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A CONTEXT–BASED APPROACH TO LINGUISTIC HEDGES

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We present a framework of L-fuzzy modifiers for L being a complete lattice. They are used to model linguistic hedges that act on linguistic terms represented by L-fuzzy sets. In the modelling process the context is taken into account by means of L-fuzzy relations, endowing the L-fuzzy modifiers with a clear inherent semantics. To our knowledge, these L-fuzzy modifiers are the first ones proposed that are suitable to perform this representation task for a lattice L different from the unit interval. In the latter case they undoubtedly outperform the traditional representations, such as powering and shifting hedges, from the semantical point of view.

Keywords: L-fuzzy modifier, L-fuzzy relation, resemblance relation, linguistic hedge, linguistic variable

1. Introduction

Computing with words had his biggest impulse with the introduction of the concept of a linguistic variable (Zadeh, 1975) and the representation of its values by means of fuzzy sets (Zadeh, 1965). In contrast to numerical variables, whose values are numbers, the values of a linguistic variable are linguistic terms that allow us to capture the vagueness present in human perceptions of the real world. A nice feature of linguistic variables is that their values are structured, which makes it possible to compute the representations of composed linguistic terms from those of their composing parts.

Although some exceptions can be found (see, e.g., Wei *et al.*, 2000) in this context a basic linguistic term is usually an adjective (e.g. beautiful). Composed terms can be generated by applying either a linguistic hedge to a term (e.g. very beautiful) or by combining two terms by means of a conjunction (e.g. very beautiful but stupid). It is clear that the class of linguistic terms obtained in this way does not correspond to the whole of a natural language, but it still covers its remarkably expressive part.

The construction of suitable fuzzy sets for the linguistic terms involved is typically one of the most difficult tasks when building an application. A good representation of the connectives and the linguistic hedges is therefore desired, since it allows for the automatic deduction of new fuzzy sets from known ones. In this paper we will focus on the latter: the representation of linguistic hedges (also called "linguistic modifiers") by fuzzy modifiers (also called "fuzzy hedges"), i.e. operators that transform a fuzzy set into another.

In the greater part of the literature and applications linguistic terms are modelled by means of fuzzy sets taking membership degrees in the unit interval $[0, 1]$. The construction of a fuzzy set corresponding to a linguistic term has a large degree of freedom: we can choose from many possible parametrized shape functions (see, e.g., Kerre, 1993) and, moreover, the parameters can often be arbitrarely chosen within some small domain, all this with negligible or no impact on the functionality of the systems which use this model of linguistic terms. The exact numerical values of the membership degrees are usually neither justifiable, nor important; only the values of 0, 0.5, and 1 have a special intuitive meaning, while the others are mostly only shots. Often this is explained away by saying that the only really important thing is that the membership degrees induce a graded ordering of "belonging to" on the set of objects, which is the power of fuzzy set theory.

Indeed, the mapping of elements of the universe to the interval $[0, 1]$ implies a crisp, linear ordering of these elements. However, there also exists incomparable information in the real world (Xu *et al.*, 1999), within which there are linguistic terms that do not correspond to a linear ordering on the universe. It is clear that $[0, 1]$-valued fuzzy set theory is inadequate to deal with noncomparable information. The key to the solution, however, was present from the very start of fuzzy set theory. In fact, in 1965, in his seminal paper Zadeh (1965) included the footnote: "*In a more general setting, the range of the membership function can be taken to be a suitable partially ordered set P.*"

The unit interval was always and still is considered to be a very natural and transparant set of membership degrees (Novák *et al.*, 1999), and it is used in a majority of applications. From the beginning, however, some attention has been paid to other partially ordered sets as well. In 1967 Goguen formally introduced the notion of an L-fuzzy set with a membership function taking values in a lattice L (Goguen, 1967). The concept of an L-fuzzy set is still studied albeit mostly on a theoretical level, with some practical interest limited mainly to type-2 fuzzy sets. Since a lot of information in the real world is incomparable, we believe that there is still a huge gap on the market of applications, to be filled by systems able to represent this kind of information. In this paper we walk in this direction, deepening the research started in (De Cock *et al.*, 2001).

As we will explain later on, the popular $[0,1]$-fuzzy modifiers—such as powering and shifting hedges—that are traditionally used to model linguistic modifiers, cannot be generalized straightforwardly to the L-fuzzy case. On the other hand, the $[0,1]$-fuzzy modifiers based on fuzzy relations as proposed in (De Cock and Kerre, 2000; 2002a) lend themselves extremely well to an L-fuzzy generalization. In this paper we will present the research bottom-up, first introducing L-fuzzy modifiers based on L-fuzzy relations, and then studying the $[0,1]$-fuzzy modifiers based on $[0,1]$-fuzzy relations as an interesting special case. We will also point out how and why they outperform the traditional representations.

2. Preliminaries

An algebraic structure $(L, \vee, \wedge)$ consisting of a non-empty set L and two binary operations $\vee$ and $\wedge$ on L is called a *lattice* iff for all a, b and c in L

(L.1) $a \wedge a = a$,
(L.1′) $a \vee a = a$,
(L.2) $a \wedge b = b \wedge a$,
(L.2′) $a \vee b = b \vee a$,
(L.3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
(L.3′) $a \vee (b \vee c) = (a \vee b) \vee c$,
(L.4) $a \wedge (a \vee b) = a$,
(L.4′) $a \vee (a \wedge b) = a$.

$(L, \vee, \wedge)$ is usually abbreviated to L, tacitly assuming the presence of the join-operation $\vee$ and the meet-operation $\wedge$. Every lattice is a partially ordered set with the ordering defined by $a \le b$ iff $a \vee b = b$ (or, equivalently, $a \le b$ iff $a \wedge b = a$), for all a and b in L. A corresponding strict ordering can be defined by $a < b$ iff ($a \le b$ and $a \neq b$). Throughout the remainder of this paper, let L denote a *complete* lattice, i.e. every subset of L has a least upper bound (supremum) and a greatest lower bound (infimum). The smallest and the greatest elements of L will be denoted by 0 and 1, respectively.

Triangular norms were originally introduced by Schweizer and Sklar (1961) in the framework of probabilistic metric spaces. $[0,1]$-fuzzy set theory eagerly uses them to represent the intersection of fuzzy sets, as well as to model connectives such as and, but and or. Several authors studied the generalization of triangular norms and implicators from the real unit interval to partially ordered sets (De Cooman and Kerre, 1994; Drossos and Navara, 1997; De Baets and Mesiar, 1999). We will comply with the following definitions:

Definition 1. *(Triangular norm)* A *triangular norm* (t-norm for short) $\mathcal{T}$ on L is an associative, commutative and increasing $L^2 - L$ mapping $\mathcal{T}$ satisfying the boundary condition $\mathcal{T}(a,1) = a$ for all a in L.

Definition 2. *(Triangular conorm)* A *triangular conorm* (t-conorm for short) $\mathcal{S}$ on L is an associative, commutative and increasing $L^2 - L$ mapping $\mathcal{S}$ satisfying the boundary condition $\mathcal{S}(a,0) = a$ for all a in L.

Definition 3. *(Residual implicator)* Let $\mathcal{T}$ be a t-norm on L. If $\mathcal{I}_\mathcal{T}$ is an $L^2 - L$ mapping such that $\mathcal{T}(a,b) \le c$ iff $a \le \mathcal{I}_\mathcal{T}(b,c)$ for all a, b and c in L, then $\mathcal{I}_\mathcal{T}$ is called the *residual implicator* on L induced by $\mathcal{T}$.

