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FUZZY–ARITHMETIC–BASED LYAPUNOV SYNTHESIS IN THE DESIGN OF STABLE FUZZY CONTROLLERS: A COMPUTING–WITH–WORDS APPROACH

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A novel approach to designing stable fuzzy controllers with perception-based information using fuzzy-arithmetic-based Lyapunov synthesis in the frame of computing with words (CW) is presented. It is shown that a set of conventional fuzzy control rules can be derived from the perception-based information using the standard-fuzzy-arithmetic-based Lyapunov synthesis approach. On the other hand, a singleton fuzzy controller can be devised by using a constrained-fuzzy-arithmetic-based Lyapunov synthesis approach. Furthermore, the stability of the fuzzy controllers can be guaranteed by means of the fuzzy version of Lyapunov stability analysis. Moreover, by introducing standard and constrained fuzzy arithmetic in CW, the "words" represented by fuzzy numbers could be efficiently manipulated to design fuzzy controllers. The results obtained are illustrated with the design of stable fuzzy controllers for an autonomous pole balancing mobile robot.

Keywords: fuzzy control, standard fuzzy arithmetic, constrained fuzzy arithmetic, Lyapunov synthesis, stability, computing with words, perception-based information, pole balancing mobile robot

1. Introduction

In many applications of fuzzy control systems, fuzzy if-then rules are *heuristically* obtained from human experts. How to systematically, rather than heuristically, design and justify a fuzzy controller has been proved to be an extremely challenging problem for the design and analysis of fuzzy control systems. Recently, several different methods to design and analyse fuzzy controllers have been proposed. The model-based fuzzy control approaches usually yield a non-fuzzy controller that will lead to the loss of linguistic interpretability, which is the most important property of fuzzy systems. For the fuzzy rules derived from a human operator, it is usually difficult to implement and hard to justify. For control knowledge acquisition, the most common problem is that the human could only express the control actions in a natural language. Thus, transferring human empirical knowledge to a controller may turn out to be a difficult task. The importance of the language and speech to human intelligence has been recognised for many years. If a computer is to implement artificial intelligence (AI), it must understand the language and speech of human intelligence as a prerequisite (Wang, 2000). Zadeh (1996; 1999) originated the phrase "computing with words (CW)." It is believed that CW is capable of delivering the quality of services in at least two very important areas (Wang, 2000; Zadeh and Kacprzyk, 1999): intelligent information systems and intelligent control systems.

In this paper, rather than considering how to control a plant, we look at the way humans devise their control strategies with the perception-based information on the plant and control objectives. We should notice that in performing control tasks, for most of the cases, humans use *perceptions* rather than measurements. The computational theory of perceptions (CTP) (Zadeh, 2001) is inspired by the remarkable human capability to operate on, and reason with, perception-based information. A basic difference between perceptions and measurements is that, in general, measurements are crisp (e.g., manipulation of numbers and symbols) whereas perceptions are fuzzy (e.g., manipulation of words and propositions drawn from a natural language).

In this paper, we look at a novel approach to design fuzzy controllers from perceptions rather than plant models using a fuzzy-type Lyapunov function (Gupta *et al.*, 1986; Margaliot and Langholz, 1999; 2000) by means of fuzzy arithmetic in the frame of CW. Classical Lyapunov synthesis suggests the design of a controller that should guarantee $\dot{V}(x) < 0$ for a Lyapunov function $V(x)$. Fuzzy Lyapunov synthesis follows the same idea but the linguistic description (perception-based information) of the plant and control objective is utilized by means of CW. The basic assumption of the fuzzy Lyapunov synthesis is that, for a Lyapunov function $V(x)$, if the linguistic value of $\dot{V}(x)$ is *Negative*, then $\dot{V}(x) < 0$, so the stability can be guaranteed. As an example, for $\dot{V}(x) = Negative \cdot Negative + Positive \cdot u$, we may choose $u =$ *Positive Big* to make $\dot{V}(x) = Negative$. But this is again a *heuristic* method! An important point addressed here is that $\dot{V}(x)$ might not be *Negative* unless there exists a set of suitable linguistic variables and their arithmetic operations to guarantee this. On the other hand, for the fuzzy Lyapunov synthesis approach proposed by Magaliot and Langholz (1999; 2000), only the sign of the fuzzy linguistic value, such as "*Negative*" or "*Positive*", was used. Its magnitude was not considered. This means it ignores the changes in states. It could be considered as a very crude estimator of the derivative. Hence, the information extracted from the perceptions could be very limited. Also, it seems difficult to derive more fuzzy rules as there are only a limited number of linguistic terms, such as *Negative* and *Positive*, which are utilised. The number of fuzzy rules is therefore limited.

