

The proposed state estimator is

$$\dot{\widehat{x}}(t) = A_0\widehat{x}(t) + Bu(t) + B\widehat{\Gamma}(t)y(t) + f(t),$$
$$\widehat{x}(0) = \widehat{x}_0, \tag{26}$$

where $\widehat{x}(t)$ is the state estimate at time t and $\widehat{\Gamma}(t)$ is the adaptive estimate of the unknown gain. In order to extract the adaptation rule for $\widehat{\Gamma}(t)$, we use the Lyapunov redesign method (Khalil, 1992; Narendra and Annaswamy, 1989), which has proved successful for finite-dimensional systems. In this section, we show that the same adaptive observer that was proposed in (Curtain *et al.*, 1997) for scalar (SISO) systems can be extended to the larger class of multivariable (MIMO) systems considered in this paper.

Let X_{-1} be the completion of X under the norm $\|\phi\|_{-1} = \|A_0^{-1}\phi\|_X$. Then X_{-1} is a Hilbert space and

$$D(A_0^*) \subset X \subset X_{-1}.$$

Theorem 2. *Consider the structurally perturbed system (1), where A_0 is the generator of an exponentially stable C_0 semigroup on X, $B \in \mathcal{L}(\mathbb{R}^m, X)$, $C \in \mathcal{L}(X, \mathbb{R}^m)$, $f(t)$ is a known exogenous signal which is locally Bochner integrable, and Γ is an unknown matrix feedback gain. If there exist a positive constant μ, $Q \in \mathcal{L}(X)$ and $L \in \mathcal{L}(D(A), X)$ or $\mathcal{L}(D(A), \mathbb{R}^m)$ satisfying the constrained Lyapunov equation for $x \in D(A)$*

$$(A_0 + \mu I)^* Qx + Q(A_0 + \mu I)x = -L^*Lx, \tag{27}$$

$$B^*Qx = Cx, \tag{28}$$

then the state estimator defined by (26) and the adaptation rule with adaptation matrix gain $G = G^T > 0$ given by

$$\dot{\widehat{\Gamma}}(t) = GCe(t)y^T(t),$$
$$\widehat{\Gamma}(0) = \widehat{\Gamma}_0 \tag{29}$$

have the following properties:

(i) If $u, y \in L^\infty_{\ell oc}(0,\infty;\mathbb{R}^m)$, then the quantities $\widehat{\Gamma}(t)$ and $Q^{\frac12}e(t)$ are bounded in norm for $t \ge 0$ and $e^{\frac{\mu}{2}t}\|Q^{\frac12}e(t)\|_X \to 0$ as $t \to \infty$.

(ii) Moreover, if $y \in L^2(0,\infty;\mathbb{R}^m)$, then $\|e(t)\|_X \to 0$ as $t \to \infty$.

(iii) If we assume that $y \in L^\infty(0,\infty,\mathbb{R}^m)$, then the estimation error $e(t) = x(t) - \widehat{x}(t)$ is bounded in norm for $t \ge 0$. Moreover, if Q is coercive, and the plant is persistently exciting, i.e., there exist T_0, δ_0 and ϵ_0 such that for each admissible gain $q \in \mathbb{R}^{m\times m}$ with (Euclidean) norm equal to 1 and each sufficiently large $t > 0$, there exits $\bar t \in [t, t+T_0]$ such that

$$\Big\|\int_{\bar t}^{\bar t+\delta_0} B\,q\,y(\tau)\,\mathrm{d}\tau\Big\|_{-1} \ge \epsilon_0,$$

then the parameter convergence

$$\widehat{\Gamma}(t) \to \Gamma \quad \text{as} \quad t \to \infty$$

holds.