If there exists a residual implicator $\mathcal{I}_\mathcal{T}$ induced by a triangular norm $\mathcal{T}$, then

$$\mathcal{I}_\mathcal{T}(x,y) = \sup\{\lambda \mid \lambda \in L \text{ and } \mathcal{T}(x,\lambda) \le y\}$$

for all x and y in X. The structure $(L, \vee, \wedge, \mathcal{T}, \mathcal{I}_\mathcal{T}, 0, 1)$ is then usually referred to as a residuated lattice. For a detailed overview of residuated lattices and their properties, we refer the reader to (Novák *et al.*, 1999). Furthermore, we also note that there is a strong connection between residuated lattices and the well-studied concept of a lattice implication algebra (Xu *et al.*, 2000), which is also a source of inspiration for properties regarding t-norms and implicators. From now on we will assume that $\mathcal{T}$ is a t-norm, with residual implicator $\mathcal{I}_\mathcal{T}$, and that X is the universe of discourse, i.e. the crisp set of objects that we want to discuss. Before arriving at the main topic of this paper, we only need to recall some definitions from L-fuzzy set theory.

Definition 4. *(L-fuzzy set)* An *L-fuzzy set* A on X is a mapping from X to L, also called the *membership function* of A. For all x in X, $A(x)$ is called the *membership degree* of x in A. The class of all L-fuzzy sets on X is denoted by $\mathcal{F}_\mathcal{L}(X)$. The *kernel* and the *support* of A are respectively defined as $\ker(A) = \{x \mid x \in X \text{ and } A(x) = 1\}$ and $\text{supp}(A) = \{x \mid x \in X \text{ and } A(x) > 0\}$.

Definition 5. *($\mathcal{T}$-intersection)* For A and B L-fuzzy sets on X, the *$\mathcal{T}$-intersection* of A and B is the L-fuzzy set on X defined by $(A \cap_{\mathcal{T}} B)(x) = \mathcal{T}(A(x), B(x))$ for x in X.

Furthermore, we will write $\mathcal{I}_{\mathcal{T}}(A, B)$ for the pointwise extension of $\mathcal{I}_{\mathcal{T}}$ to L-fuzzy sets, i.e. $\mathcal{I}_{\mathcal{T}}(A, B)(x) = \mathcal{I}_{\mathcal{T}}(A(x), B(x))$ for all x in X.

Definition 6. *(Inclusion)* For A and B L-fuzzy sets on X, A is included in B iff $A(x) \leq B(x)$ for all x in X. As usual, we denote this by $A \subseteq B$.

Definition 7. *(L-fuzzy relation)* An L-fuzzy relation R on X is an L-fuzzy set on $X \times X$, i.e. an element of $\mathcal{F}_{\mathcal{L}}(X \times X)$. Furthermore,

1. R is *reflexive* iff $R(x, x) = 1$ for all x in X.
2. R is *symmetrical* iff $R(x, y) = R(y, x)$ for all x and y in X.
3. R is *$\mathcal{T}$-transitive* iff $\mathcal{T}(R(x, y), R(y, z)) \leq R(x, z)$, for all x, y and z in X.
4. R is a *fuzzy $\mathcal{T}$-equivalence* relation iff R satisfies 1, 2 and 3.

For all y in X, the *R-foreset* of y is the L-fuzzy set Ry on X defined by $Ry(x) = R(x, y)$ for all x in X.

From now on, we will also refer to $[0, 1]$-fuzzy set theory ($[0, 1]$-fuzzy sets, $[0, 1]$-fuzzy relations, etc.) as fuzzy set theory (fuzzy sets, fuzzy relations, etc.), tacitly assuming that the membership values are taken from the real unit interval. The class of all fuzzy sets on X will be denoted by $\mathcal{F}(X)$.

3. *L*-fuzzy modifiers based on *L*-fuzzy relations

The direct image $R(A)$ of a fuzzy set A under a fuzzy relation R is a well-known concept in fuzzy set theory (Kerre, 1993). Although many researchers might not be familiar with the terminology "direct image", they will still recognize the formula as a result of the compositional rule of inference, often referred to as the unary composition of A and R (usually denoted by $A \circ R$). In (De Cock *et al.*, 2000) it is shown that this and other kinds of images of fuzzy sets under fuzzy relations have a remarkably high amount of applications, ranging from fuzzy databases over fuzzy rough sets and fuzzy morphology for image processing, to the representation of linguistic modifiers. The "ingredients" of the fuzzy images are fuzzy logical operators such as a t-norm and an implicator, as well as the fuzzy relations themselves. Since these ingredients can be easily generalized to the L-fuzzy case as we recalled in the previous section, all kinds of L-fuzzy relational images can be defined as well. Note that this opens the door to widening all the applications mentioned above to domains of incomparable information. However, in this paper we focus only on the representation of linguistic modifiers by L-fuzzy modifiers.

Definition 8. *(L-fuzzy modifier)* An *L-fuzzy modifier* on X is a mapping from $\mathcal{F}_{\mathcal{L}}(X)$ to $\mathcal{F}_{\mathcal{L}}(X)$.

Definition 9. *(L-fuzzy modifiers based on L-fuzzy relations)* For every L-fuzzy relation R on X, the *L-fuzzy modifiers* $R^{\clubsuit}$ and $R^{\heartsuit}$ are defined as

$$R^{\clubsuit}(A)(y) = \sup_{x \in X} \mathcal{T}\big(R(x, y), A(x)\big),$$

$$R^{\heartsuit}(A)(y) = \inf_{x \in X} \mathcal{I}_{\mathcal{T}}\big(R(x, y), A(x)\big),$$

respectively, for all A in $\mathcal{F}_{\mathcal{L}}(X)$ and for all y in X.

For ease of presentation, we do not mention the triangular norm and residual implicator used in the shorthand notation of fuzzy modifiers. Note that we can rewrite the formulae above as

$$R^{\clubsuit}(A)(y) = \sup_{x \in X} (Ry \cap_{\mathcal{T}} A)(x),$$

$$R^{\heartsuit}(A)(y) = \inf_{x \in X} \mathcal{I}_{\mathcal{T}}(Ry, A)(x).$$

They can be interpreted as the degree to which Ry and A overlap, and the degree to which Ry is included in A, respectively. The assumption that the lattice L of membership degrees under study is complete guarantees the existence of the supremum and the infimum in the formulae above. Here $\clubsuit$ and $\heartsuit$ can be seen as operators that act on an L-fuzzy relation R and an L-fuzzy set A, and turn them into an L-fuzzy set, namely $R^{\clubsuit}(A)$ and $R^{\heartsuit}(A)$, respectively. In (Orłowska and Radzikowska, 2001), $\clubsuit$ and $\heartsuit$ are studied under the name "fuzzy information operators." The following two propositions regarding entailment are given in (Orłowska and Radzikowska, 2001):

Proposition 1. (Entailment) *R is a reflexive L-fuzzy relation on X iff for all A in $\mathcal{F}_{\mathcal{L}}(X)$*

$$R^{\heartsuit}(A) \subseteq A \subseteq R^{\clubsuit}(A).$$

Proposition 2. (Inverse entailment) *R is a $\mathcal{T}$-transitive L-fuzzy relation on X iff for all A in $\mathcal{F}_{\mathcal{L}}(X)$*

$$R^{\heartsuit}\big(R^{\heartsuit}(A)\big) \supseteq R^{\heartsuit}(A),$$

$$R^{\clubsuit}\big(R^{\clubsuit}(A)\big) \subseteq R^{\clubsuit}(A).$$

Corollary 1. (Idempotency) *If R is a reflexive, $\mathcal{T}$-transitive L-fuzzy relation on X, then for all A in $\mathcal{F}_{\mathcal{L}}(X)$*

$$R^{\heartsuit}\big(R^{\heartsuit}(A)\big) = R^{\heartsuit}(A),$$
$$R^{\clubsuit}\big(R^{\clubsuit}(A)\big) = R^{\clubsuit}(A).$$

The following propositions are generalizations of the ones studied in (De Cock and Kerre, 2002a).

