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CONTINUITY OF SOLUTIONS OF RICCATI EQUATIONS FOR THE DISCRETE–TIME JLQP

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The continuity of the solutions of difference and algebraic coupled Riccati equations for the discrete-time Markovian jump linear quadratic control problem as a function of coefficients is verified. The line of reasoning goes through the use of the minimum property formulated analogously to the one for coupled continuous Riccati equations presented by Wonham and a set of comparison theorems.

Keywords: coupled algebraic Ricatti equations, jump parameter system, quadratic control, stochastic stabilizability, observability, robustness, sensitivity

1. Introduction

The continuity of various types of Riccati equations has been considered in various contexts in the last decade. In (Czornik, 1996; 2000; Delchamps, 1980; Faibusovich, 1986; Lancaster and Rodman, 1995; Rodman, 1980), the authors examine the continuity of continuous-time algebraic Riccati equations under different conditions. Czornik and Sragovich (1995) consider the situation when the coefficients of a continuous-time algebraic Riccati equation tend to coefficients for which the solution does not exist. The continuity of discrete algebraic equations is shown in (Chen, 1985). In (Chojnowska-Michalik *et al.*, 1992) the continuity in the uniform operator topology of the solution of the Riccati equations in Hilbert space is verified.

In the discrete-time Markovian jump linear quadratic control problem on a finite time interval the following set of coupled Riccati difference equations appears (Chizeck *et al.*, 1986):

$$P_{k+1}(P,j) = \widetilde{A}'(j)\widetilde{F}_k(j)\widetilde{A}(j) - \widetilde{A}'(j)\widetilde{F}_k(j)\widetilde{B}(j) \times \left(R(j) + \widetilde{B}'(j)\widetilde{F}_k(j)\widetilde{B}(j)\right)^{-1} \times \widetilde{B}'(j)\widetilde{F}_k(j)\widetilde{A}(j) + Q(j), \tag{1}$$

$j \in S,\ k = 0, 1, \ldots$, where

$$\widetilde{F}_k(j) = \sum_{i\in S} \widetilde{p}(j,i) P_k(P,i) \tag{2}$$

with terminal conditions $P_0(P,j) = P \geq 0$. Here $\widetilde{A}(j) \in \mathbb{R}^{n,n}, \widetilde{B}(j) \in \mathbb{R}^{n,m}, Q(j) \in \mathbb{R}^{n,n},\ Q(j) \geq 0$, $R(j) \in \mathbb{R}^{m,m},\ R(j) > 0,\ \widetilde{p}(j,i) \in R,\ \widetilde{p}(j,i) \geq 0$, $\sum_{i\in S} \widetilde{p}(j,i) = 1,\ j \in S$, and S is a finite set that consists of $|S|$ elements. In the case of an infinite time interval the difference Riccati equations become the following coupled algebraic Riccati equations:

$$P(j) = \widetilde{A}'(j)\widetilde{F}(j)\widetilde{A}(j) - \widetilde{A}'(j)\widetilde{F}(j)\widetilde{B}(j) \times \left(R(j) + \widetilde{B}'(j)\widetilde{F}(j)\widetilde{B}(j)\right)^{-1} \times \widetilde{B}'(j)\widetilde{F}(j)\widetilde{A}(j) + Q(j), \tag{3}$$

where

$$\widetilde{F}(j) = \sum_{i\in S} \widetilde{p}(j,i) P(i). \tag{4}$$

The objective of this paper is to show that the solutions of both the differential (1) and algebraic (3) equations are continuous functions of their coefficients.

2. Main Result

We follow the notation of (Abou-Kandil *et al.*, 1995):

$$A(j) = \sqrt{p(j,j)}\widetilde{A}(j), B(j) = \sqrt{\widetilde{p}(j,j)}\widetilde{B}(j)R^{-1/2}(j), p(i,j) = \frac{\widetilde{p}(i,j)}{\widetilde{p}(j,j)}, \quad i,\, j \in S.$$