Proof. (i) Consider the dynamics of the state error

$$\dot e(t) = A_0e(t) + B\Gamma y(t) - B\widehat{\Gamma}(t)y(t)$$
$$= A_0 e(t) + B\widetilde{\Gamma}(t)y(t), \tag{30}$$

$$e(0) = x_0 - \widehat{x}_0 = e_0. \tag{31}$$

The dynamics for the parameter error $\widetilde{\Gamma}(t) = \Gamma - \widehat{\Gamma}(t)$ become

$$\dot{\widetilde{\Gamma}}(t) = -GCe(t)y^T(t),$$
$$\widetilde{\Gamma}(0) = \Gamma - \widehat{\Gamma}_0 = \widetilde{\Gamma}_0. \tag{32}$$

First we need to examine the well-posedness of the coupled system (30), (32), which is, in fact, a linear time-dependent system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} e(t) \\ \widetilde{\Gamma}(t)\end{bmatrix} = \begin{bmatrix} A_0 & B[\,\cdot\,]y(t) \\ -GC[\,\cdot\,]y^T(t) & 0 \end{bmatrix}\begin{bmatrix} e(t) \\ \widetilde{\Gamma}(t)\end{bmatrix}. \tag{33}$$

The perturbation term $D(t) : X \oplus \mathbb{R}^{m\times m} \to X \oplus \mathbb{R}^{m\times m}$ is given by

$$D(t) = \begin{bmatrix} 0 & B[\,\cdot\,]y(t) \\ -GC[\,\cdot\,]y^T(t) & 0 \end{bmatrix}. \tag{34}$$

So, if $y \in L^{\infty}_{\ell oc}(0, t_1; \mathbb{R}^m)$, (33) has a unique solution given by

$$\begin{bmatrix} e(t) \\ \widetilde{\Gamma}(t)\end{bmatrix} = U(t,0)\begin{bmatrix} e_0 \\ \widetilde{\Gamma}_0\end{bmatrix}, \tag{35}$$

where $U(t,s)$ is a mild evolution operator (Curtain and Pritchard, 1978) defined for $0 \le s \le t \le t_1$. In fact, $y(t)$ defined by (1) will always be in $C(0, t_1; \mathbb{R}^m)$ for $u, f \in L_p(0, t_1; \mathbb{R}^m)$, $p = 1, 2$ or ∞. In general, $e(t)$ will not be in $D(A_0)$, even if $e_0 \in D(A_0)$. Sufficient conditions for $e(t) \in D(A_0)$ are that $y(\cdot) \in C^1(0, t_1; \mathbb{R}^m)$ and $e_0 \in D(A_0)$, which are very strong. However, we assume this initially to facilitate the Lyapunov argument. We examine the asymptotic properties of (35) using the following Lyapunov functional for $\binom{e}{\Gamma}$

$$V(e, \widetilde{\Gamma}) = \langle e, Qe\rangle + \mathrm{Tr}\left\{\widetilde{\Gamma}^T G^{-1}\widetilde{\Gamma}\right\}, \tag{36}$$

where Q is the solution to (27). Since $e(t) \in D(A_0)$, we may differentiate V along solutions of (33) for $0 \le t \le t_1$ to obtain

$$\begin{aligned} \dot V(e, \widetilde{\Gamma}) &= \langle A_0 e + B\widetilde{\Gamma}y, Qe\rangle + \langle Qe, A_0 e + B\widetilde{\Gamma}y\rangle \\ &\quad + 2\mathrm{Tr}\left\{\dot{\widetilde{\Gamma}}^T G^{-1}\widetilde{\Gamma}\right\} \\ &= -\|Le\|^2 - 2\mu\langle e, Qe\rangle - 2(Ce)^T\widetilde{\Gamma}y \\ &\quad + 2\mathrm{Tr}\left\{y(Ce)^T\widetilde{\Gamma}\right\} \\ &\quad \text{using (27), (28) and (32)} \\ &= -\|Le\|^2 - 2\mu\langle e, Qe\rangle \text{ using } b^T a = \mathrm{Tr}(ab^T). \end{aligned} \tag{37}$$

We now integrate (37) from $t = 0$ to $t = t_1$ to obtain

$$\begin{aligned} &\langle e(t_1), Qe(t_1)\rangle + \mathrm{Tr}\left\{\widetilde{\Gamma}^T(t_1)G^{-1}\widetilde{\Gamma}(t_1)\right\} \\ &\quad + \int_0^{t_1}\|Le(t)\|^2\,\mathrm{d}t + 2\mu\int_0^{t_1}\langle Qe(t), e(t)\rangle\,\mathrm{d}t \\ &\qquad = \langle e_0, Qe_0\rangle + \mathrm{Tr}\left\{\widetilde{\Gamma}_0^T G^{-1}\widetilde{\Gamma}_0\right\}. \end{aligned} \tag{38}$$