To solve the above problems, a fuzzy Lyapunov synthesis approach in connection with fuzzy numbers and their arithmetic operations was investigated in our previous study (Zhou, 2001; Zhou and Ruan, 2002). However, the standard fuzzy arithmetic does not take into account all the available information, and the obtained results are more imprecise than necessary or, in some cases, even incorrect. On the other hand, the perception-based information used for fuzzy controller design is not always reliable. To overcome the above deficiencies, in this paper the constrained fuzzy arithmetic (Klir, 1997; Klir and Pan, 1998) is introduced for "word" manipulation of the fuzzy-arithmetic-based Lyapunov function.

The theory of fuzzy numbers was introduced by Nahmias (1977), Dubois and Prade (1982), and many others. The concept of a fuzzy number led to what has come to be called fuzzy arithmetic. In the Foreword of the first book on fuzzy arithmetic theory and applications by Kaufmann and Gupta (1991), Professor L.A. Zadeh wrote: "As a language, fuzzy arithmetic may be expressed in linguistic terms, making it possible to *compute with words* rather than numbers. Furthermore, the membership function of a fuzzy number may be fuzzy set valued, leading to the concept of a fuzzy number of type 2 or, equivalently, an ultrafuzzy number. In this way, the fuzziness of a fuzzy number provides an additional degree of freedom for representing various types of uncertainty as nonuniform possibility distributions over the real line." The research conducted in this paper is highly motivated by Professor Zadeh's inspirational comments on fuzzy numbers and CW. We found that it is possible to systematically, rather than heuristically, design a fuzzy controller modelled on perception-based information by means of both standard and constrained fuzzy arithmetic in the domain of CW.

In the following section, a brief introduction of the standard fuzzy arithmetic in the framework of CW is given. In Section 3, an inverted pendulum balancing system is used as a benchmark to demonstrate a systematic method to design a fuzzy controller from perception-based information using the standard-fuzzy-arithmetic-based Lyapunov synthesis approach. In Section 4, a deficiency of the standard fuzzy arithmetic in fuzzy controller design is identified, and the constrained-fuzzy-arithmetic-based Lyapunov approach is proposed. The practical implementation of fuzzy control to the pole-balancing mobile robot is given in Section 5 to verify the proposed method. This is followed by some discussions and concluding remarks.

2. Standard Fuzzy Arithmetic Operations for CW

As was mentioned in the Introduction, Computing with Words (CW) provides a mathematical model for natural language theory. Its foundation lies in the concepts of fuzzy sets and fuzzy logic (Zadeh, 1996; 1999). In CW, the objects of computing are words rather than numbers, with words playing the role of labels of granules. Let us look at the following example of reasoning with perceptions (Zhou and Ruan, 2002):

Perceptions: (propositions expressed in a natural language)

$p1 =$ Motor 1 is *slow*,

$p2 =$ Motor 2 is *a few rpw faster* than Motor 1,

$p3 =$ Motor 3 is *a few rpw slower* than Motor 1.

Conclusion: (propositions expressed in a natural language)

$q1 =$ Motor 2 is (*slow* + *few*),

$q2 =$ Motor 3 is (*slow* $-$ *few*).

In this example, *slow* and *few* could be expressed as fuzzy numbers; $+$ and $-$ could be fuzzy arithmetic operations.

When the fuzzy numbers represent linguistic concepts, such as *big*, *small* and so on, as interpreted in a particular context, the resulting constructs are usually called linguistic variables (Klir and Yuan, 1995; Zadeh, 1973). Each linguistic variable is fully characterized by a quintuple (v, T, X, g, m) in which v is the name of the variable, T is the set of linguistic terms of v that refers to base variable linguistic terms whose values range over a universal set X, g is a syntactic rule for generating linguistic terms, and is a semantic rule that assigns to each linguistic term $t \in T$ its meaning, $m(t)$, which is a fuzzy set on X (i.e., m: $T \to F(X)$).