Using these abbreviations, we can rewrite (1) and (3) as

$$P_{k+1}(P,j) = A^{'}(j)F_k(j)A(j) - A^{'}(j)F_k(j)B(j) \times \left(I + B^{'}(j)F_k(j)B(j)\right)^{-1} \times B^{'}(j)F_k(j)A(j) + Q(j), \quad (5)$$

where

$$F_k(j) = \sum_{i\in S} p(j,i)P_k(P,i), \quad (6)$$

and

$$P(j) = A^{'}(j)F(j)A(j) - A^{'}(j)F(j)B(j)\left(I + B^{'}(j)F(j)B(j)\right)^{-1} \times B^{'}(j)F(j)A(j) + Q(j), \quad (7)$$

where

$$F(j) = \sum_{i\in S} p(j,i)P(i). \quad (8)$$

An easy computation shows that (1) and (3) can be rewritten as

$$P_{k+1}(P,j) = (A(j) - B(j)L_k(j))^{'} F_k(j) \times (A(j) - B(j)L_k(j)) + L_k^{'}(j)L_k(j) + Q(j), \quad (9)$$

where

$$L_k(j) = \left(I + B^{'}(j)F_k(j)B(j)\right)^{-1}B^{'}(j)F_k(j)A(j), \quad (10)$$

$F_k(j)$ is given by (6) and

$$P(j) = (A(j) - B(j)L(j))^{'} F(j)\big(A(j) - B(j)L(j)\big) + L^{'}(j)L(j) + Q(j). \quad (11)$$

Here

$$L(j) = \left(I + B^{'}(j)F(j)B(j)\right)^{-1}B^{'}(j)F(j)A(j) \quad (12)$$

and $F(j)$ is given by (8).

Throughout the paper we will denote by $\|X\|$ the operator norm of a matrix X. In our future deliberations, we will make the following assumptions about the coefficients $A(j), B(j), Q(j)$:

(A) There exists $\varepsilon > 0$ such that for all $\widehat{A}(j) \in \mathbb{R}^{n,n}$, $\widehat{B}(j) \in \mathbb{R}^{n,m}$, $\widehat{Q}(j) \in \mathbb{R}^{n,n}$, $\widehat{Q}(j) \geq 0$ such that $\|A(j) - \widehat{A}(j)\| < \varepsilon$, $\|B(j) - \widehat{B}(j)\| < \varepsilon$, $\|Q(j) - \widehat{Q}(j)\| < \varepsilon$, eqn. (11) with $A(j)$, $B(j)$, $Q(j)$ replaced by $\widehat{A}(j)$, $\widehat{B}(j)$, $\widehat{Q}(j)$, respectively, has a unique solution $\widehat{P}(j)$, $j \in S$.

(B) The solution $P_k(P,j)$ of (9) converges as $k \to \infty$ and

$$\lim_{k\to\infty} P_k(\widehat{P}, j) = P(j), \quad j \in S \quad (13)$$

for any initial value $\widehat{P}$.

(C) The matrices $A(j) - B(j)L(j)$, $j \in S$ are stable (i.e. all eigenvalues have absolute values less than 1).

Various conditions under which assumptions (A), (B) and (C) hold, and the connection between the stability of the matrices $A(j) - B(j)L(j)$, $j \in S$ and that of the original closed-loop system (1) are discussed in (Abou-Kandil *et al.*, 1995; Bourles *et al.*, 1990; Chizeck *et al.*, 1986).

Now we formulate the discrete-time version of the minimum property (Wonham, 1971, p. 193) of the solution of (1) and (3).

Theorem 1. (Minimum property) *Let* $\bar{P}_k(P,j)$ *and* $\bar{P}(j)$, $j \in S$ *be the solutions of (9) and (11) with* $L_k(j)$ *and* $L(j)$ *replaced by an arbitrary matrix* $\bar{L}(j) \in \mathbb{R}^{n,m}$, *respectively. Then* $P_k(P,j) \leq \bar{P}_k(P,j)$ *and* $P(j) \leq \bar{P}(j)$.

As in (Wonham, 1971, p. 193), the proof is a straightforward consequence of the fact that $L(j)$ given by (12) minimizes the right-hand side of (11) regarded as a function of $L(j)$.

Now we can formulate the continuity results.