Notice that although we have assumed that $e_0 \in D(A_0)$ and $y \in C^1(0, t_1; \mathbb{R}^m)$ to derive (38), all terms make perfectly good sense for $e_0 \in X$ and $y \in C(0, t_1; \mathbb{R}^m)$. Moreover, (35) and the facts that $\sup_{0\le s\le t\le t_1}\|U(t,s)\| < \infty$ and that $D(A_0)$ is dense in X show that (38) can be extended to all $e_0 \in X$. We now extend (38) to all $y \in L^{\infty}_{\ell oc}(0, t; \mathbb{R}^m)$ by appealing to Lemma A1 in Appendix, which shows that if we approximate y by a sequence $y_n \in C^1(0, t_1; \mathbb{R}^m)$ satisfying

$$\int_0^{t_1}\left\|y(s) - y_n(s)\right\|^2\mathrm{d}s \to 0 \;\text{ as }\; n \to \infty,$$

then there holds

$$\sup_{0\le s\le t\le t_1}\left\|U(t,s) - U_n(t,s)\right\| \to 0 \;\text{ as }\; n \to \infty. \tag{39}$$

So the respective solutions to (33) satisfy

$$\sup_{0\le s\le t\le t_1}\left(\left\|e(t) - e_n(t)\right\|_X + \left\|\widetilde{\Gamma}(t) - \widetilde{\Gamma}_n(t)\right\|\right) \to 0 \quad \text{as } n \to \infty \tag{40}$$

and this suffices to show that (38) holds for any $y \in L^{\infty}_{\ell oc}(0, t_1; \mathbb{R}^m)$ and $e_0 \in X$. This implies that $\widetilde{\Gamma} \in L^{\infty}(0, \infty; \mathbb{R}^{m\times m})$ and $Q^{\frac12}e \in L^{\infty}(0, \infty; X)$.

Next, we define $q(t) := \|Q^{\frac12}e(t)\|^2$ and deduce the following from (38):

$$q(t_1) + 2\mu\int_0^{t_1} q(s)\,\mathrm{d}s \le q(0) + \mathrm{Tr}\left\{\widetilde{\Gamma}_0^T G^{-1}\widetilde{\Gamma}_0\right\} = V(0). \tag{41}$$

Equation (41) and the Bellman-Gronwall Lemma imply that $q(t_1) \le e^{-2\mu t_1}V(0)$ or, equivalently,

$$\left\|Q^{\frac12}e(t_1)\right\|^2 \le e^{-2\mu t_1}V(0). \tag{42}$$

Now t_1 can be chosen arbitrarily large and so $\|e^{\frac{\mu}{2}t}Q^{\frac12}e(t)\| \to 0$ as $t \to \infty$.

(ii) If $y \in L^2(0, \infty; \mathbb{R}^m)$, then it follows from (i) that the forcing term $B\widetilde{\Gamma}(t)y(t)$ in (30) is in $L^2(0, \infty; X)$. Note that for all $\tau \in [0, t]$

$$\begin{aligned} e(t) &= T(t-\tau)\Big[T(\tau)x(0) + \int_0^{\tau} T(\tau - s)B\widetilde{\Gamma}(s)y(s)\,\mathrm{d}s\Big] \\ &\quad + \int_{\tau}^{t} T(t-s)B\widetilde{\Gamma}(s)y(s)\,\mathrm{d}s. \end{aligned}$$

Since $T(t)$ is exponentially stable and B has finite rank, this implies that $\|e(t)\|_X \to 0$ as $t \to \infty$. This follows from the asymptotic property of the convolution of two $L^2(0, \infty)$ functions (see Titchmarsh, 1962).

(iii) Since $\widetilde{\Gamma}(t)$ and $y(t)$ are uniformly bounded in norm for $t \ge 0$ and A_0 generates an exponentially stable semigroup, (30) shows that $e(t)$ is uniformly bounded in norm for $t \geq 0$.

