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THE ASYMPTOTICAL STABILITY OF A DYNAMIC SYSTEM WITH STRUCTURAL DAMPING

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A dynamic system with structural damping described by partial differential equations is investigated. The system is first converted to an abstract evolution equation in an appropriate Hilbert space, and the spectral and semigroup properties of the system operator are discussed. Finally, the well-posedness and the asymptotical stability of the system are obtained by means of a semigroup of linear operators.

Keywords: dynamic system, evolution equation, asymptotic stability

1. Introduction

We shall be concerned with the following system of partial differential equations with initial and boundary conditions:

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} + \eta \dfrac{\partial^5 u(x,t)}{\partial t \partial x^4} \\ \qquad + \dfrac{\partial^2}{\partial x^2}\Big(p(x)\dfrac{\partial^2 u(x,t)}{\partial x^2}\Big) = f(t,u), \\ \dfrac{\partial^2 u(x,t)}{\partial x^2}\Big|_{x=0,l} = \dfrac{\partial}{\partial x}\Big(p(x)\dfrac{\partial^2 u(x,t)}{\partial x^2}\Big)\Big|_{x=0,l} = 0, \\ u(x,0) = \varphi_0(x), \quad \dfrac{\partial u(x,t)}{\partial t}\Big|_{t=0} = \psi_0(x). \end{cases} \tag{1}$$

So far, we have been concerned with clamped beam equations in (Hou and Tsui, 1998; 1999; 2000; 2003), in which the systems are different from (1). The system (1) stands for a typical beam equation with two free ends (Komkov, 1978; Köhne, 1978; Li and Zhu, 1988), where $u(x,t)$ is the transverse displacement of the point x at the time t, l is the length of the beam, $p(x)$ is the bending rigidity at the point x, and $f(t,u)$ represents the controlled moment of the system.

Suppose that $p(x) \in C^2[0,l]$, and $0 < p_0 \le p(x) \le p_1 < +\infty$, where p_0 and p_1 are constants. Now, we take $L^2[0,l]$ as the state space, with the inner product and norm respectively defined as follows:

$$\langle f, g\rangle_0 = \int_0^l f(x)\overline{g(x)}\,\mathrm{d}x, \quad f,g \in L^2[0,l],$$

$$\|f\|_0^2 = \int_0^l |f(x)|^2\,\mathrm{d}x, f \in L^2[0,l].$$

Let $H_1 = \operatorname{span}\{1,x\}$. Then $L^2[0,l] = H_1 \oplus H_2$, where H_2 is the orthogonal complement of H_1 in $L^2[0,l]$. Suppose that P_1 is the operator of projection onto H_1 and $I - P_1$ is the operator of projection onto H_2, so that the system (1) can be rewritten as

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = P_1 f(t,u), \\ u(x,0) = P_1\varphi_0, \quad \dfrac{\partial u(x,t)}{\partial t}\Big|_{t=0} = P_1\psi_0. \end{cases} \tag{2}$$

It is clear that the solution of (2) can be described as

$$u^{(1)}(x,t) = a_1 + a_2 x + a_3 t + a_4 t x, \tag{3}$$

where a_1, a_2, a_3 and a_4 are determined by $P_1\varphi_0$, $P_1\psi_0$, and $P_1 f(t,u)$.

For the system (1) in H_2, we have

$$\begin{cases} \dfrac{\partial^2 (x,t)}{\partial t^2} + \eta \dfrac{\partial^5 u(x,t)}{\partial t \partial x^4} \\ \qquad + \dfrac{\partial^2}{\partial x^2}\Big(p(x)\dfrac{\partial^2 u(x,t}{\partial x^2}\Big) = (I-P_1)f(t,u), \\ \dfrac{\partial^2 u(x,t)}{\partial x^2}\Big|_{x=0,l} = \dfrac{\partial}{\partial x}\Big(p(x)\dfrac{\partial^2 u(x,t)}{\partial x^2}\Big)\Big|_{x=0,l} = 0, \\ u(x,0) = (I-P_1)\varphi_0, \quad \dfrac{\partial u(x,t)}{\partial t}\Big|_{t=0} = (I-P_1)\psi_0. \end{cases} \tag{4}$$

If we denote by $u^{(2)}(x,t)$ the solution of (4), then the solution of the system (1) can be represented as

$$u(x,t) = u^{(1)}(x,t) \oplus u^{(2)}(x,t). \tag{5}$$

It should be noted that the form of $u^{(1)}(x,t)$ is given by (3), and $u^{(2)}(x,t)$ will play a key role in investigating the solution of the system (1).