Given a fuzzy set A and a real number $\alpha \in [0, 1]$, the crisp set ${}^{\alpha}A = \{x \in \mathbb{R} \mid A(x) \geq \alpha\}$ is called the α-cut of A. The crisp set $\mathrm{Supp}\,(A) = \{x \in \mathbb{R} \mid A(x) > 0\}$ is called the support of A. In the following, some relevant concepts and notation on fuzzy numbers and their arithmetic operations (Kaufman and Gupta, 1991; Klir and Yuan, 1995) are briefly introduced.

Definition 1. A fuzzy number A is a fuzzy set in $\mathbb{R}$ that is convex and normal. Recall that A is convex if for any $x_1, x_2 \in X \subset \mathbb{R}$, and $\lambda \in [0, 1]$, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$, and A is normal if $\mathrm{Sup}_{x\in X}\mu_A(x) = 1$.

The requirement of convexity implies that the points of the real line with the highest membership values are clustered around a given interval (or point). This fact allows us to easily understand the semantics of a fuzzy number by looking at its distribution and to associate it with a properly descriptive syntactic label. On the other hand, the requirement of normality implies that among the points of the real line with the highest membership value, there exists at least one which is completely comparable with the predicate associated with the considered fuzzy number (Kaufman and Gupta, 1991).

In this paper, the discussion will be based on the triangular fuzzy numbers (TFNs) as shown in Fig. 1. We can represent this type of TFN by a triple $A = \langle a, b, c\rangle$ (see Fig. 2), where its α-cut is ${}^{\alpha}A = [a+(b-a)\alpha, c-(c-b)\alpha]$. In Fig. 1, we have $\mathrm{PB} = \langle 2, 3, 4\rangle$, $\mathrm{PM} = \langle 1, 2, 3\rangle$, $\mathrm{PS} = \langle 0, 1, 2\rangle$, $\mathrm{ZE} = \langle -1, 0, 1\rangle$, $\mathrm{NS} = \langle -2, -1, 0\rangle$, $\mathrm{NM} = \langle -3, -2, -1\rangle$, $\mathrm{NB} = \langle -4, -3, -2\rangle$.

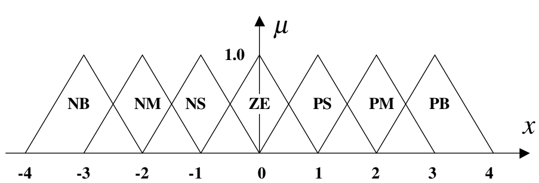

Fig. 1. Triangular fuzzy numbers with seven terms: PB (Positive Big), PM (Positive Medium), PS (Positive Small), Z (Zero), NS (Negative Small), NM (Negative Medium), NB (Negative Big).

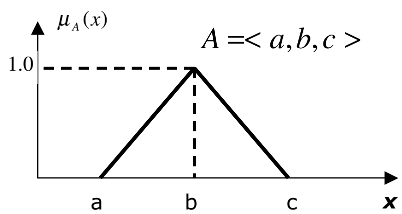

Fig. 2. A triangular membership function $A = \langle a, b, c\rangle$.

To deal with linguistic variables, we need not only the various set theoretic operations, but also arithmetic operations on linguistic variables and, specifically, fuzzy numbers in this paper. There are two common ways of defining fuzzy arithmetic operations (Klir and Yuan, 1995; Klir, 1997). One is based on the extension principle of fuzzy set theory and the other on the α-cut representation.

Definition 2. Let A and B denote linguistic variables (fuzzy numbers), and let $* \in \{+, -, \cdot, /\}$, which denotes any of the four basic arithmetic operations. Employing the extension principle, the arithmetic operations on fuzzy sets A and B are defined by

$$\mu_{A*B}(z) = \sup_{z=x*y} \min\big(\mu_A(x), \mu_B(y)\big) \qquad (1)$$

for all $z \in \mathbb{R}$. More specifically, the four arithmetic operations are defined as follows:

$$\mu_{A+B}(z) = \sup_{z=x+y} \min\big(\mu_A(x), \mu_B(y)\big),$$

$$\mu_{A-B}(z) = \sup_{z=x-y} \min\big(\mu_A(x), \mu_B(y)\big),$$

$$\mu_{A\cdot B}(z) = \sup_{z=x\cdot y} \min\big(\mu_A(x), \mu_B(y)\big),$$

$$\mu_{A/B}(z) = \sup_{z=x/y} \min\big(\mu_A(x), \mu_B(y)\big) \quad (0 \notin {}^{\alpha}B).$$

By employing the α-cut representation, arithmetic operations on fuzzy intervals are defined in terms of the well-established arithmetic operations on closed intervals of real numbers.