The parameter convergence is proven by applying the results in Section 3 of (Baumeister *et al.*, 1997). Our persistent excitation condition coincides with the one in Definition 3.3 of (Baumeister *et al.*, 1997). The results in (Baumeister *et al.*, 1997) are based on the fact that

$$\left\|e(\bar{t}+\delta_0)\right\|_{-1} \geq \left\| \int_{\bar{t}}^{\bar{t}+\delta_0} B\widetilde{\Gamma}(\tau)y(\tau)\,\mathrm{d}\tau \right\|_{-1} - \left\|e(\bar{t})\right\|_{-1} - \left\| \int_{\bar{t}}^{\bar{t}+\delta_0} A_0 e(\tau)\,\mathrm{d}\tau \right\|_{-1}$$

and that for $0 \leq t_1 \leq \tau$

$$\begin{aligned} \left|\widetilde{\Gamma}(t_1) - \widetilde{\Gamma}(\tau)\right| &= \left| \int_{t_1}^{\tau} GCe(t)y^T(t)\,\mathrm{d}t \right| \\ &\leq |G|\,\|C\|\,|y|_\infty\,|\tau - t_1|^{\frac{1}{2}} \sqrt{\int_{t_1}^{\tau} \|e(t)\|_X^2\,\mathrm{d}t}. \end{aligned}$$

Our assertion then simply follows from the corresponding results to Lemmas 3.5–3.6 and Theorem 3.4 in (Baumeister *et al.*, 1997). ■

Remark 1. The assumption $y \in L^2(0,\infty)$ in (ii) can be verified when $A_0 + B\Gamma C$ generates an exponentially stable C_0 semigroup on X and $u,\ f \in L^2(0,\infty)$. The assumption $y \in L^\infty(0,\infty)$ in (iii) can be verified when $A_0 + B\Gamma C$ generates an exponentially stable C_0 semigroup on X and $u,\ f \in L^\infty(0,\infty)$.

4. Adaptive Compensators

In this section, we propose an adaptive compensator for the perturbed plant (1) where $f(t)$ is a known exogenous signal. We obtain results for $f(t)$ a periodic signal and for $f \in L^2(0,\infty;X)$. First we apply output injection to obtain a modified control problem:

$$u(t) = u_2 - \widehat{\Gamma}(t)y(t). \tag{43}$$

This has the advantage of producing the new estimator dynamics

$$\begin{aligned} \dot{\widehat{x}}(t) &= A_0\widehat{x}(t) + Bu_2(t) + f(t), \\ \widehat{x}(0) &= \widehat{x}_0 \end{aligned} \tag{44}$$

and the same error dynamics (33) for $\binom{e(t)}{\widehat{\Gamma}(t)}$ as before.

So it remains to design a controller $u_2(t)$ for the system (44).

We use the LQR control design from (Prato and Ichikawa, 1988). Suppose that (A_0, B, C_2) is exponentially stabilizable and detectable and $0 < R = R^T \in \mathcal{L}(\mathbb{C}^m)$. We seek to minimize the average cost

$$J(u) = \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \left(\|C_2\widehat{x}(t)\|^2 + \|R^{-\frac{1}{2}}u(t)\|^2 \right) \mathrm{d}t \tag{45}$$

over all controls satisfying $\overline{\lim}_{T\to\infty} \frac{1}{T}\int_0^T \|u(t)\|^2\,\mathrm{d}t < \infty$ and for which the corresponding closed loop trajectory is bounded on $t \geq 0$. They showed that if $f(t)$ is periodic, the minimizing control law is given by

$$u_2(t) = -R^{-1}B^*\Big(P\widehat{x}(t) + r(t)\Big), \tag{46}$$

where $P = P^* \in \mathcal{L}(X)$ is the solution to the Riccati equation for $x \in D(A_0)$

$$A_0^*Px + PA_0x - PBR^{-1}B^*Px + C_2^*C_2x = 0 \tag{47}$$

and $r(t)$ is the solution to

$$\begin{aligned} \dot{r}(t) &= \left(A_0^* - PBR^{-1}B^*\right) r(t) - Pf(t), \\ & r(t) \to 0 \text{ as } t \to \infty. \end{aligned} \tag{48}$$

Equation (48) has the solution

$$r(t) = \int_t^\infty T_P^*(s-t)Pf(s)\,\mathrm{d}s, \tag{49}$$

where $T_P(t)$ is the exponentially stable C_0-semigroup generated by $A - BR^{-1}B^*P$.