We now define differential operators A and T as follows:

$$
\begin{aligned}
&A\varphi = \big(p(x)\varphi''(x)\big)'', \quad \varphi \in D(A),\\
&D(A) = \big\{\varphi \mid \varphi \in H_2, \varphi''(0) = \varphi''(l) = 0,\\
&\qquad \big(p(x)\varphi''\big)'\big|_{x=0,l} = 0, \big(p(x)\varphi''(x)\big)'' \in H_2\big\},\\
&T\varphi = \eta\varphi''''(x), \quad \varphi \in D(T), \quad D(T) = D(A).
\end{aligned}
$$

From the definitions of A and T it can be seen that $H_1 = \mathrm{span}\,\{1, x\}$ is the null space of A, and both A and T are positive definite self-adjoint operators in H_2, and there is a greatest positive number λ such that

$$
\langle A\varphi, \varphi\rangle_0 \geq \lambda\|\varphi\|_0^2, \quad \varphi \in D(A). \tag{6}
$$

It is easy to show that

$$
\frac{p_0}{\eta}T \leq A \leq \frac{p_1}{\eta}T. \tag{7}
$$

In fact, integrating by parts and taking account of the definitions of A and T as well as the boundary conditions, we have

$$
\begin{aligned}
\langle A\varphi, \varphi\rangle_0 &= \int_0^l (p(x)\varphi''(x))''\overline{\varphi(x)}\,\mathrm{d}x\\
&= \int_0^l (p(x)\varphi''(x))'\overline{\varphi'(x)}\,\mathrm{d}x\\
&= \int_0^l p(x)\varphi''(x)\overline{\varphi''(x)}\,\mathrm{d}x.
\end{aligned}
$$

From the inequalities $0 < p_0 \leq p(x) \leq p_1 < \infty$ it follows that

$$
\begin{aligned}
p_0\int_0^l \varphi''(x)\overline{\varphi''(x)}\,\mathrm{d}x &\leq \langle A\varphi, \varphi\rangle_0\\
&\leq p_1\int_0^l \varphi''(x)\overline{\varphi''(x)}\,\mathrm{d}x.
\end{aligned}
$$

In other words,

$$
p_0\|\varphi''\|_0^2 \leq \langle A\varphi, \varphi\rangle_0 \leq p_1\|\varphi''\|_0^2.
$$

Similarly, we have

$$
\begin{aligned}
\langle T\varphi, \varphi\rangle_0 &= \langle \eta\varphi'''', \varphi\rangle_0 = \langle \eta\varphi''', \varphi'\rangle_0\\
&= \langle \eta\varphi'', \varphi''\rangle_0 = \eta\|\varphi''\|_0^2
\end{aligned}
$$

and

$$
\varphi''\|_0^2 = \frac{1}{\eta}\langle T\varphi, \varphi\rangle_0.
$$

Hence

$$
\Big\langle \frac{p}{\eta}T\varphi, \varphi\Big\rangle_0 \leq \langle A\varphi, \varphi\rangle_0 \leq \Big\langle \frac{p_1}{\eta}T\varphi, \varphi\Big\rangle_0
$$

and therefore

$$
\frac{p_0}{\eta}T \leq A \leq \frac{p_1}{\eta}T.
$$

In terms of the operators A and T we can rewrite the system (4) as follows:

$$
\begin{cases}
\dfrac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \dfrac{\mathrm{d}}{\mathrm{d}t}(Tu) + Au = (I - P_1)f(t, u),\\
u(0) = (I - P_1)\varphi_0, \quad \dfrac{\mathrm{d}u(x,t)}{\mathrm{d}t}\Big|_{t=0} = (I - P_1)\psi_0.
\end{cases} \tag{8}
$$

Let us now introduce the Hilbert space $H = H_2 \times H_2$ equipped with the general inner product. Set

$$
\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad u_1 = A^{\frac{1}{2}}u, \quad u_2 = \frac{\mathrm{d}u}{\mathrm{d}t},
$$

$$
\mathcal{A} = \begin{bmatrix} 0 & A^{\frac{1}{2}} \\ -A^{\frac{1}{2}} & -T \end{bmatrix}, \quad D(\mathcal{A}) = D(A^{\frac{1}{2}}) \times D(A),
$$

$$
\vec{F}(t, \vec{u}) = \begin{bmatrix} 0 \\ (I - P_1)f(t, u) \end{bmatrix},
$$

$$
\vec{u}_0 = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} A^{\frac{1}{2}}(I - P_1)\varphi_0 \\ (I - P_1)\psi_0 \end{bmatrix}.
$$

Then the evolution equation (8), or the original system (1), is equivalent to the following first-order evolution equation:

$$
\begin{cases}
\dfrac{\mathrm{d}\vec{u}(t)}{\mathrm{d}t} = \mathcal{A}\vec{u}(t) + \vec{F}(t, \vec{u}),\\
\vec{u}(0) = \vec{u}_0,
\end{cases} \tag{9}
$$

and the corresponding equation is given by

$$
\begin{cases}
\dfrac{\mathrm{d}\vec{u}(t)}{\mathrm{d}t} = \mathcal{A}\vec{u}(t),\\
\vec{u}(0) = \vec{u}_0.
\end{cases} \tag{10}
$$

2. Main Results

We shall first discuss the semigroup properties of the operator $\mathcal{A}$, and then investigate the well-posedness and asymptotical stability of the system (1). The following results will be obtained:

Theorem 1. *The linear operator $\mathcal{A}$ in the system (9) is the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying*

$$
\|T(t)\| \leq Me^{-\delta t}, \quad t \geq 0,
$$

where M and δ are positive constants.