Theorem 2. *Assume that the sequence* $(A^{(l)}(j), B^{(l)}(j), Q^{(l)}(j))$, $A^{(l)}(j) \in \mathbb{R}^{n,n}$, $B^{(l)}(j) \in \mathbb{R}^{n,m}$, $Q^{(l)}(j) \in \mathbb{R}^{n\times n}$, $j \in S$, $l \in N$ *is such that the limits of* $A^{(l)}(j), B^{(l)}(j), Q^{(l)}(j)$ *as* $l \to \infty$, *exist for each* $j \in S$,

$$A(j) = \lim_{l\to\infty} A^{(l)}(j), \quad B(j) = \lim_{l\to\infty} \lim B^{(l)}(j), \quad Q(j) = \lim_{k\to\infty} Q^{(l)}(j), \quad j \in S, \quad (14)$$

and the boundary system

$$A(j), B(j), Q(j), p(i,j); \quad i, j \in S$$

satisfies Assumptions (A)–(C). Then

$$\lim_{l\to\infty} P_k^{(l)}(P,j) = P_k(P,j), \quad j \in S,$$

where $P_k^{(l)}(P,j)$, $j \in S$, *are the solutions of the equations*

$$P_{k+1}^{(l)}(P,j) = \left(A^{(l)}(j)\right)' F_k^{(l)}(j)A^{(l)}(j) - \left(A^{(l)}(j)\right)' F_k^{(l)}(j)B^{(l)}(j) \times \left(I + \left(B^{(l)}(j)\right)' F_k^{(l)}(j)B^{(l)}(j)\right)^{-1} \times \left(B^{(l)}(j)\right)' F_k^{(l)}(j)A^{(l)}(j) + Q^{(l)}(j), \quad (15)$$

where

$$F_k^{(l)}(j) = \sum_{i\in S} p(j,i)P_k^{(l)}(P,i).$$

Proof. Fix $M > 0$. We first show that

$$\max_{k\le M, l\in N} \|P_k^{(l)}(P,j)\| \le c(P,M). \tag{16}$$

Fix $k \le M$ and let $\bar{P}_k^{(l)}(P,j)$, $j\in S$, $l\in N$ be the solution of (9) with $A(j), B(j), Q(j)$ and $L_k(j)$ replaced by $A^{(l)}(j), B^{(l)}(j), Q^{(l)}(j)$ and $L(j)$, respectively, where $L(j)$ is given by (12). By the minimum property it is sufficient to prove that (16) holds for $\bar{P}_k^{(l)}(P,j)$. Since $A(j) - B(j)L(j)$ is stable, (14) shows that there exist positive constants $a > 0, 1 > b > 0$ such that

$$\left\|\left(X^{(l)}(j)\right)^k\right\| < ab^k, \quad k \in N \tag{17}$$

for all $l \in N$, where $X^{(l)}(j) = A^{(l)}(j) - B^{(l)}(j)L(j)$. From (17) and (9) with $A(j)$, $B(j)$, $Q(j)$ and $L_k(j)$ replaced by $A^{(l)}(j), B^{(l)}(j), Q^{(l)}(j)$ and $L(j)$, respectively, (16) follows immediately.

Since the sequences $A^{(l)}(j), B^{(l)}(j), Q^{(l)}(j)$, $F_k^{(l)}(j)$ are bounded as functions of l, an easy computation shows that

$$\begin{aligned}
&\Big\|A'(j)F_k(j)B(j)\big(I + B'(j)F_k(j)B(j)\big)^{-1} \\
&\quad\times B'(j)F_k(j)A(j) - \big(A^{(l)}(j)\big)'F_k^{(l)}(j)B^{(l)}(j) \\
&\quad\times\Big(I + \big(B^{(l)}(j)\big)'F_k^{(l)}(j)B^{(l)}(j)\Big)^{-1} \\
&\quad\times\big(B^{(l)}(j)\big)'F_k^{(l)}(j)A^{(l)}(j)\Big\| \\
&\le c_1(P,M)\|A(j) - A^{(l)}(j)\| \\
&\quad + c_2(P,M)\|B(j) - B^{(l)}(j)\| \\
&\quad + c_3(P,M)\sum_{i\in S}\|P_k(P,i) - P_k^{(l)}(P,i)\|,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\Big\|A'(j)F_k(j)A(j) - \big(A^{(l)}(j)\big)'F_k^{(l)}(j)A^{(l)}(j)\Big\| \\
&\quad\le c_4(P,M)\|A(j) - A^{(l)}(j)\| \\
&\qquad + c_5(P,M)\sum_{i\in S}\|P_k(P,i) - P_k^{(l)}(P,i)\|,
\end{aligned} \tag{19}$$

for some non-negative constants $c_1(P,M)$, $c_2(P,M)$, $c_3(P,M)$, $c_4(P,M)$ and $c_5(P,M)$. Taking into account (18), (19), and subtracting (15) from (5), we get