The closed loop trajectory converges exponentially fast to the periodic solution

$$p(t) = \int_{-\infty}^t T_P(t-s)\Big(f(s) - BR^{-1}B^*r(s)\Big)\,\mathrm{d}s, \tag{50}$$

i.e.,

$$\lim_{t\to\infty} e^{\nu t}\|\widehat{x}(t,t_0) - p(t)\|_X = 0, \tag{51}$$

where ν is the decay rate of $T_P(t)$.

In the case of a constant exogenous signal, i.e., $f(t) = f_0$, we get

$$p(t) = -\Big(A_0 - BR^{-1}B^*P\Big)^{-1} A_0^* \Big(A_0 - BR^{-1}B^*P\Big) f_0.$$

We note that for the case when $f \in L^2(0,\infty;X)$, the feedback control law

$$u_2(t) = -R^{-1}B^*P\widehat{x}(t)$$

ensures that $\|x(t)\| \to 0$ as $t \to \infty$ as argued in Theorem 3.1, (ii), see also Lemma 12 in (Oostveen and Curtain, 1998).

We propose the following adaptive compensator for the case of a known periodic exogenous input:

$$\dot{\widehat{x}}(t) = \left(A_0 - BR^{-1}B^*P\right)\widehat{x}(t) - BR^{-1}B^*r(t) + f(t),$$
$$\widehat{x}(0) = \widehat{x}_0,$$
$$\dot{r}(t) = \left(A_0^* - PBR^{-1}B^*\right)r(t) - Pf(t),$$
$$r(t) \to 0 \text{ as } t \to \infty,$$
$$u(t) = -R^{-1}B^*\left(P\widehat{x}(t) + r(t)\right) - \widehat{\Gamma}(t)y(t),$$
$$\dot{\widehat{\Gamma}}(t) = GCe(t)y^T(t), \quad \widehat{\Gamma}(0) = \widehat{\Gamma}_0$$

for our structurally perturbed plant (1).

In Section 3 we showed that

$$e^{\frac{\mu}{2}t}\left\|Q^{\frac{1}{2}}\left(x(t) - \widehat{x}(t)\right)\right\|_X \to 0 \text{ as } t \to \infty,$$

independently of the choice of the control. So combining this with the results in this section, we conclude that for the case of a known periodic input $f(t)$

$$e^{\beta t}\left\|Q^{\frac{1}{2}}\left(p(t) - \widehat{x}(t)\right)\right\|_X \to 0 \text{ as } t \to \infty,$$

where $\beta = \min(\nu, \mu/2)$ and $p(t)$ is given by (50).

5. Examples and Numerical Results

We present some numerical results for the three examples considered in Section 2. For each of these examples, there exists a solution $Q \in \mathcal{L}(X)$ satisfying (27) for a certain $\mu > 0$, and in all three cases Q is invertible. Consequently, we can conclude that for the adaptive observer and adaptation rule (2)

$$e^{\mu t}\left\|\left(x(t) - \widehat{x}(t)\right)\right\|_X \to 0$$

and with the adaptive compensator of Section 4

$$\left\|e^{\beta t}\left(x(t) - p(t)\right)\right\|_X \to 0.$$

All the computations described below were carried out on a Digital Personal Workstation *433 au-Series* in the Mechanical Engineering Department at Worcester Polytechnic Institute. A finite element Galerkin approximation scheme based on spline elements was used for the spatial discretization of the two PDEs similar to the one developed in (Baumeister *et al.*, 1997). The resulting finite dimensional ODE systems were integrated in time using a Fehlberg fourth-fifth Runge-Kutta method. The delay system in Example 3 was discretized using the method presented in the paper by Ito and Kappel (1991). The resulting evolution (finite dimensional) system was similarly integrated using the Runge-Kutta code `rkf45.f`.