Theorem 2. *If $f(t,u) : [0,T] \times L^2[0,l] \to L^2[0,l]$ is continuous in t for any $T > 0$ on $[0,T]$ and uniformly Lipschitz continuous in u on $L^2[0,l]$, then for every $\vec{u}_0 \in H$, the evolution equation (9) has a unique weak solution in $C([0,T];H)$. Moreover, the mapping $\vec{u}_0 \to \vec{u}$ is Lipschitz continuous.*

Theorem 3. *Suppose that $f(t,u)$ meets the conditions of Theorem 2 with the Lipschitz constant N satisfying*

$$N < \frac{\delta}{M}\sqrt{\lambda},$$

where M and δ are the same as those in Theorem 1, and λ is the same as that in the equality (6). Then the solution of the system (9) (and therefore the solution of the system (1)) is exponentially stable.

3. Proofs of the Main Results

In this section, we shall prove Theorems 1–3. To prove Theorem 1, we shall first prove the following lemmas.

Lemma 1. *If λ is a complex number with Re $\lambda \geq 0$, then $(\lambda^2 + \lambda T + A)^{-1}$ exists and is bounded.*

Proof. Clearly, the result is true for $\lambda = 0$. If $\lambda \neq 0$, then for any $x \in D(A)$, we let $\lambda = \sigma + i\tau$, $\sigma \geq 0$. Consequently,

$$\begin{aligned}
&\Big\|\Big(\lambda + T + \frac{1}{\lambda}A\Big)x\Big\|\,\|x\| \\
&\quad\geq \Big|\Big\langle\Big(\lambda + T + \frac{1}{\lambda}A\Big)x, x\Big\rangle\Big|\ \Big|\sigma\|x\|^2 + \langle Tx, x\rangle \\
&\qquad + \frac{\sigma}{\sigma^2+\tau^2}\langle Ax, x\rangle + i[\tau\|x\|^2 \\
&\qquad - \frac{\tau}{\sigma^2+\tau^2}\langle Ax, x\rangle]\Big| \geq \langle Tx, x\rangle \geq \omega\|x\|^2,
\end{aligned}$$

where $\omega > 0$ is the smallest eigenvalue of T.

Since $x \in D(A)$, it can be seen that

$$\begin{aligned}
&\Big\langle\Big(\lambda + T + \frac{1}{\lambda}A\Big)x, x\Big\rangle \\
&\quad = \sigma\|x\|^2 + \langle Tx, x\rangle + \frac{\sigma}{\sigma^2+\tau^2}\langle Ax, x\rangle \\
&\qquad + i\Big[\tau\|x\|^2 - \frac{\tau}{\sigma^2+\tau^2}\langle Ax, x\rangle\Big],
\end{aligned}$$

and

$$\mathrm{Re}\Big\langle -\Big(\lambda + T + \frac{1}{\lambda}A\Big)x, x\Big\rangle \leq -\langle Tx, x\rangle \leq -\omega\|x\|^2.$$

It follows that the numerical range of $-(\lambda + T + \frac{1}{\lambda}A)$ has the form

$$\begin{aligned}
&V\Big(-\Big(\lambda + T + \frac{1}{\lambda}A\Big)\Big) \\
&\quad = \Big\{-\Big\langle\Big(\lambda + T + \frac{1}{\lambda}A\Big)x, x\Big\rangle : \|x\| = 1, \\
&\qquad\qquad x \in D\Big(\lambda + T + \frac{1}{\lambda}A\Big)\Big\} \subseteq \{\lambda \mid \mathrm{Re}\,\lambda \leq -\omega\}.
\end{aligned}$$

This implies that $0 \in \rho(-(\lambda + T + \frac{1}{\lambda}A))$ (see (Balaskrishnan, 1981)), and so $0 \in \rho(\lambda^2 + \lambda T + A)$. Thus, $(\lambda^2 + \lambda T + A)^{-1}$ exists and is bounded. ■

Lemma 2. *If λ is a complex number with $\mathrm{Re}\,\lambda \geq 0$, $\lambda \neq 0$, then $(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}})^{-1}$ exists and is bounded.*

Proof. First, it should be noted that $A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ can be extended to a bounded linear operator on H_2, for every $x \in H_2$, $\lambda = \sigma + i\tau$, $\sigma \geq 0$. Since

$$\begin{aligned}
&\Big\|\Big(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)x\Big\|\|x\| \\
&\quad\geq \Big|\Big\langle\Big(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)x, x\Big\rangle\Big| \\
&\quad= \Big|\frac{\sigma}{\sigma^2+\tau^2}\|x\|^2 + \sigma\langle A^{-1}x, x\rangle + \langle A^{-\frac{1}{2}}TA^{-\frac{1}{2}}x, x\rangle \\
&\qquad + i\Big[\frac{-\tau}{\sigma^2+\tau^2}\|x\|^2 + \tau\langle A^{-1}x, x\rangle\Big]\Big| \\
&\quad\geq \frac{\sigma}{\sigma^2+\tau^2}\|x\|^2 + \sigma\langle A^{-1}x, x\rangle + \langle A^{-\frac{1}{2}}TA^{-\frac{1}{2}}x, x\rangle \\
&\quad\geq \frac{\eta}{p_1}\|x\|^2,
\end{aligned}$$

the operator $\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ is invertible. We also see that its image is dense in H_2. In fact, if $y_0 \in H_2$, then

$$\Big\langle\Big(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)x, y_0\Big\rangle = 0, \quad x \in H_2.$$

Noticing that $\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ is self-adjoint, we have

$$\Big\langle x, \Big(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)y_0\Big\rangle = 0, \quad x \in H_2.$$