Proposition 3. (Behaviour w.r.t. the kernel) *For all R in $\mathcal{F}_{\mathcal{L}}(X \times X)$, for all A in $\mathcal{F}_{\mathcal{L}}(X)$, and for all y in X, the following holds:*

1. *If $\ker(A) \cap \ker(Ry) \neq \emptyset$ then $y \in \ker(R^{\clubsuit}(A))$.*
2. *If $\mathrm{co}(\ker(A)) \cap \ker(Ry) \neq \emptyset$ then $y \notin \ker(R^{\heartsuit}(A))$.*
3. *If $Ry \subseteq A$ then $y \in \ker(R^{\heartsuit}(A))$.*

Proof.

1. According to the condition, there exists an x_0 in X such that $A(x_0) = 1$ and $R(x_0, y) = 1$. Hence $R^{\clubsuit}(A)(y) \geq \mathcal{T}(R(x_0, y), A(x_0)) = 1$.
2. According to the condition, there exists an x_0 in X such that $A(x_0) < 1$ and $R(x_0, y) = 1$. Hence $R^{\heartsuit}(A)(y) \leq \mathcal{I}_{\mathcal{T}}(R(x_0, y), A(x_0)) < 1$.
3. The result follows from Definition 6 and

$$\mathcal{I}_{\mathcal{T}}(x, y) = 1 \text{ iff } x \leq y. \qquad \blacksquare$$

Proposition 4. (Behaviour w.r.t. the support) *For all R in $\mathcal{F}_{\mathcal{L}}(X \times X)$, for all A in $\mathcal{F}_{\mathcal{L}}(X)$, and for all y in X the following holds:*

1. *If $\mathrm{supp}(A) \cap \mathrm{supp}(Ry) \neq \emptyset$ then we have $y \in \mathrm{supp}(R^{\clubsuit}(A))$, provided that $\mathcal{T}$ has no zero divisors (i.e. $a \neq 0$ and $b \neq 0$ imply $\mathcal{T}(a, b) \neq 0$).*
2. *If $\mathrm{co}(\mathrm{supp}(A)) \cap \ker(Ry) \neq \emptyset$ then we have $y \notin \mathrm{supp}(R^{\heartsuit}(A))$.*

Proof.

1. According to the condition, there exists an x_0 in X such that $A(x_0) > 0$ and $R(x_0, y) > 0$. Since $\mathcal{T}$ has no zero divisors, $0 < \mathcal{T}(R(x_0, y), A(x_0)) \leq R^{\clubsuit}(A)(y)$.
2. According to the condition, there exists an x_0 in X such that $A(x_0) = 0$ and $R(x_0, y) = 1$. Hence $0 = \mathcal{I}_{\mathcal{T}}(R(x_0, y), A(x_0)) = R^{\heartsuit}(A)(y)$. $\blacksquare$

All of these properties have an important interpretation when representing linguistic modifiers by L-fuzzy modifiers, as we will discuss later on.

4. Representing Linguistic Hedges

In a fuzzy set theoretical context a linguistic term is usually represented by a fuzzy set A on a universe X, characterized by an $X - [0, 1]$ mapping (for simplicity also denoted by A), which is called the membership function. Hence for every x in X, $A(x)$ is the membership degree of x in the fuzzy set A and it may vary between 0 and 1. This graded approach makes fuzzy set theory extremely suitable to model linguistic terms which are often inherently vague. As indicated in the Introduction, however, the representation of a term by a $[0, 1]$-fuzzy set forces total ordering on the objects of the universe, which might not be desirable if the universe contains incomparable objects. Imagine, e.g., that a man cannot say which one of two women is prettier than the other. Assigning to both of them a membership degree between 0 and 1 in the fuzzy set 'pretty' inevitably implies ordering among these women w.r.t. being pretty. At first sight the best way out is to give them the same degree of membership. However, that this is only a naive solution becomes apparent when the older sister of the first woman comes along and turns out to be prettier than her younger sibling but incomparable with the second woman. To overcome this kind of problems, we may represent linguistic terms by L-fuzzy sets, since in this case the membership degrees are not necessarily linearly ordered.

As we already mentioned in the Introduction, constructing proper fuzzy sets is a difficult part of establishing a working application which usually involves expert knowledge. Fortunately, the concept of a fuzzy set is straightforward and can be grasped quite easily by experts of all kinds of domains, but still the construction of membership functions often remains rather *ad hoc* and subjective. Having a representation for some linguistic hedges such as very, more or less, extremely, ... can facilitate the task. Indeed, using these representations one can significantly increase the number of available fuzzy sets automatically. During the last three decades, many $[0, 1]$-fuzzy modifiers were proposed for the representation of linguistic hedges acting on terms represented by $[0, 1]$-fuzzy sets (see (Kerre and De Cock, 1999) for an overview). Almost all of them can be categorized as modifiers with pure premodification, pure postmodification, or a combination of both.

Definition 10. *(Pre- and postmodification)* A fuzzy modifier m on X is *decomposable in a pre- and a post-modifier* if there exists an $X - X$ mapping t and a $[0, 1] - [0, 1]$ mapping r such that for all A in $\mathcal{F}(X)$

$$m(A) = r \circ A \circ t.$$

In other words,

$$(\forall x \in X)\ \big(m(A)(x) = r\big(A(t(x))\big)\big)\,,$$

where t is called the premodifier of m, while r is called the postmodifier of m. If t is the identity $X - X$ mapping, then m is called a *modifier with pure postmodification*. If, on the other hand, r is the identity $[0,1] - [0,1]$ mapping, then m is called a *modifier with pure premodification*.

A very popular class of modifiers with pure postmodification are the powering hedges P_α defined by

$$P_\alpha(A)(x) = A(x)^\alpha$$

for A in $\mathcal{F}(X)$, x in X, and α a positive real number (Zadeh, 1972). Well-known modifiers with pure premodification are shifting modifiers, usually defined on $\mathbb{R}$ by

$$S_\alpha(A)(x) = A(x - \alpha)$$

for A in $\mathcal{F}(\mathbb{R})$, x in $\mathbb{R}$, and α a real number (Lakoff, 1973; Hellendoorn, 1990). The shifting modifiers are usually defined on $\mathbb{R}$ because, just like all modifiers with premodification, they need to perform an operation t on the universe of discourse. Since the set of real numbers is equipped with many well-known operations, it is a popular candidate. In many fuzzy control systems nowadays the universe is numerical (i.e. a suitable subset of $\mathbb{R}$). Allowing and studying non-numerical universes as well might open the door to an uncultivated area of applications, especially those in which the input and output are neither given by, nor intended for a measuring instrument or a machine. In such universes, however, suitable operations do not always grow on trees. Furthermore, if such a universe contains incomparable information, the application of shifting hedges becomes even less straightforward.