$$\begin{aligned}
&\|P_{k+1}(P,j) - P_{k+1}^{(l)}(P,j)\| \\
&\quad\le c_6(P,N)\sum_{i\in S}\|P_k(P,i) - P_k^{(l)}(P,i)\| \\
&\qquad + f(P,M,l),
\end{aligned} \tag{20}$$

where $c_6(P,M) = c_3(P,M) + c_5(P,M)$ and

$$\begin{aligned}
&f(P,N,l) \\
&\quad = \big(c_1(P,M) + c_4(P,M)\big)\max_{j\in S}\|A(j) - A^{(l)}(j)\| \\
&\qquad + c_2(P,M)\max_{j\in S}\|B(j) - B^{(l)}(j)\| \\
&\qquad + \max_{j\in S}\|Q(j) - Q^{(l)}(j)\|.
\end{aligned}$$

Note that

$$\lim_{l\to\infty} f(P,M,l) = 0. \tag{21}$$

Set $Y(P,k,l) = \sum_{i\in S}\|P_k(P,i) - P_k^{(l)}(P,i)\|$. From (20) we have

$$Y(P,k+1,l) \le |S|\, c_6(P,N)Y(P,k,l) + |S|\, f(P,N,l). \tag{22}$$

Since $Y(P,0,l) = 0$, (22) shows that

$$Y(P,k,l) \le |S|\, f(P,M,l)\sum_{\nu=0}^{l-1}\big(|S|\, c_6(P,N)\big)^{\nu},$$

and

$$\lim_{l\to\infty} Y(P,k,l) = 0, \tag{23}$$

by (21). Formula (23) makes it obvious that $\lim_{l\to\infty} P_k^{(l)}(P,j) = P_k(P,j)$, $j\in S$. ■

Theorem 3. *Let Assumptions (A)–(C) be fulfilled. Then there exists $l_0 \in N$ such that for all $l \ge l_0$ the coupled Riccati equation*

$$\begin{aligned}
P^{(l)}(j) &= \big(A^{(l)}(j)\big)'F^{(l)}(j)A^{(l)}(j) \\
&\quad - \big(A^{(l)}(j)\big)'F^{(l)}(j)B^{(l)}(j) \\
&\quad \times\big(I + \big(B^{(l)}(j)\big)'F^{(l)}(j)B^{(l)}(j)\big)^{-1} \\
&\quad \times\big(B^{(l)}(j)\big)'F^{(l)}(j)A^{(l)}(j) + Q^{(l)}(j),
\end{aligned} \tag{24}$$

where

$$F^{(l)}(j) = \sum_{i\in S} p(j,i)P^{(l)}(i), \tag{25}$$

has a solution $P^{(l)}(j),\ j \in S$, *and*

$$\lim_{l\to\infty} P^{(l)}(j) = P(j), \quad j \in S,$$

$P(j),\ j \in S$ *being the solutions of (7).*

Proof. The existence of l_0 is ensured by Assumption (A). Assume now that $l > l_0$. An analysis similar to that in the proof of (16) shows that

$$\sup_{l\in N} \left\|P^{(l)}(j)\right\| < \infty, \quad j \in S.$$

By Assumption (A) we know that the matrices $A^{(l)}(j) - B^{(l)}(j)L^{(l)}(j),\ j \in S$ are stable, where $L^{(l)}(j) = (I + (B^{(l)}(j))'F^{(l)}(j)B^{(l)}(j))^{-1}(B^{(l)}(j))'F^{(l)}(j)A^{(l)}(j)$ and $F^{(l)}(j)$ are given by (25). Let $\widetilde{P}_k^{(l)}(P,j)$ be the solution of (9) with $A(j), B(j), Q(j)$ and $L_k(j)$ replaced by $A^{(l)}(j), B^{(l)}(j), Q^{(l)}(j)$ and $L(j)$, respectively, where $L(j)$ is given by (12). From the minimum property Assumption (A) we have

$$P^{(l)}(j) \le P_k^{(l)}(P,j) \le \widetilde{P}_k^{(l)}(P,j)$$

and

$$\left\|P_k^{(l)}(P,j) - P^{(l)}(j)\right\| \le \left\|\widetilde{P}_k^{(l)}(P,j) - P^{(l)}(j)\right\|. \quad (26)$$