Since $\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ is invertible, $y_0 = 0$, and therefore the range of $(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}})$ is dense in H_2. Thus, $(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}})^{-1}$ exists and is bounded. ■

Lemma 3. *If λ is a complex number with $\mathrm{Re}\,\lambda \geq 0$, $\lambda \neq 0$, then the resolvent of $\mathcal{A}$ can be expressed by*

$$R(\lambda, \mathcal{A}) = \frac{1}{\lambda}\begin{bmatrix} R_{11}, & R_{12} \\ R_{21}, & R_{22} \end{bmatrix},$$

where

$$R_{11} = I - \frac{1}{\lambda^2}\Big(\frac{1}{\lambda^2} + A^{-1} + \frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)^{-1},$$

$$R_{12} = \frac{1}{\lambda}\Big(\frac{1}{\lambda^2} + A^{-1} + \frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)^{-1}A^{-\frac{1}{2}}$$

$$R_{21} = -\frac{1}{\lambda}A^{-\frac{1}{2}}\Big(\frac{}{\lambda^2} + A^{-1} + \frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)^{-1},$$

$$R_{22} = A^{-\frac{1}{2}}\Big(\frac{1}{\lambda^2} + A^{-1} + \frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)^{-1}A^{-\frac{1}{2}}.$$

Proof. From Lemma 2 we know that $R(\lambda,\mathcal{A})$ is a bounded linear operator on H and the expression for $R(\lambda,\mathcal{A})$ can be obtained by a direct calculation. ■

Lemma 4. *If λ is complex number with $\operatorname{Re}\lambda \geq 0$ and $\lambda \neq 0$, the family of the operators with the parameter λ*

$$\begin{aligned} F(\lambda) &= \frac{1}{\lambda}\Big(\frac{1}{\lambda^2} + A^{-1} + \frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)^{-1} \\ &= \Big(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)^{-1} \end{aligned}$$

is uniformly bounded.

Proof. Let

$$Z_\lambda = \Big(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)^{-1}x, \quad x \in H_2.$$

Then $\{\|Z_\lambda\|\}$ is bounded for all λ. Otherwise, there is a λ_0 such that

$$\lim_{\lambda\to\lambda_0}\|Z_\lambda\| = +\infty.$$

As regards the inner product of the sequence $y_\lambda = Z_\lambda/\|Z_\lambda\|$ with $\lambda = \sigma + i\tau$, we have

$$\begin{aligned} &\Big\langle\Big(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)y_\lambda, y_\lambda\Big\rangle \\ &\quad = \frac{\sigma}{\sigma^2+\tau^2} + \sigma\langle A^{-1}y_\lambda, y_\lambda\rangle + \langle A^{-\frac{1}{2}}TA^{-\frac{1}{2}}y_\lambda, y_\lambda\rangle \\ &\qquad + i\Big[\frac{-\tau}{\sigma^2+\tau^2} + \tau\langle A^{-1}y_\lambda, y_\lambda\rangle\Big]. \end{aligned} \tag{11}$$

Obviously, the real part of the right-hand side (11) is greater than $\eta/p_0 > 0$. On the other hand,

$$\lim_{\lambda\to\lambda_0}\Big(\frac{1}{\lambda} + \lambda A^{-1} + A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)y_\lambda = \lim_{\lambda\to\lambda_0}\frac{x}{\|Z_\lambda\|} = 0,$$

so that a contradiction occurs. Hence $\{\|Z_\lambda\|\}$ is uniformly bounded for every $x \in H_2$, and the result of this lemma follows from the Principle of Uniform Boundedness. ■

Lemma 5. *If λ is complex number with $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$, and there is a $\lambda_0 > 0$ such that if $|\lambda| \geq \lambda_0$, then $(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1})^{-1}$ is uniformly bounded.*

Proof. For every $x \in H_2$, it is easy to see that

$$\begin{aligned} &\Big\|\Big(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\Big)x\Big\|^2 \\ &= \Big\langle\Big(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\Big)x, \Big(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\Big)x\Big\rangle \\ &= \frac{1}{|\lambda|^2}\|x\|^2 + \frac{\overline{\lambda}}{\lambda}\langle x, A^{-1}x\rangle + \frac{\lambda}{\overline{\lambda}}\langle A^{-1}x, x\rangle \\ &\quad + |\lambda|^2\|A^{-1}x\|^2 + \overline{\lambda}\langle TA^{-1}x, A^{-1}x\rangle \\ &\quad + +\lambda\langle A^{-1}x, TA^{-1}x\rangle + \frac{1}{\lambda}\langle x, TA^{-1}x\rangle \\ &\quad + \frac{1}{\overline{\lambda}}\langle TA^{-1}x, x\rangle + \|TA^{-1}x\|^2 \\ &\geq \frac{1}{\lambda}\langle x, TA^{-1}\rangle + \frac{1}{\overline{\lambda}}\langle TA^{-1}x, x\rangle + \|TA^{-1}x\|^2. \end{aligned} \tag{12}$$

Since TA^{-1} is bounded, there is a $\lambda_0 > 0$, such that if $|\lambda| \geq \lambda_0$, the right-hand side of the above inequality has the form

$$\begin{aligned} &\frac{1}{\lambda}\langle x, TA^{-1}x\rangle + \frac{1}{\overline{\lambda}}\langle TA^{-1}x, x\rangle + \|TA^{-1}x\|^2 \\ &\qquad \geq \frac{1}{4}\|TA^{-1}x\|^2 \geq \frac{1}{4}\delta_0^2\|x\|^2, \end{aligned} \tag{13}$$

where $\delta_0 > 0$, and the last inequality is due to the invertibility of TA^{-1}. This follows from

$$\Big\|\Big(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\Big)x\Big\| \geq \frac{1}{2}\delta_0\|x\|. \tag{14}$$

Hence $(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1})$ is invertible.