Unfortunately, modifiers with pure postmodification cannot simply save the day either. Unlike those with pure premodification, they act on the membership degrees themselves. The real unit interval is equipped with a vast amount of well-known operations, but arbitrary lattices are not. What is more, we do not know a variant of the powering operation on an arbitrary lattice. Needless to say, a combination of both kinds of modifiers only increases the difficulties since one has to find solutions for the problems on both the levels (in the universe and in the set of membership values).

When computing the degree to which y is very A, for example, powering modifiers (and all modifiers with pure postmodification) only look at the degree to which y is A. They completely ignore the other objects of the universe and their degree of belonging to A. Shifting modifiers do not even look at the degree to which y is A, but only to the degree to which some other object z is A (if the membership function of A is increasing (resp. decreasing) then z will be to the left (resp. to the right) of y). In this paper we advocate the use of fuzzy modifiers that look at the degree to which y is A, but which also take into account the objects in the *context* of y. The context of y is defined as the objects that are related to y by some relation R that models an approximate equality. This relation is intrinsically fuzzy because the approximate equality is a vague concept. Hence, as a natural result, the context of y will be an L-fuzzy set.

4.1. Resemblance Relations

The approximate equality is a vague concept: the transition between *being* approximately equal and *not being* approximately equal is not abrupt but gradual. Every object is approximately equal to itself to the highest degree (i.e. 1), but two objects can also be approximately equal to degree 1 even if they are not exactly equal. Furthermore, the greater the distance between two objects, the less they are approximately equal, and vice versa. Consider, e.g. children in a primary school. Those of the same class are usually approximately equal w.r.t. the age (i.e. “of the same age”) although a very small minority of them are born on exactly the same day. Those of the first and final years are not usually approximately equal (i.e. approximately equal to degree 0) w.r.t. their age. On the other hand, children in two successive years might be considered as being of the same age to some degree which is smaller than 1 but greater than 0.

In (De Cock and Kerre, 2001; 2002b) it is shown that fuzzy $\mathcal{T}$-equivalence relations are not suitable to model this kind of approximate equality. An important evidence for this statement, related to the Poincaré paradox (Poincaré, 1904), arises when objects can be approximately equal to degree 1 without being exactly equal. If $(x_1, x_2, x_3, \ldots, x_n)$ is a chain in the universe such that two successive elements are approximately equal to degree 1 (i.e. $E(x_i, x_{i+1}) = 1$), one can prove that x_1 and x_n are approximately equal to degree 1 when modelling E by means of a fuzzy $\mathcal{T}$-equivalence relation. Usually this is not desired. Imagine, e.g., that the elements of a chain are girls that can be ranked in beauty from the ugliest one to the prettiest one. Although girls standing next to each other might be approximately equal to degree 1 in beauty, it is likely that there is a big difference between the first and the last one. A similar reasoning can be made when ranking the children of the primary school from the youngest to the eldest one.

Since this paradoxal result is due to the $\mathcal{T}$-transitivity, we suggest to omit this condition in any case when looking for an L-fuzzy relation to model the approximate equality. If the universe X is equipped with a meaningful $\mathcal{S}$-pseudo-metric d, $\mathcal{T}$-transitivity can be replaced by a condition based on it, reflecting the above mentioned duality between the distance and the approximate equality.

Definition 11. *($\mathcal{S}$-pseudo-metric)* Let $\mathcal{S}$ be a t-conorm on L. An $X^2 - L$ mapping d is called an *$\mathcal{S}$-pseudo-metric* on X iff for all x, y and z in X:

(SPM.1) $d(x,x) = 0$,
(SPM.2) $d(x,y) = d(y,x)$,
(SPM.3) $\mathcal{S}(d(x,y), d(y,z)) \geq d(x,z)$.

The couple (X, d) is called an *$\mathcal{S}$-pseudo-metric space*.

The class of $[0,1]$-valued $\mathcal{S}_L$-pseudo-metrics (based on the Łukasiewicz t-conorm $\mathcal{S}_L$, defined as $\mathcal{S}_L(x,y) = \min(x+y, 1)$ for all x and y in $[0,1]$) coincides with the class of $[0,1]$-valued pseudo-metrics.

Definition 12. *(Resemblance relation)* An L-fuzzy relation R on an $\mathcal{S}$-pseudo-metric space (X,d) is called a *resemblance relation* iff for all x, y, z and u in X

(R.1) $R(x,x) = 1$,
(R.2) $d(x,y) \leq d(z,u)$ implies $R(x,y) \geq R(z,u)$.

Note that condition (R.2) implies the symmetry of R. If it is not straightforwardly clear which meaningful $\mathcal{S}$-pseudo-metric is available on the universe, a mapping g from X to a $\mathcal{S}$-pseudo-metric space $(\mathcal{M}, d)$ might do the trick, replacing condition (R.2) by

$$d\big(g(x), g(y)\big) \leq d\big(g(z), g(u)\big) \text{ implies } R(x,y) \geq R(z,u).$$

As is pointed out in (Klawonn, 2002), the definition of the resemblance relation covers a broad class of fuzzy relations. It is certainly not as restrictive as that of the fuzzy $\mathcal{T}$-equivalence relation, in the sense that some clear intuitive examples of the approximate equality which cannot be modelled by fuzzy $\mathcal{T}$-equivalence relations can be represented by resemblance relations. Whether resemblance relations are strong enough to enforce a meaningful concept of approximate equality (Bodenhofer, 2002) highly depends on the meaningfulness of the $\mathcal{S}$-pseudo-metric d used in the definition. From now on, when we use the term "resemblance relation", we assume that it is an acceptable intuitive model of the approximate equality.

If R is a resemblance relation on X then for all y in X, the R-foreset of y, i.e. Ry, is the L-fuzzy set of objects resembling y. As we will illustrate in the following, Ry is a suitable context of y when modelling the weakening hedges more or less and roughly and the intensifying hedges very and extremely by means of the L-fuzzy modifiers based on L-fuzzy relations presented in Section 3. Our goal is to establish representations for these linguistic hedges that respect the semantic entailment (Lakoff 1973), i.e. such that for all A in $\mathcal{F}_{\mathcal{L}}(X)$

$$\text{extremely } A \subseteq \text{very } A \subseteq A \subseteq \text{more or less } A \subseteq \text{roughly A}, \quad (1)$$

which is often assumed in the literature on fuzzy set theory (Novák and Perfilieva, 1999).

4.2. Weakening Hedges: 'More or Less' and 'Roughly'

We could say that somebody is more or less adult "if he resembles an adult". Likewise, a park is more or less large "if it resembles a large park". In general, y is more or less A if y resembles an x that is A. Hence we can say that y is more or less A if the intersection of A and Ry is not empty, or to state it in a more fuzzy manner: y is more or less A to the degree to which Ry and A overlap, i.e.

$$\text{more or less } A(y) = R^{\clubsuit}(A)(y).$$

The semantics of this representation becomes even clearer if A is a crisp singleton, i.e. $A(z) = 1$ for some z in X and $A(x) = 0$ for all other x in X. We also denote this by $A = \{z\}$. In this case for all y in X, more or less $A(y)$ is equal to $R(z,y)$. In other words y is more or less $\{z\}$ to the degree to which z resembles y. Due to the reflexivity of R (every object is approximately equal to itself to the highest degree) and Proposition 1, the relevant inclusion of (1) holds, i.e.

$$A \subseteq \text{more or less } A,$$

meaning that every object that is A is also more or less A to the same or to a higher degree. If R would have been $\mathcal{T}$-transitive, Proposition 2 would hold as well, implying that more or less more or less A would be the same as more or less A. As a corollary of the reflexivity of R, for every non $\mathcal{T}$-transitive resemblance relation R we have $R^{\clubsuit}(A) \subset R^{\clubsuit}(R^{\clubsuit}(A))$. Note that this is due to the reflexivity and the non-transitivity of R (cf. Propositions 1 and 2). This makes $R^{\clubsuit}(R^{\clubsuit}(A))$ a suitable candidate to model roughly A such that

$$\text{more or less } A \subseteq \text{roughly } A$$

is satisfied. The meaning of this representation of roughly A comes down to: y is roughly A if y resembles an x that is more or less A.