Set $M_k^{(l)}(P,j) = \widetilde{P}_k^{(l)}(P,j) - P^{(l)}(j)$. An easy computation shows that

$$M_{k+1}^{(l)}(P,j) = \big(\widetilde{X}^{(l)}(j)\big)' F_k^{(l)}(j)\widetilde{X}^{(l)}(j), \quad (27)$$

where $\widetilde{X}^{(l)}(j) = A^{(l)}(j) - B^{(l)}(j)L^{(l)}(j)$, $F_k^{(l)}(j) = \sum_{i\in S} p(j,i)M_k^{(l)}(P,i)$. The stability of the matrices $\widetilde{X}^{(l)}(j)$ implies that there exist positive constants $\widetilde{a}, \widetilde{b}$ such that

$$\left\|\big(\widetilde{X}^{(l)}(j)\big)^k\right\| < \widetilde{a}\widetilde{b}^k, \quad k \in \mathbb{N}. \quad (28)$$

Fix $\varepsilon > 0$. By (26)–(28) and the bound on the sequence $M_k^{(l)}(P,j),\ l \in \mathbb{N}$, there exists $T_1(\varepsilon) > 0$ such that

$$\left\|P_k^{(l)}(P,j) - P^{(l)}(j)\right\| < \frac{\varepsilon}{3} \quad (29)$$

for $l \in \mathbb{N},\ k > T_1(\varepsilon)$. Assumption (B) implies that there is a $T_2(\varepsilon) > 0$ such that

$$\left\|P_k(P,j) - P(j)\right\| < \frac{\varepsilon}{3} \quad (30)$$

for any $k > T_2(\varepsilon)$. Let $T(\varepsilon) = \max\{T_1(\varepsilon), T_2(\varepsilon)\}$. According to Theorem 2 there is $l_0(\varepsilon, T(\varepsilon))$ such that

$$\left\|P_{T(\varepsilon)}^{(l)}(P,j) - P_{T(\varepsilon)}(j)\right\| < \frac{\varepsilon}{3} \quad (31)$$

for $l > l_0(\varepsilon, T(\varepsilon))$. Finally,

$$\begin{aligned}\left\|P^{(l)}(j) - P(j)\right\| &\le \left\|P^{(l)}(j) - P_{T(\varepsilon)}^{(l)}(P,j)\right\| \\ &\quad + \left\|P_{T(\varepsilon)}^{(l)}(P,j) - P_{T(\varepsilon)}(j)\right\| \\ &\quad + \left\|P_{T(\varepsilon)}(j) - P(j)\right\| < \varepsilon,\end{aligned}$$

which completes the proof. ■

Having in mind the equivalence of Cauchy's and Heine's definitions of limits, we can reformulate Theorems 2 and 3 as follows:

Corollary 1. *If Assumptions (A)–(C) are satisfied, then for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $\widehat{A}(j) \in \mathbb{R}^{n,n}$, $\widehat{B}(j) \in \mathbb{R}^{n,m}$, $\widehat{Q}(j) \in \mathbb{R}^{n,n}$, $\widehat{Q}(j) \ge 0$, $\|A(j) - \widehat{A}(j)\| < \delta$, $\|B(j) - \widehat{B}(j)\| < \delta$, $\|Q(j) - \widehat{Q}(j)\| < \delta$ *we have* $\|\widehat{P}_k(P,j) - P_k(P,j)\| < \varepsilon$ *and* $\|\widehat{P}(j) - P(j)\| < \varepsilon$, *where* $P_k(P,j)$ *and* $P(j)$ *are solutions of (5) and (11), respectively, and* $P_k(P,j)$ *and* $P(j)$ *are solutions of (5) and (11) with* $A(j)$, $B(j)$ *and* $Q(j)$ *replaced by* $\widehat{A}(j)$, $\widehat{B}(j)$ *and* $\widehat{Q}(j)$, *respectively.*

3. Conclusions

In this paper the continuity of solutions of algebraic and difference Riccati equations as functions of coefficients is verified. Continuity is important in applications to problems of adaptive control of stochastic systems, see, e.g., (Chen, 1985; Czornik, 1996). It may also be useful for a sensitivity (robustness) analysis of linear systems with jumps. Since the problem of solutions for coupled Riccati equations is not trivial, see, e.g., (Chizeck *et al.*, 1986), we hope that the material presented in this paper may play a role in establishing efficient numerical algorithms.

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