Next, we shall show by contradiction that the range of $(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1})$ is dense in H_2. If the range of $(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1})$ is not dense in H_2, there is a $y_0 \in H_2$, $y_0 \neq 0$ such that

$$\Big\langle\Big(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1}\Big)x, y_0\Big\rangle = 0, \quad x \in H_2.$$

This implies

$$\Big\langle\Big(\frac{A}{\lambda} + T + \lambda\Big)y, y_0\Big\rangle = 0, \quad y \in D(A),$$

where $y = A^{-1}x$.

In view of Lemma 1, $(\frac{1}{\lambda}A + T + \lambda)^{-1}$ is a bounded linear operator, and its range is dense in H_2. Hence $y_0 = 0$, but this contradicts $y_0 \neq 0$. Thus the range of $(\frac{1}{\lambda} + TA^{-1} + \lambda A^{-1})$ is dense in H_2. If $|\lambda| \geq \lambda_0$, $\operatorname{Re}\lambda \neq 0$, and for a fixed $x \in H_2$ we set

$$Z_\lambda = \Big(\frac{1}{\lambda} + \lambda A^{-1} + TA^{-1}\Big)^{-1}x, \quad |\lambda| \geq \lambda_0,$$

then it can be shown that $\{\|Z_\lambda\|\}$ is bounded. Otherwise, there is a sequence $\{\lambda_n\}$ with $|\lambda_n| \geq \lambda_0$ and $\operatorname{Re}\lambda_n \geq 0$ such that

$$\lim_{n\to\infty} \|Z_{\lambda_n}\| = \infty,$$

and

$$\Big(\frac{1}{\lambda_n}+\lambda_n A^{-1}+TA^{-1}\Big)\frac{Z_{\lambda_n}}{\|Z_{\lambda_n}\|} = \frac{x}{\|Z_{\lambda_n}\|} \to 0, \quad n\to\infty.$$

Let $y_n = Z_{\lambda_n}/\|Z_{\lambda_n}\|$. From (13) it follows that

$$\Big\|\Big(\frac{1}{\lambda_n} + TA^{-1} + \lambda_n A^{-1}\Big)y_n\Big\| \geq \frac{\delta_0}{2}\|y_n\| = \frac{\delta_0}{2} > 0,$$

which contradicts (15). Hence $\{\|Z_\lambda\|\}$ is bounded, for every $x \in H_2$. From the Principle of Uniform Boundedness it follows that $(\frac{1}{\lambda} + \lambda A^{-1} + TA^{-1})^{-1}$ is uniformly bounded for $|\lambda| \geq \lambda_0$ and $\operatorname{Re}\lambda \geq 0$. ■

Lemma 6. *Under the condition of Lemma 5, if* $|\lambda| \geq \lambda_0$ *and* $\operatorname{Re}\lambda \geq 0$*, the family of operators with* λ

$$A^{-\frac{1}{2}}\Big(\frac{1}{\lambda^2} + A^{-1} + \frac{1}{\lambda}A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\Big)^{-1}A^{-\frac{1}{2}} = \Big(\frac{1}{\lambda^2}A + \frac{1}{\lambda}T + I\Big)^{-1}$$

is uniformly bounded.

Proof. If $|\lambda| \geq \lambda_0$ and $\operatorname{Re}\lambda \geq 0$, from Lemma 5 we have

$$\Big(\frac{1}{\lambda^2}A + \frac{1}{\lambda}T + I\Big)^{-1} = A^{-1}\Big(\frac{1}{\lambda^2} + \frac{1}{\lambda}TA^{-1} + A^{-1}\Big)^{-1}.$$

Thus, the result of Lemma 6 is concluded by virtue of Lemma 5. ■

Proof of Theorem 1. Since

$$\mathcal{A} = \begin{bmatrix} 0 & A^{\frac{1}{2}} \\ -A^{\frac{1}{2}} & -T \end{bmatrix}$$

and A and T are positive definite self-adjoint operators, we can easily verify that $(i\mathcal{A})^* = i\mathcal{A}$. From the celebrated Stone Theorem (Pazy, 1983) it follows that $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ on H. On the other hand, we can see that $0 \in \rho(\mathcal{A})$ gives us

$$\mathcal{A}^{-1} = \begin{bmatrix} -A^{-\frac{1}{2}}TA^{-\frac{1}{2}} & -A^{-\frac{1}{2}} \\ A^{\frac{1}{2}} & 0 \end{bmatrix}.$$

If $\operatorname{Re}\lambda \geq 0$ and $\lambda \neq 0$, we can show that the resolvent $R(\lambda, \mathcal{A})$ of $\mathcal{A}$ satisfies

$$\|R(\lambda,\mathcal{A})\| \leq \frac{M}{|\lambda|}, \quad 1 \leq M < \infty. \tag{15}$$

In fact, from Lemma 3 we deduce that $R(\lambda, \mathcal{A})$ is an analytic function of λ on the right-half complex plane. According to the analyticity of $R(\lambda, \mathcal{A})$, it suffices to show that if $|\lambda| \geq \lambda_0 > 0$ and $\operatorname{Re}\lambda \geq 0$, then $\|R(\lambda,\mathcal{A})\| \leq M_1/|\lambda|$. However, this can be easily obtained by Lemmas 4–6.