Definition 3. Let A and B denote fuzzy sets, and let $* \in \{+, -, \cdot, /\}$, which denotes any of the four basic arithmetic operations. Then, we define a fuzzy set on $\mathbb{R}$, $A*B$, by the following equation:

$${}^{\alpha}(A * B) = \big\{x * y \mid \langle x, y\rangle \in {}^{\alpha}A \times {}^{\alpha}B\big\}, \qquad (2)$$

where ${}^{\alpha}A$ and ${}^{\alpha}B$ are the α-cuts of fuzzy sets A and B, $\alpha \in (0, 1]$. When the operation is division of A and B, it is required that $0 \notin {}^{\alpha}B$ for any $\alpha \in (0, 1]$.

Let ${}^{\alpha}A = {}^{\alpha}[\underline{a}, \bar{a}]$ and ${}^{\alpha}B = {}^{\alpha}[\underline{b}, \bar{b}]$. The individual arithmetic operations on the α-cuts of fuzzy sets A and B can be defined as follows:

$${}^{\alpha}[\underline{a}, \bar{a}] + {}^{\alpha}[\underline{b}, \bar{b}] = {}^{\alpha}[\underline{a} + \underline{b}, \bar{a} + \bar{b}],$$

$${}^{\alpha}[\underline{a}, \bar{a}] - {}^{\alpha}[\underline{b}, \bar{b}] = {}^{\alpha}[\underline{a} - \bar{b}, \bar{a} - \underline{b}],$$

$$\begin{aligned}{}^{\alpha}[\underline{a}, \bar{a}] \cdot {}^{\alpha}[\underline{b}, \bar{b}] = {}^{\alpha}\big[&\min(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}),\\ &\max(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b})\big],\end{aligned}$$

$${}^{\alpha}[\underline{a}, \bar{a}]/{}^{\alpha}[\underline{b}, \bar{b}] = {}^{\alpha}[\underline{a}, \bar{a}] \cdot {}^{\alpha}[1/\bar{b}, 1/\underline{b}] \quad \big(\text{if } 0 \notin {}^{\alpha}[\underline{b}, \bar{b}]\big).$$

A problem of concern at this point is: If A and B are fuzzy numbers, is $A * B$ also a fuzzy number? In other words, is $A * B$ convex and normal? The following theorem gives a positive answer to the above question (Kaufman and Gupta, 1991).

Theorem 1. *Let $* \in \{+, -, \cdot, /\}$, and let A and B denote continuous fuzzy numbers. Then the fuzzy set $A * B$ defined by Definition 2 is also a continuous fuzzy number.*

Theorem 1 can guarantee that the manipulation of the “words” in the framework of fuzzy numbers and their arithmetic operations is consistent.

In designing and analysing a fuzzy controller by means of fuzzy numbers and their arithmetic operations, another key issue is how to compare fuzzy numbers. As an example, for fuzzy Lyapunov synthesis, to guarantee that $\dot{V}(x) = f_1(x_1) - f_2(x_2) < 0$, we need to compare the linguistic values of $f_1(x_1)$ and $f_2(x_2)$, that is, to compare fuzzy numbers. The issue of comparing fuzzy numbers is closely connected to the applications of fuzzy set theory in decision theory (Matarazzo and Munda, 2001).