Example 4. As was already mentioned in Section 2, we can choose in this case $\mu = \pi^2 - \epsilon$ in (7), and define the operators Q and L via (8). Alternatively, when Lemma 1 is used, we have $Q = I$ and $L = 0$ with the same μ. The input operator $b(x)$ was chosen as

$$b(\xi) = \begin{cases} 1 & \text{on } [0, 1/2), \\ 0 & \text{elsewhere.} \end{cases}$$

The unknown gain was chosen as $\Gamma = 1$, and as initial conditions we chose $z(\xi, 0) = \sin(\pi\xi)$ and $\widehat{z}(\xi, 0) = \cos(2\pi\xi) - 1$. The exogenous input was $f(\xi, t) = 50\chi_{[0,1]}(\xi)\sin(2\pi t)$. The initial guess for $\widehat{\Gamma}(0) = 0$ with an adaptive gain of $G = 20$. Figure 1(a) depicts the time evolution of the output state error $Ce(t) = \int_0^1 (z(\xi, t) - \widehat{z}(\xi, t))\,\mathrm{d}\xi$. The convergence to zero is achieved within 0.5 seconds. The parameter estimate $\widehat{\Gamma}(t)$ (dashed) and the actual value of $\Gamma = 1$ are depicted in Fig. 1(b). Parameter convergence is achieved in 4 seconds. ♦

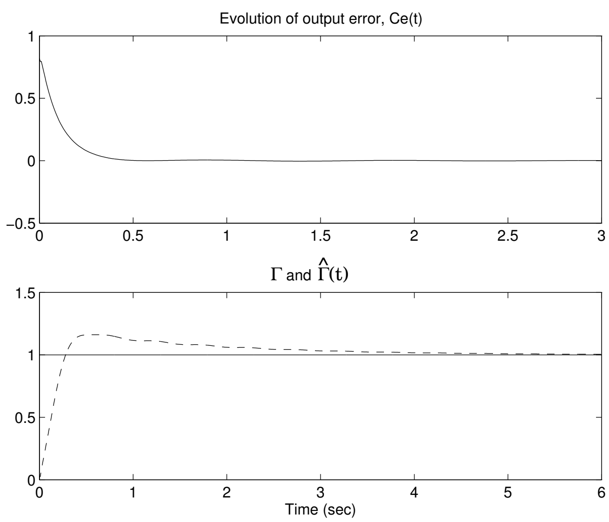

Fig. 1. Evolution of (a) output error and (b) parameter estimate $\widehat{\Gamma}(t)$ (dashed) – actual parameter Γ (solid).

Example 5. Equations (7) and (8) can be satisfied with $\mu = \frac{\alpha^2}{4}\pi^2 - \epsilon$, where the parameter $\alpha = 0.2$. Initial conditions were set as $z(\xi, 0) = \sin(\pi\xi)$ and $\widehat{z}(\xi, 0) = -0.25\sin(\pi\xi)$. A constant in space and time exogenous function is implemented as $f(\xi, t) = 50\chi_{[0,1]}(\xi)$ and $\widehat{\Gamma}(0) = 0$ with $G = 2$. With these values of initial conditions, it is observed in Fig. 2(a) that the output state error converges to zero in 3.5 seconds. Furthermore, the parameter $\widehat{\Gamma}(t)$ converges to the actual value $\Gamma = 1$ in 4 seconds as shown in Fig. 2(b).

Example 6. The plant parameters were chosen as $a = 3$, $b = 1$. In this case the solution to the constrained Lya-

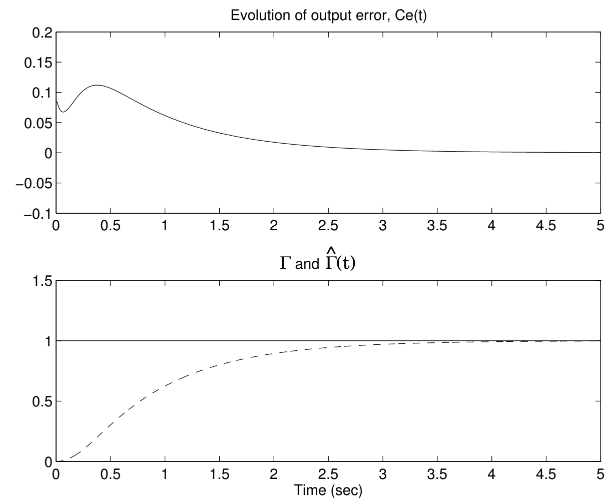

Fig. 2. Evolution of (a) output error and (b) parameter estimate $\widehat{\Gamma}(t)$ (dashed) – actual parameter Γ (solid).