Since $\rho(\mathcal{A}) \supset \{\lambda \mid \operatorname{Re}\lambda \geq 0\}$, $\rho(\mathcal{A})$ being an open set on the complex plane, there is a constant $\epsilon > 0$ such that

$$\sigma(\mathcal{A}) \subset \big\{\lambda \mid \operatorname{Re}\lambda \leq -\epsilon\big\}$$

and therefore from the stability theorem of the analytic semigroup (Pazy, 1983) we conclude that there is a constant $\delta > 0$ such that

$$\|T(t)\| \leq Me^{-\delta t}, \quad t \geq 0.$$

The proof of Theorem 1 is thus complete. ■

Proof of Theorem 2. For $\vec{F}(t,\vec{u}) = [0,(I-P_1)f(t,u)]^T \in H$ in (9), we have $\|\vec{F}(t,\vec{u}\| = \|(I-P_1)f(t,u)\|$. Since $I - P_1$ is a bounded linear operator, and $f(t,u)$ is continuous in t on $[0,T]$ and uniformly Lipschitz continuous in u on $L^2[0,l]$, $\vec{F}(t,\vec{u})$ has the same properties as $f(t,u)$. Applying Theorem 1.2 of (Pazy, 1983) yields Theorem 2. ■

In order to prove Theorem 3, we first introduce a continuous function space $C[0,+\infty)$ equipped with the norm

$$\|g\|_m = \max_{t\geq 0}|g(t)| < +\infty, \quad g \in C[0,+\infty),$$

and define the linear operator K through

$$Kg(t) = \int_0^t e^{-\delta(t-s)}g(s)\,\mathrm{d}s,$$

where δ is the same as in Theorem 1.

We see that K is a bounded linear operator on $C[0,+\infty)$. In fact,

$$|Kg(t)| \leq \int_0^t e^{\delta(t-s)}|g(s)|\,\mathrm{d}s \leq \|g\|_m \int_0^t e^{-\delta(t-s)}\,\mathrm{d}s = \|g\|_m\frac{1}{\delta}(1-e^{-\delta t}) \leq \frac{1}{\delta}\|g\|_m$$

for any $t \geq 0$. Thus we have

$$\|Kg\|_m = \max_{t\geq 0}|Kg(t)| \leq \frac{1}{\delta}\|g\|_m,$$

and

$$\|K\|_m \leq \frac{1}{\delta}. \tag{16}$$

Proof of Theorem 3. From Theorem 1 and (Pazy, 1983) we know that the evolution equation (9) has a unique solution

$\vec{u}(x,t)$, and hence the system (4) has a unique solution $\vec{u}(x,t)$. We now decompose $u(x,t)$ as follows:

$$\vec{u}(x,t) = \vec{\xi}(x,t) + \vec{\eta}(x,t),$$

where $\vec{\xi}(x,t)$ and $\vec{\eta}(x,t)$ satisfy

$$\begin{cases} \dfrac{\partial^2 \vec{\xi}(x,t)}{\partial t^2} + \eta \dfrac{\partial^5 \vec{\xi}(x,t)}{\partial t \partial x^4} + \dfrac{\partial^2}{\partial x^2}\Big(p(x)\dfrac{\partial^2 \vec{\xi}(x,t)}{\partial x^2}\Big) = 0, \\ \dfrac{\partial^2 \vec{\xi}(x,t)}{\partial x^2}\Big|_{x=0,l} = \dfrac{\partial}{\partial x}\Big(p(x)\dfrac{\partial^2 \vec{\xi}(x,t)}{\partial x^2}\Big)\Big|_{x=0,l} = 0, \\ \vec{\xi}(x,0) = (I-P_1)\varphi_0, \dfrac{\partial \vec{\xi}(x,t)}{\partial t}\Big|_{t=0} = (I-P_1)\psi_0, \end{cases} \tag{17}$$

and

$$\begin{cases} \dfrac{\partial^2 \vec{\eta}(x,t)}{\partial t^2} + \eta \dfrac{\partial^5 \vec{\eta}(x,t)}{\partial t \partial x^4} + \dfrac{\partial^2}{\partial x^2}\Big(p(x)\dfrac{\partial^2 \vec{\eta}(x,t)}{\partial x^2}\Big) \\ \qquad\qquad = (I-P_1)f(t,u(x,t)), \\ \dfrac{\partial^2 \vec{\eta}(x,t)}{\partial x^2}\Big|_{x=0,l} = \dfrac{\partial}{\partial x}\Big(p(x)\dfrac{\partial^2 \vec{\xi}(x,t)}{\partial x^2}\Big)\Big|_{x=0,l} = 0, \\ \vec{\eta}(x,0) = 0, \quad \dfrac{\partial \vec{\eta}(x,t)}{\partial t}\Big|_{t=0} = 0, \end{cases} \tag{18}$$

respectively. Here $u(x,t)$ in $f(t,u)$ is the solution of the system (4).