4.3. Intensifying Hedges: 'Very' and 'Extremely'

For the representation of very and extremely in an L-fuzzy framework, we suggest an analogous scheme to that presented above for more or less and roughly. Indeed: if all men resembling Alberik in height are tall, then Alberik must be very tall. Likewise, a woman is very beautiful "if all women resembling her are beautiful". In general: y is very A if all x resembling y are A. Hence y is very A if Ry is included in A. To state it in a more fuzzy manner: y is very A to the degree to which Ry is included in A, i.e.

$$\text{very } A(y) = R^{\heartsuit}(A)(y).$$

Under the natural assumption that R is reflexive, the semantic entailment (1) holds:

$$\text{very } A \subseteq A.$$

Imposing $\mathcal{T}$-transitivity on R would again lead to the counter-intuitive result that very very A would have the same meaning as very A. For a non $\mathcal{T}$-transitive resemblance relation R, however, we have $R^{\heartsuit}(R^{\heartsuit}(A)) \subset R^{\heartsuit}(A)$. Therefore we propose to model extremely A as $R^{\heartsuit}(R^{\heartsuit}(A))$, meaning that y is extremely A if every x resembling y is very A. Of course, this representation satisfies

$$\text{extremely } A \subseteq \text{very } A.$$

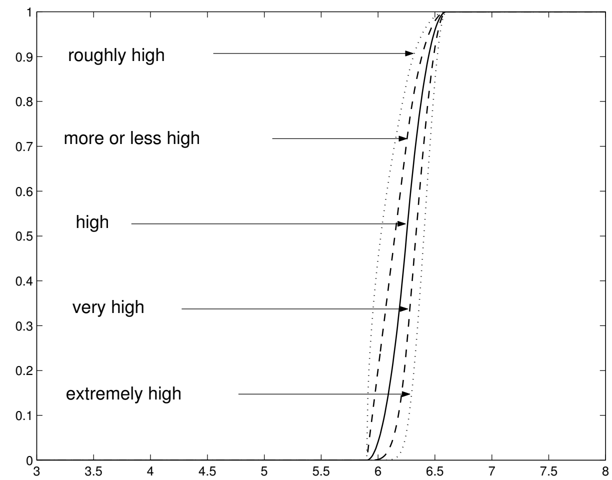
4.4. Examples

This section is entirely devoted to examples that will convince the reader of the great and uniform power of the framework of L-fuzzy modifiers presented above. In the first example we will show how the framework can lead to very nice results when representing modified linguistic terms by fuzzy sets on a numerical universe, as commonly used in applications such as fuzzy control and fuzzy expert systems. The use of traditional fuzzy modifiers such as powering and shifting hedges is well studied for this case. We will show, however, that our representation can outperform them on the semantical level on the one hand and on the uniformity and ease in application on the other.

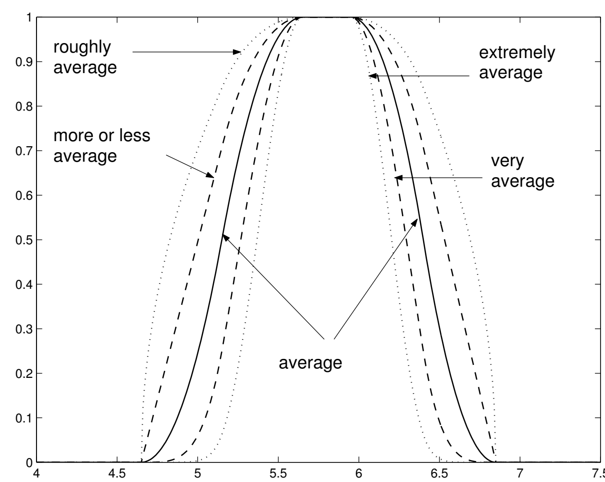
When turning to non-numerical universes, traditional fuzzy modifiers based on premodification become difficult to use due to a lack of suitable operators on the universe under study. In the second example we will show that the fuzzy modifiers based on resemblance relations can be applied with the same ease as in the case of numerical universes, and that they have a greater semantical power than all fuzzy modifiers with pure postmodification.

In the third example we will once again deal with a non-numerical universe, but furthermore we will generalize the set of membership values from $[0, 1]$ to a complete lattice L, allowing us to deal with information that is incomparable w.r.t. the linguistic terms being modelled. To our knowledge, the L-fuzzy modifiers based on L-fuzzy relations are the first ones suitable to perform this representation task.

Example 1. In (du Bois *et al.*, 2002) an approach towards the automatic generation of absenteeism reports is presented. One of the variables under study is the sickness percentage, i.e. the percentage of employees in a company that have reported sick. To describe the values of this variable, linguistic terms and their corresponding fuzzy sets were constructed. Membership functions like the ones presented in Figs. 1(a) and (b) for the basic terms high and average were generated from historical data using fuzzy clustering techniques. In the same figures the use of powering hedges is demonstrated, namely: $P_{0.25}(A)$ to represent roughly A, $P_{0.5}(A)$ for more or less A, $P_2(A)$ for very A, and $P_4(A)$ for extremely A. We would like to mention that in the literature not everybody agrees that an intensifying hedge such as very should be applied to a medium term such as average (Novák *et al.*, 1999). To some people very average might even have a negative connotation. For this reason this term was not considered in (du Bois *et al.*, 2002), but in the present paper we study it for the sake of completeness and to demonstrate the power of our framework of fuzzy modifiers based on fuzzy relations.



(a)



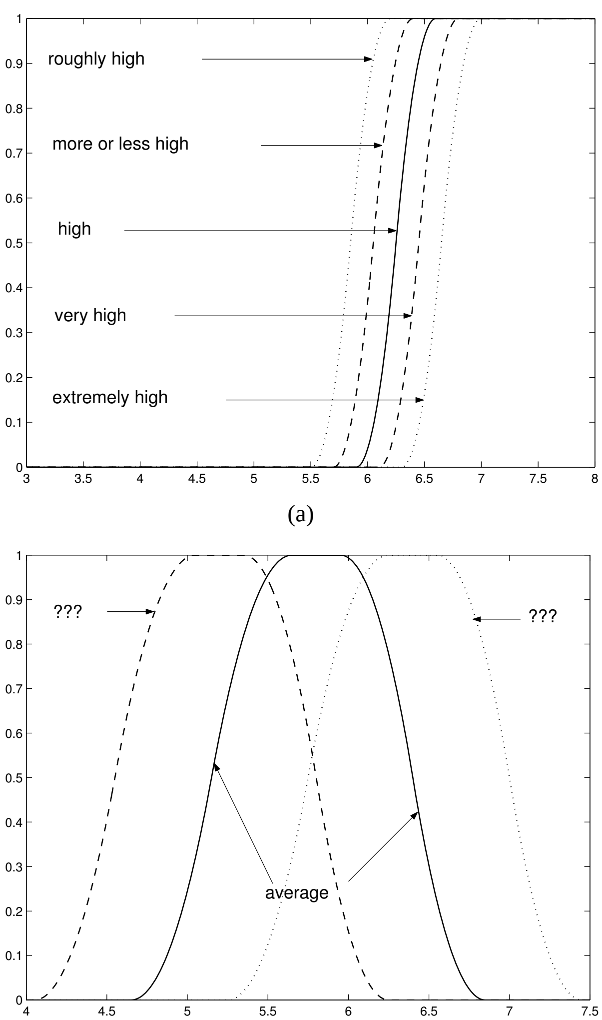
(b)

Fig. 1. Powering hedges applied (a) to high and (b) to average.