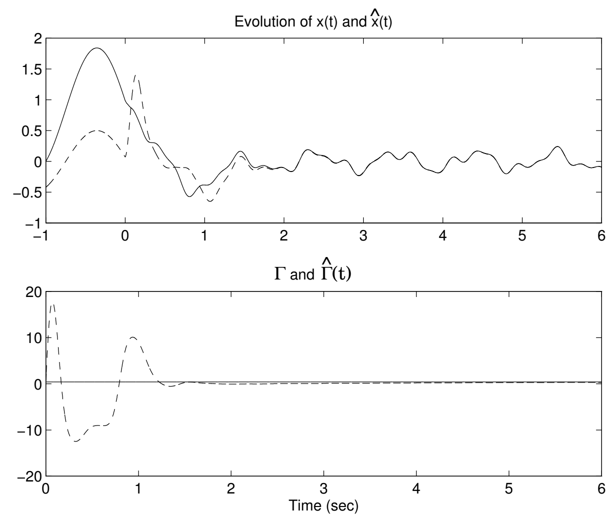

Fig. 3. Evolution of (a) plant state $x(t)$ (solid) and state estimate $\widehat{x}(t)$ (dashed); (b) parameter estimate $\widehat{\Gamma}(t)$ (dashed) – actual parameter Γ (solid).

punov equations (13), (14) is

$$Q = \begin{bmatrix} I & 0 \\ 0 & (3 \pm \sqrt{8})I \end{bmatrix},$$

which is boundedly invertible.

The actual value of the parameter was $\Gamma = 0.4$ with the initial condition for its estimate chosen as $\widehat{\Gamma}(0) = 0.2$. The initial state was set at $x(t-1) = \sin(4t-1) - \sin(-1)$ and the state estimate as $\widehat{x}(t-1) = 0.5\sin(4t-1)$; thus $x(0) = \sin(3) + \sin(1)$ and $\widehat{x}(0) = 0.5\sin(3)$. Here we had $f(t) = 0$ for the exogenous signal and chose an adaptive gain of $G = 500$. It is observed from Fig. 3(a) that the state estimate converges to the plant state in about 2 seconds. Parameter convergence is also achieved in about 5 seconds. For numerical results for a multivariable example the reader is directed to (Demetriou *et al.*, 1998).

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Appendix

Lemma A1. *Suppose that A generates a C_0-semigroup on the Hilbert space X and consider the mild evolution operator $U(t,s)$ generated by $A+\sum_{i=1}^k D_i y_i(t)$, $D_i \in \mathcal{L}(X)$ and $y_i \in L_\infty(0,t_1)$. Let $U_n(t,s)$ be the evolution operator generated by $A+\sum_i^k D_i y_i^n(t)$, where for each i $y_i^n(t)$ is a sequence of functions in $C^1(0,t_1)$ satisfying*

$$\int_0^{t_1} \left\|y_i^n(t)-y_i(t)\right\|^2 \mathrm{d}t \to 0 \quad \textit{as } n\to\infty.$$

There holds

$$\sup_{0\le s\le t\le t_1} \left\|U(t,s)-U_n(t,s)\right\|_{\mathcal{L}(X)} \to \infty \quad \textit{as } n\to\infty.$$

Proof. We only give a detailed proof for $k=1$, since the arguments extend readily to any finite k. We recall from (Curtain and Zwart, 1995) the defining equations for $U(t,s)$ and $U_n(t,s)$:

$$U(t,s)x = T(t-s) + \int_s^t T(t-\alpha)Dy(\alpha)U(\alpha,s)x\,\mathrm{d}\alpha \tag{A1}$$

and

$$U_n(t,s)x = T(t-s) + \int_s^t T(t-\alpha)Dy_n(\alpha)U_n(\alpha,s)x\,\mathrm{d}\alpha, \tag{A2}$$

and the estimate

$$\|U(\alpha,s)\| \le Me^{(\alpha-s)(\omega+\mu)}, \tag{A3}$$

where

$$\|T(t)\| \le Me^{\omega t}, \quad t\ge 0, \tag{A4}$$

and

$$\mu = M\|D\|\|y\|_{L^\infty(0,t_1)} > 0.$$

Consider the following estimates obtained using (A1) and (A2):