From Theorem 1 it should be noted that the system (17) in H is equivalent to the system (10), whose solution $\vec{\xi}(x,t) = T(t)\vec{\xi}_0$ satisfies

$$\|\vec{\xi}(x,t)\| \le M e^{-\delta t}\|\vec{\xi}_0\|. \tag{19}$$

It is obvious that the system (18) in H is equivalent to the system

$$\begin{cases} \dfrac{\mathrm{d}\vec{\eta}(t)}{\mathrm{d}t} = \mathcal{A}\vec{\eta}(t) + \vec{F}(t,\vec{u}), \\ \vec{\eta}(0) = \vec{\eta}_0 = 0, \end{cases}$$

where

$$\vec{\eta} = (\eta_1, \eta_2)^T, \quad \eta_1 = A^{\frac{1}{2}}\eta, \quad \eta_2 = \frac{\mathrm{d}\eta}{\mathrm{d}t}$$

and

$$\vec{F}(t,\vec{u}) = \big[0, (1-P_1)f(t,u)\big]^T = \big[0, (I-P_1)f(t,\xi+\eta)\big]^T.$$

Since $\mathcal{A}$ generates a C_0 semigroup $T(t)$ on H and $\vec{\eta}_0 = 0$, we have

$$\vec{\eta}(x,t) = \int_0^t T(t-s)\vec{F}\big(s,\vec{u}(s)\big)\,\mathrm{d}s,$$

and from Theorem 1 it follows that

$$\begin{aligned} \|\vec{\eta}(x,t)\| &\le M\int_0^t e^{-\delta(t-s)} \quad \|\vec{F}(s,\vec{u}(s)\|\,\mathrm{d}x \\ &= M\int_0^t e^{-\delta(t-s)}\|(I-P_1)f(x,u(s))\|_0\,\mathrm{d}s \\ &\le M\int_0^t e^{-\delta(t-s)}\|f(s,u(s))\|_0\,\mathrm{d}s \\ &\le MN\int_0^t e^{-\delta(t-s)}\|u(s)\|_0\,\mathrm{d}s \\ &\le MN\int_0^t e^{-\delta(t-s)}(\|\xi(s)\|_0 \\ &\qquad + \|\eta(s)\|_0)\,\mathrm{d}s. \end{aligned} \tag{20}$$

By virtue of (6) and the definition of the inner product described before, we have

$$\begin{aligned} \|\vec{\xi}\|^2 &= \langle \xi_1,\xi_1\rangle + \langle \xi_2,\xi_2\rangle = \Big\langle A^{\frac{1}{2}}\xi, A^{\frac{1}{2}}\xi\Big\rangle_0 + \|\xi_2\|^2 \\ &= \langle A\xi,\xi\rangle_0 + \|\xi_2\|^2 \ge \lambda\|\xi\|_0^2 + \|\xi_2\|^2 \ge \lambda\|\xi\|_0^2, \end{aligned}$$

and so

$$\|\xi\|_0 \le \frac{1}{\sqrt{\lambda}}\|\vec{\xi}\|. \tag{21}$$

Similarly, it can be shown that $\|\vec{\eta}\|^2 \ge \lambda\|\eta\|_0^2$ and

$$\|\eta\|_0 \le \frac{1}{\sqrt{\lambda}}\|\vec{\eta}\|. \tag{22}$$

Combining (19)–(22) leads to

$$\begin{aligned} \|\vec{\eta}(x,t)\| &\le MN\int_0^t e^{-\delta(t-s)}\big(\|\xi(s)\|_0 + \|\eta(s)\|_0\big)\mathrm{d}s \\ &\le \frac{MN}{\sqrt{\lambda}}\int_0^t e^{-\delta(t-s)}\|\vec{\xi}(s)\|\,\mathrm{d}s \\ &\quad + \frac{MN}{\sqrt{\lambda}}\int_0^t e^{-\delta(t-s)}\|\vec{\eta}(s)\|\,\mathrm{d}s \\ &\le \frac{M^2N}{\sqrt{\lambda}}e^{-\delta t}\|\vec{\xi}_0\| \\ &\quad + \frac{MN}{\sqrt{\lambda}}\int_0^t e^{-\delta(t-s)}\|\vec{\eta}(s)\|\,\mathrm{d}s \\ &= \frac{M^2N}{\sqrt{\lambda}}te^{-\delta t}\|\vec{\xi}_0\| + \frac{MN}{\sqrt{\lambda}}K\big(\|\vec{\eta}(s)\|\big), \end{aligned}$$

and therefore

$$\Big(I - \frac{MN}{\sqrt{\lambda}}K\Big)\|\vec{\eta}(s)\| \le \frac{M^2N}{\sqrt{\lambda}}te^{-\delta t}\|\vec{\xi}_0\|. \tag{23}$$

Looking at (21) with the assumption that $N < \delta\sqrt{\lambda}/M$ and (16), we find

$$\Big\|\frac{MN}{\sqrt{\lambda}}K\Big\|_m < 1.$$

Hence $(I - \frac{MN}{\sqrt{\lambda}}K)$ is invertible, and

$$\Big(I - \frac{MN}{\sqrt{\lambda}}K\Big)^{-1} = \sum_{n=0}^{\infty}\Big(\frac{MN}{\sqrt{\lambda}}\Big)^n K^n. \tag{24}$$