Although the presented representation by means of powering hedges satisfies (1), it has an important and well-known drawback from the intuitive point of view (Lakoff, 1973; Hellendoorn, 1990; Kerre, 1993), namely that it keeps the kernel and the support. In other words,

$\ker(A) = \ker(P_\alpha(A))$ and $\mathrm{supp}(A) = \mathrm{supp}(P_\alpha(A))$ for all A in $\mathcal{F}(X)$ and for all positive real numbers α. As an immediate consequence using this kind of representation, it is not possible to distinct between percentages that are high to degree 1 and percentages that are very high to degree 1, although to an occupational physician a percentage of 6.5 might seem high to degree 1 but very high only to a lower degree, e.g. 0.7. It is easy to see that all fuzzy modifiers with pure postmodification have the same shortcoming: either they keep the kernel and/or the support, or they turn it into the empty set and the universe, respectively.

Shifting hedges do not have this property and are therefore more suitable to model linguistic terms like more or less high and very high (see Fig. 2(a)). However, if the membership function to which they are applied is neither increasing nor decreasing, they do not satisify (1), as can be seen from Fig. 2(b). In our opinion, the inappropriate behaviour of powering hedges w.r.t. the kernel and the support and its influence on the shape of the membership function on the one hand and the limited applicability of shifting hedges on the other are due to the fact that they are technical operators that do not take the context into account.



(a)

(b)

Fig. 2. Shifting hedges applied (a) to high and (b) to average.

In our approach, when determining the degree to which x belongs to a modified fuzzy set, we make use of the context of x. In this example, the context of x might be a Π-membership function centred around x, e.g. $\Pi(x-a, x-b, x+b, x+a, .)$. We recall that a Π-membership function on $\mathbb{R}$ is characterized by four real parameters and defined by

$$\Pi(\alpha,\beta,\gamma,\delta,x) = \begin{cases} S(\alpha,(\alpha+\beta)/2,\beta,x), & x \le \beta, \\ 1, & \beta \le x \le \gamma, \\ 1 - S(\gamma,(\gamma+\delta)/2,\delta,x), & \gamma \le x \end{cases}$$

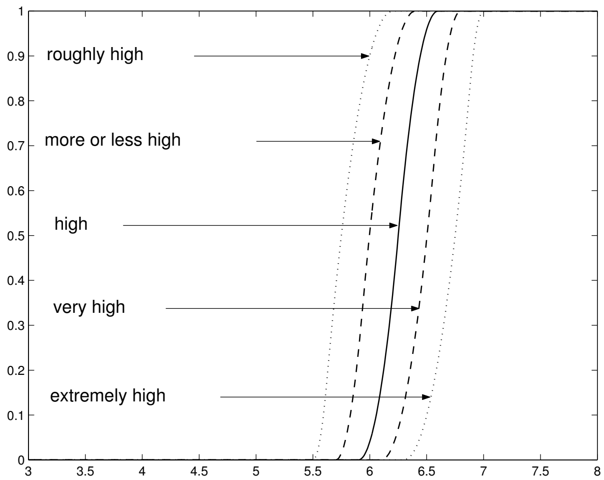
for all x in $\mathbb{R}$. It is assumed that $\alpha \le \beta \le \gamma \le \delta$. Hence it is the union of an S-membership function and its complement:

$$S(\alpha,\beta,\gamma,x) = \begin{cases} 0, & x \le \alpha, \\ \dfrac{2(x-\alpha)^2}{(\gamma-\alpha)^2}, & \alpha \le x \le \beta, \\ 1 - \dfrac{2(x-\gamma)^2}{(\gamma-\alpha)^2}, & \beta \le x \le \gamma, \\ 1, & \gamma \le x. \end{cases}$$

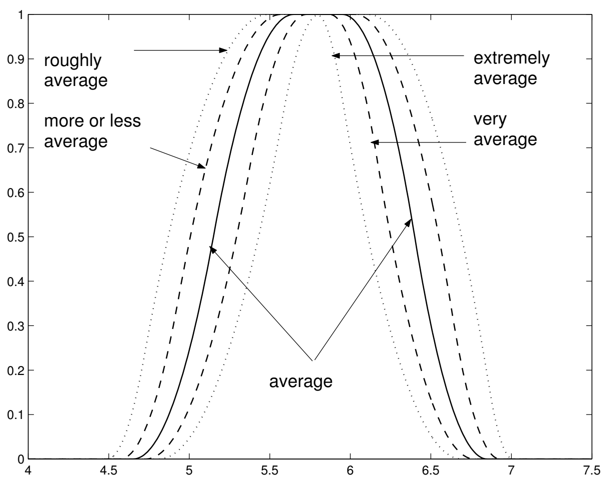
Furthermore, we need a t-norm and its residual implicator to model the intersection and the inclusion which is present in the definition of fuzzy modifiers based on fuzzy relations. In Table 1 the most popular ones are recalled. Figure 3 presents the membership functions generated using the fuzzy modifiers based on fuzzy relations using $\mathcal{T}_{\mathrm{L}}$ and $\mathcal{I}_{\mathcal{T}_{\mathrm{L}}}$. In all cases the kernels and the supports are changed. This is due to Propositions 3 and 4

Table 1. Some t-norms on $[0,1]$ and their residual implicators.

$\mathcal{T}_{\mathrm{M}}(x,y) = \min(x,y)$	$\mathcal{I}_{\mathcal{T}_{\mathrm{M}}}(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise} \end{cases}$
$\mathcal{T}_{\mathrm{P}}(x,y) = x \cdot y$	$\mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y/x & \text{otherwise} \end{cases}$
$\mathcal{T}_{\mathrm{L}}(x,y) = \max(x+y-1,0)$	$\mathcal{I}_{\mathcal{T}_{\mathrm{L}}}(x,y) = \min(1-x+y,1)$



(a)



(b)

Fig. 3. Fuzzy relation based hedges applied (a) to high and (b) to average.

and a proper choice of the resemblance relation that allows these propositions "to do their work." Furthermore, it can be applied to all kinds of membership functions.