$$\begin{aligned}
&\left\|U(t,s)-U_n(t,s)\right\| \\
&\le \int_s^t \left\|T(t-\alpha)D\right\| |y(\alpha)-y_n(\alpha)| \left\|U(\alpha,s)\right\| \mathrm{d}\alpha \\
&\quad + \int_s^t \left\|T(t-\alpha)D\right\| \|y_n\|_{L_\infty} \left\|U(\alpha,s)-U_n(\alpha,s)\right\| \mathrm{d}\alpha \\
&\le \|D\| \int_s^t Me^{\omega(t-\alpha)} Me^{(\omega+\mu)(\alpha-s)} |y(\alpha)-y_n(\alpha)|\,\mathrm{d}\alpha \\
&\quad + \|D\|\,\|y_n\|_{L_\infty} \int_s^t Me^{\omega(t-\alpha)} \left\|U(\alpha,s)-U_n(\alpha,s)\right\| \mathrm{d}\alpha.
\end{aligned}$$

Defining $f_n(t,s) = e^{-\omega(t-s)}\|U(t,s) - U_n(t,s)\|$, we obtain

$$f_n(t,s) \le M^2\|D\| \int_s^t e^{\mu\alpha}e^{-\mu s}|y(\alpha) - y_n(\alpha)|\,\mathrm{d}\alpha + \|D\|\,\|y_n\|_{L_\infty}\int_s^t f_n(\alpha,s)\,\mathrm{d}\alpha$$

$$\le M^2\|D\| \left(\int_s^t |y(\alpha) - y_n(\alpha)|^2\,\mathrm{d}\alpha\right)^{\frac{1}{2}} \times \left(\int_s^t e^{2\mu(\alpha-s)}\,\mathrm{d}\alpha\right)^{\frac{1}{2}} + \|D\|\,\|y_n\|_{L_\infty}\int_s^t f_n(\alpha,s)\,\mathrm{d}\alpha$$

$$= C_1\left|e^{2\mu(t-s)} - 1\right|^{\frac{1}{2}} \left(\int_s^{t_1}|y(\alpha) - y_n(\alpha)|^2\,\mathrm{d}\alpha\right)^{\frac{1}{2}} + C_2\int_s^t f_n(\alpha,s)\,\mathrm{d}\alpha,$$

where C_1 and C_2 only depend on t_1.

Thus

$$f_n(t,s) \le 2C_1e^{\mu(t-s)}\|y-y_n\|_{L_2(0,t_1)}+C_2\int_s^t f_n(\alpha,s)\,\mathrm{d}s$$

and differentiating this inequality with respect to t for fixed s yields

$$\frac{\mathrm{d}f_n}{\mathrm{d}t}(t,s) \le 2C_1\mu e^{\mu(t-s)}\|y - y_n\|_{L_2(0,t_1)} + C_2 f_n(t,s)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-C_2t}f_n(t,s)\right) \le 2C_1\mu e^{\mu(t-s)}e^{-C_2t}\|y-y_n\|_{L_2(0,t_1)}.$$

We integrate from t to s noting that $f_n(s,s) = 0$ to obtain

$$e^{-C_2t}f_n(t,s) \le 2C_1\mu e^{-\mu s}\int_s^t e^{(\mu-C_2)\beta}\,\mathrm{d}\beta\,\|y - y_n\|_{L_2(0,t_1)} = \frac{2C_1\mu}{\mu-C_2}e^{-\mu s}\left(e^{(\mu-C_2)t}-e^{(\mu-C_2)s}\right)\|y-y_n\|_{L_2(0,t_1)}$$

and

$$f_n(t,s) \le \frac{2C_1\mu}{\mu - C_2}\left(e^{\mu(t-s)} - e^{C_2(t-s)}\right)\|y-y_n\|_{L_2(0,t_1)}$$

and

$$\|U(t,s) - U_n(t,s)\| \le \frac{2C_1\mu}{\mu - C_2}\left[e^{(\omega+\mu)(t-s)}-e^{(\omega+C_2)(t-s)}\right]\|y-y_n\|_{L_2(0,t_1)},$$

which proves our claim.

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