Analysing the definition of K, we see that K is a monotonically increasing operator on $C[0,+\infty)$, and so is $(I - \frac{MN}{\sqrt{\lambda}}K)^{-1}$ based on (24). Now, multiply two sides of (23) by $(I - \frac{MN}{\sqrt{\lambda}}K)^{-1}$ to find

$$\|\vec{\eta}\| \le \frac{M^2N}{\sqrt{\lambda}}\|\vec{\xi_0}\|\sum_{n=0}^{\infty}\Big(\frac{MN}{\sqrt{\lambda}}\Big)^n K^n(te^{-\delta t}).$$

Since

$$\begin{aligned} K(te^{\delta t}) &= \int_0^t e^{-\delta(t-s)} s e^{-\delta s}\,\mathrm{d}s \\ &= e^{-\delta t}\int_0^t s\,\mathrm{d}s = \frac{t^2}{2!}e^{-\delta t}, \end{aligned}$$

it can be shown by induction that

$$K^n(te^{-\delta t}) = \frac{t^{n+1}}{(n+1)!}e^{-\delta t}.$$

Thus

$$\begin{aligned} \|\vec{\eta}(x,t)\| &\le \frac{M^2N}{\sqrt{\lambda}}\|\vec{\xi_0}\|\sum_{n=0}^{\infty}\Big(\frac{MN}{\sqrt{\lambda}}\Big)^n\frac{t^{n+1}}{(n+1)!}e^{-\delta t} \\ &= \frac{M^2N}{\sqrt{\lambda}}\frac{\sqrt{\lambda}}{MN}\|\vec{\xi_0}\|\sum_{n=0}^{\infty}\Big(\frac{MN}{\sqrt{\lambda}}\Big)^{n+1}\frac{t^{n+1}}{(n+1)!}e^{-\delta t} \\ &< M\|\vec{\xi_0}\|e^{-\delta t}\,e^{\frac{MN}{\sqrt{\lambda}}t} = M\|\vec{\xi_0}\|e^{-(\delta-\frac{MN}{\sqrt{\lambda}})t}. \end{aligned}$$

Since

$$N < \frac{\delta}{M}\sqrt{\lambda}, \quad \delta - \frac{MN}{\sqrt{\lambda}} > 0,$$

let

$$\alpha = \delta - \frac{MN}{\sqrt{\lambda}}.$$

Then

$$\|\vec{\eta}(x,t)\| \le M\|\vec{\xi_0}\|e^{-\alpha t}. \tag{25}$$

As for $\vec{u}(x,t) = \vec{\xi}(x,t) + \vec{\eta}(x,t)$, we have

$$\begin{aligned} \vec{u} &= (u_1,u_2)^T = \Big(A^{\frac12}u, \frac{\mathrm{d}u}{\mathrm{d}t}\Big)^T \\ &= \Big(A^{\frac12}(\xi+\eta), \frac{\mathrm{d}\xi}{\mathrm{d}t} + \frac{\mathrm{d}\eta}{\mathrm{d}t}\Big)^T \\ &= \Big(A^{\frac12}\xi, \frac{\mathrm{d}\xi}{\mathrm{d}t}\Big)^T + \Big(A^{\frac12}\eta, \frac{\mathrm{d}\xi}{\mathrm{d}t}\Big)^T \\ &= (\xi_1,\xi_2)^T + (\eta_1,\eta_2)^T = \vec{\xi} + \vec{\eta} \end{aligned}$$

and

$$\|\vec{u}(x,t)\| = \|\vec{\xi}(x,t) + \vec{\eta}(x,t)\| \le \|\vec{\xi}(x,t)\| + \|\vec{\eta}(x,t)\|.$$

Combining (19) and (25) gives

$$\|\vec{u}(x,t)\| \le M\|\vec{\xi_0}\|e^{-\delta t} + M\|\vec{\xi_0}\|e^{-\alpha t}.$$

Since $0 < \alpha < \delta$, we have

$$\|\vec{u}(x,t)\| \le 2M\|\vec{\xi_0}\|e^{-\alpha t}, \quad t \ge 0.$$

From $\vec{\xi_0} = \vec{u}_0$ it follows that

$$\|\vec{u}(x,t)\| \le 2M\|\vec{u}_0\|e^{-\alpha t}, \quad t \ge 0.$$

This implies that the solution $\vec{u}(x,t)$ to the evolution equation (9) is exponentially stable, and thus the solution to the original system (1) is also exponentially stable. The proof is complete. ■

4. Conclusion

The beam equation with two free ends (1) was studied by means of functional analysis and semigroups of linear operators. First, the system (1) was converted to an abstract evolution equation (9). Second, the properties of the system operator $\mathcal{A}$ were investigated and a significant result that $\mathcal{A}$ generates a C_0-semigroup $T(t)$ with exponential decay property that $\|T(t)\| \le Me^{-\delta t}$ $(M > 0, \delta > 0)$ was derived (Theorem 1). Then, the well-posedness of the system (1) was discussed (Theorem 2) using the semigroup technique. Finally, the exponential stability of the system (1) was proved under appropriate conditions (Theorem 3). In further research, concrete designs of the controllers for this system to be asymptotically stable would be quite significant.

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