Example 2. In the universe of fairytale characters

$$X = \{\text{snow white, witch, wolf, dwarf, prince, little-red-riding-hood}\},$$

fuzzy sets beautiful, average and ugly are given:

	snow white	witch	wolf	dwarf	prince	red-hood
beautiful	1.00	0.00	0.00	0.10	0.80	0.50
average	0.00	0.30	0.00	0.70	0.20	0.50
ugly	0.00	0.70	1.00	0.20	0.00	0.00

For g the $X - [0,1]^3$ mapping defined by

$$g(x) = \big(\text{beautiful}(x), \text{average}(x), \text{ugly}(x)\big),$$

for all x in X and d the pseudo–metric on $[0,1]^3$ defined by

$$d\big((x_1,y_1,z_1),(x_2,y_2,z_2)\big) = \max\big(|x_1-x_2|, |y_1-y_2|, |z_1-z_2|\big)$$

for all (x_1,y_1,z_1) and (x_2,y_2,z_2) in $[0,1]^3$ we can model an approximate equality by the resemblance relation E on X with the following matrix representation:

E	snow white	witch	wolf	dwarf	prince	red-hood
snow white	1.00	0.00	0.00	0.00	1.00	0.50
witch	0.00	1.00	1.00	0.50	0.00	0.00
wolf	0.00	1.00	1.00	0.00	0.00	0.00
dwarf	0.00	0.50	0.00	1.00	0.00	0.88
prince	1.00	0.00	0.00	0.00	1.00	1.00
red-hood	0.50	0.00	0.00	0.88	1.00	1.00

Using $\mathcal{T}_P$ and $\mathcal{I}_{\mathcal{T}_P}$, the membership degrees in the fuzzy sets representing some modified terms are as follows:

	snow-white	witch	wolf	dwarf	prince	red-hood
more or less beautiful	1.00	0.05	0.00	0.44	1.00	0.80
more or less average	0.25	0.35	0.30	0.70	0.50	0.61
more or less ugly	0.00	1.00	1.00	0.35	0.00	0.18
very beautiful	0.80	0.00	0.00	0.00	0.50	0.11
very average	0.00	0.00	0.00	0.57	0.00	0.00
very ugly	0.00	0.40	0.70	0.00	0.00	0.00

Note that the witch who is ugly to degree 0.7 is more or less ugly to degree 1, because of her resemblance to the wolf who is ugly to degree 1. This result can never be achieved with powering hedges.

Example 3. One's favorite "ingredients" for a dessert might be chocolate, vanilla ice and marzipan. Depending on the absence or the presence of one or more of these ingredients, a dessert can be called delicious to some lower or higher degree. Since one likes all of the three ingredients, it is clear that adding one of them makes the dessert plate more delicious. For example, a vanilla ice topped with a bit of chocolate is considered to be more delicious than a plain vanilla ice. Adding some marzipan makes it even more delicious. On the other hand, it could be hardly impossible to say whether a vanilla ice and chocolate dessert is less or more delicious than a vanilla ice and marzipan dessert. Note that we cannot solve this by stating that they are delicious to the same degree, because the ordering discussed above would then imply that vanilla ice and marzipan is more delicious than chocolate, although we have no actual ground to assume that this is true.

Due to the incomparability of the deliciousness of some of the dessert plates $[0,1]$-fuzzy set theory is inadequate to model the term delicious in the universe of desserts X. We propose to represent this term by means of an L-fuzzy set A on X, using membership degrees from the lattice with the Hasse-diagram depicted in Fig. 4. For convenience, we will use the abbreviations C (chocolate), V (vanilla ice) and M (marzipan). Furthermore, a concatenation of ingredient symbols refers to a combined plate (e.g. CV refers to a chocolate and vanilla ice dessert):

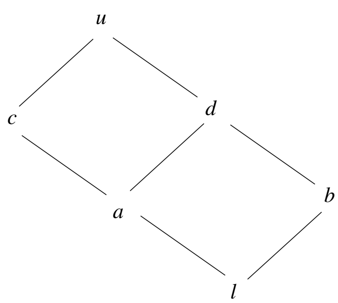


Fig. 4. Lattice $L = \{l, a, b, c, d, u\}$.

$$A = \big\{(V, a), (C, b), (CV, d), (VM, c), (CVM, u)\big\}$$

with the universe of the dessert plates given by $X = \{C, V, M, CV, VM, CVM\}$.

Table 2 represents a reflexive and symmetrical L-fuzzy relation R that models an approximate equality on X. Every dessert plate is considered to be approximately equal to itself to the highest degree u (reflexivity). Furthermore, if two dessert plates are not exactly equal but adding one ingredient to one of them results in the other dessert plate, they are still considered to be approximately equal to degree c. For example, $R(C, CV) = c$, $R(CVM, VM) = c$, etc. Otherwise, they are considered as not approximately equal, i.e. approximately equal to the lowest degree l. Table 3 represents the t-norm $\mathcal{T}_\wedge$ and the implicator $\mathcal{I}_\mathcal{T}$ induced by it on the lattice L is depicted in Figure 4.

Table 2. L-fuzzy relation R modelling the approximate equality on X.

R	V	C	CV	VM	CVM
V	u	l	c	c	l
C	l	u	c	l	l
CV	c	c	u	l	c
VM	c	l	l	u	c
CVM	l	l	c	c	u

Table 3. t-norm and implicator on $L = \{l, a, b, c, d, u\}$.

$\mathcal{T}_\wedge$	l	a	b	c	d	u
l	l	l	l	l	l	l
a	l	a	l	a	a	a
b	l	l	b	l	b	b
c	l	a	l	c	a	c
d	l	a	b	a	d	d
u	l	a	b	c	d	u

$\mathcal{I}_\mathcal{T}$	l	a	b	c	d	u
l	u	u	u	u	u	u
a	b	u	b	u	u	u
b	c	c	u	c	u	u
c	b	d	b	u	d	u
d	l	c	b	c	u	u
u	l	a	b	c	d	u

Now the membership degrees of all dessert plates in the L-fuzzy sets $R^{\clubsuit}(A)$ and $R^{\heartsuit}(A)$ can be determined. As an example, we compute

$$\begin{aligned} R^{\clubsuit}(A)(V) &= \sup\Big(\mathcal{T}_\wedge\big(R(V,V), A(V)\big), \\ &\quad \mathcal{T}_\wedge\big(R(C,V), A(C)\big), \mathcal{T}_\wedge\big(R(CV,V), A(CV)\big), \\ &\quad \mathcal{T}_\wedge\big(R(VM,V), A(VM)\big), \\ &\quad \mathcal{T}_\wedge\big(R(CVM,V), A(CVM)\big)\Big) \\ &= \sup\big(\mathcal{T}_\wedge(u,a), \mathcal{T}_\wedge(l,b), \mathcal{T}_\wedge(c,d), \mathcal{T}_\wedge(c,c), \mathcal{T}_\wedge(l,u)\big) \\ &= \sup(a, l, a, c, l) = c, \end{aligned}$$

$$\begin{aligned} R^{\heartsuit}(A)(CV) &= \inf\Big(\mathcal{I}_\mathcal{T}\big(R(V,CV), A(V)\big), \\ &\quad \mathcal{I}_\mathcal{T}\big(R(C,CV), A(C)\big), \mathcal{I}_\mathcal{T}\big(R(CV,CV), A(CV)\big), \\ &\quad \mathcal{I}_\mathcal{T}\big(R(VM,CV), A(VM)\big), \\ &\quad \mathcal{I}_\mathcal{T}\big(R(CVM,CV), A(CVM)\big)\Big) \\ &= \inf\big(\mathcal{I}_\mathcal{T}(c,a), \mathcal{I}_\mathcal{T}(c,b), \mathcal{I}_\mathcal{T}(u,d), \mathcal{I}_\mathcal{T}(l,c), \mathcal{I}_\mathcal{T}(c,u)\big) \\ &= \inf(d, b, d, u, u) = b. \end{aligned}$$

Computing all membership degrees, we obtain

$$\text{more or less } A = R^{\clubsuit}(A) = \big\{(V,c), (C,d), (CV,u), (VM,c), (CVM,u)\big\},$$

$$\text{very } A = R^{\heartsuit}(A) = \big\{(V,a), (C,b), (CV,b), (VM,a), (CVM,d)\big\}.$$

The semantic entailment clear: $R^{\clubsuit}$ (used to model the weakening hedge more or less) keeps or increases the original membership degrees, while $R^{\heartsuit}$ (used to represent the intensifying hedge very) corresponds to a reduction.

5. Conclusion and Future Research

L-fuzzy modifiers based on L-fuzzy relations prove to be powerful tools for the representation of linguistic hedges. They provide a general framework that is far more applicable than the traditional approaches, and even in the cases where traditional fuzzy hedges can be used, they are still clearly outperformed by the fuzzy relation based modifiers on the semantic level. The generalisation to a complete lattice L allows us to use them to deal with incomparable information, thereby paving the way to a new area of applications. A crucial aspect of these new modifiers are the L-fuzzy relations on which they are based. Important topics of future research will therefore be the construction of resemblance relations, especially in lattices different from the real unit interval, as well as the search for L-fuzzy relations suitable to model other kinds of hedges such as those that are not linearly ordered, as the ones discussed in this paper.

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References

Bodenhofer U. (2002): *A note on approximate equality versus the Poincaré paradox*. — Fuzzy Sets Syst., (to appear).

De Baets B. and Mesiar R. (1999): *Triangular norms on product lattices*. — Fuzzy Sets Syst., Vol. 104, No. 1, pp. 61–75.

De Cock M. and Kerre E. E. (2000): *A new class of fuzzy modifiers*. — Proc. *ISMVL2000* (30th IEEE Int. Symp. *Multiple-Valued Logic*), Computer Society, Cortland (USA), pp. 121–126.

De Cock M. and Kerre E. E. (2001): *Approximate equality is no fuzzy equality*. — Proc. *EUSFLAT2001* (Int. Conf. *European Society for Fuzzy Logic and Technology*), Leicester (UK), pp. 369–371.

De Cock M. and Kerre E. E. (2002a): *Fuzzy modifiers based on fuzzy relations*. — Inf. Sci., (to appear).

De Cock M. and Kerre E. E. (2002b): *On (un)suitable fuzzy relations to model approximate equality*. — Fuzzy Sets Syst., (to appear).

De Cock M., Nachtegael M. and Kerre E. E. (2000): *Images under Fuzzy relations: A Master-Key to Fuzzy Applications*, In: Intelligent Techniques and Soft Computing in Nuclear Science and Engineering (D. Ruan, H. A. Abderrahim, P. D'hondt and E. E. Kerre, Eds.) — Singapore: World Scientific.

De Cock M., Žabokrtský Z. and Kerre E. E. (2001): *Representing linguistic hedges by L-fuzzy modifiers*. — Proc. *CIMCA'01* (Int. Conf. *Computational Intelligence for Modelling Control and Automation*), Las Vegas (USA), pp. 64–72 (CD-ROM).

De Cooman G. and Kerre E. E. (1994): *Order norms on bounded partially ordered sets*. — J. Fuzzy Math., Vol. 2, No. 2, pp. 281–310.

du Bois N., De Cock M., Kerre E. E. and Babuška R. (2002): *A fuzzy set theoretical approach to the automatic generation of absenteeism analyses in natural language*. — Proc. *IPMU2002* (9th Int. Conf. *Information Processing and Management of Uncertainty in Knowledge-Based Systems*), Annecy (France), Vol. III, pp. 1961–1968.

Drossos C. and Navara M. (1997): *Matrix composition of t-norms*, In: Enriched Lattice Structures for Many-Valued and Fuzzy Logics, Proceedings of 18th Linz Seminar on Fuzzy Set Theory (E.P. Klement and S. Gottwald, Eds.). — Luiz, Johannes Kepler Univ., pp. 95–100.

Goguen J. (1967): *L-fuzzy sets*. — J. Math. Anal. Applic., Vol. 18, No. 1, pp. 145–174.

Hellendoorn H. (1990): *Reasoning with Fuzzy Logic*. — Ph. D. Thesis, T.U. Delft (the Netherlands).

Kerre E. E. (Ed.) (1993): *Introduction to the basic principles of fuzzy set theory and some of its applications*. — Communication and Cognition, Gent.

Kerre E. E. and De Cock M. (1999): *Linguistic modifiers: An overview*, In: Fuzzy Logic and Soft Computing (Chen G., Ying M. and Cai K.-Y., Eds.). — Dordrecht: Kluwer Academic Publishers, pp. 69–85.

Klawonn F. (2002): *Should fuzzy equality and similarity satisfy transitivity? Comments on the paper by M. De Cock and E. Kerre*. — Fuzzy Sets Syst., (to appear).

Lakoff G. (1973): *Hedges: A study in meaning criteria and the logic of fuzzy concepts*. — J. Phil. Logic, Vol. 2, No. 4, pp. 458–508.

Novák V. and Perfilieva I. (1999): *Evaluating linguistic expressions and functional fuzzy theories in fuzzy logic*, In: Computing with Words in Information/Intelligent Systems 1: Foundations (Zadeh L. A. and Kacprzyck J., Eds.). — Heidelberg: Springer-Verlag, pp. 383–406.

Novák V., Perfilieva I. and Močkoř J. (1999): *Mathematical Principles of Fuzzy Logic*. — Boston: Kluwer.

Orlowska E. and Radzikowska A. (2001): *Information relations and operators based on double residuated lattices*. — Proc. *RELMICS'6* (6-th Int. Workshop *Relational Methods in Computer Science*), Aisterwijk (the Netherlands), pp. 169–186.

Poincaré H. (1904): *La Valeur de la Science*. — Paris: Flammarion.

Schweizer B. and Sklar A. (1961): *Associative functions and statistical triangle inequalities*. — Publ. Math. Debrecen, Vol. 8, pp. 169–186.

Wei Q., Chen G. and Wets G. (2000): *Modifying fuzzy association rules with linguistic hedges*. — 19th Int. Meeting of the North American Fuzzy Information Processing Society, Atlanta (USA), pp. 397–401.

Xu Y., Ruan D. and Liu J. (1999): *Uncertainty automated reasoning in intelligent learning of soft knowledge*. — Proc. ICST Workshop *Information and Communication Technology for Teaching and Training*, Gent (Belgium), pp. 29–45.

Xu Y., Ruan D. and Liu J. (2000): *Approximate reasoning based on lattice-valued propositional logic* L_{vpl}, In: Fuzzy IF-THEN Rules in Computational Intelligence, Theory and Applications (Ruan D. and Kerre E. E., Eds.) — Boston: Kluwer Academic Publishers, pp. 81–105

Zadeh L. A. (1965): *Fuzzy sets*. — Inf. Contr., Vol. 8, No. 3, pp. 338–353.

Zadeh L. A. (1972): *A fuzzy-set-theoretic interpretation of linguistic hedges*. — J. Cybern., Vol. 2, No. 3, pp. 4–34.

Zadeh L. A. (1975): *The concept of a linguistic variable and its application to approximate reasoning I, II, III*. — Inf. Sci., Vol. 8, No. 3, pp. 199–249, No. 4, pp. 301–357; Vol. 9, No. 1, pp. 43–80.