The linear ordering of real numbers does not extend to fuzzy numbers, but the fuzzy numbers can be ordered partially in a natural way and this partial ordering forms a distributive lattice. The values of linguistic variables in most applications are defined by fuzzy numbers that are comparable. The lattice $\langle \mathbb{R}, \mathrm{MIN}, \mathrm{MAX} \rangle$ can also be expressed as the pair $\langle \mathbb{R}, \prec \rangle$, where $\prec$ is a partial ordering defined as (Klir and Yuan, 1995):

Definition 4. For fuzzy numbers A and B, $A \prec B$ iff $\mathrm{MIN}\,(A, B) = A$ or $\mathrm{MAX}\,(A, B) = B$, where

$$\mu_{\mathrm{MIN}\,(A,B)}(z) = \sup_{z=\min(x,y)} \min\big(\mu_A(x), \mu_B(y)\big),$$

$$\mu_{\mathrm{MAX}(A,B)}(z) = \sup_{z=\max(x,y)} \min\big(\mu_A(x), \mu_B(y)\big)$$

for all $z \in \mathbb{R}$.

Using Definition 4 and the TFNs given in Fig. 1, we can prove that $\mathrm{NM} \prec \mathrm{NS} \prec \mathrm{ZE} \prec \mathrm{PS} \prec \mathrm{PM}$. We may conclude that the "words" represented by fuzzy numbers are comparable. It is a basic requirement for the “words” manipulation of the fuzzy controller design. The following definition gives a linguistic approximation of the “words.” It also provides a foundation for the CW version of the fuzzy controller design.

Definition 5. For fuzzy numbers A and A', $A \cong A'$ iff $\mathrm{Core}\,(A) = \mathrm{Core}\,(A')$, where $\mathrm{Core}\,(A) = \{x | \mu_A(x) = 1\}$ and $\mathrm{Core}\,(A') = \{x \mid \mu_{A'}(x) = 1\}$.

Applying Definitions 2 and 5, it can be seen that $\mathrm{Core}\,(NB + \mathrm{PM}) = \mathrm{Core}\,(\mathrm{NS})$. From Definition 5, we have $NB + \mathrm{PM} \cong \mathrm{NS}$ (see Fig. 3). More general standard fuzzy arithmetic operating results of $C \cong A + B$ and $C \cong A - B$ are given in Tables 1 and 2, respectively. They will be used to design a fuzzy controller using perception-based information via the fuzzy Lyapunov synthesis, the CW version of the classical Lyapunov synthesis method. As an example, from Tables 1 and 2, we have $\mathrm{PS} - \mathrm{PM} + \mathrm{NS} \cong \mathrm{NM}$ and $\mathrm{NS} + \mathrm{PM} - \mathrm{NM} \cong \mathrm{PB}$.

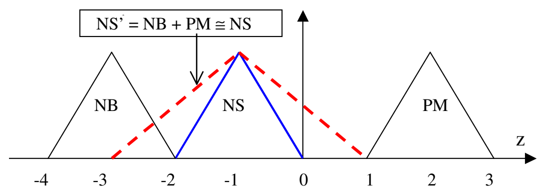

Fig. 3. Illustration of the fuzzy arithmetic operation.

Table 1. Results of $C \cong A + B$.

$C \cong A + B$		A				
		NM	NS	ZE	PS	PM
B	NM	NL	NB	NM	NS	ZE
	NS	NB	NM	NS	ZE	PS
	ZE	NM	NS	ZE	PS	PM
	PS	NS	ZE	PS	PM	PB
	PM	ZE	PS	PM	PB	PL

Table 2. Results of $C \cong A - B$.

$C \cong A - B$		A				
		NM	NS	ZE	PS	PM
B	NM	ZE	PS	PM	PB	PL
	NS	NS	ZE	PS	PM	PB
	ZE	NM	NS	ZE	PS	PM
	PS	NB	NM	NS	ZE	PS
	PM	NL	NB	NM	NS	ZE

3. Standard-Fuzzy-Arithmetic-Based Lyapunov Synthesis

The inverted pendulum is frequently used as a benchmark dynamic nonlinear plant for evaluating a control algorithm or a combination of control algorithms. The state variables are $x_1 = \theta$ (the pendulum’s angle), and $x_2 = \dot{\theta}$ (the pendulum’s angular velocity). The system’s dynamic equations are described as follows (Slotine and Li, 1991):

$$\begin{cases} \dot{x}_1 = x_2 = F_1(x), \\ \dot{x}_2 = f(x_1, x_2) + g(x_1, x_2)u = F_2(x), \end{cases} \tag{3}$$

where

$$f(x_1, x_2) = \frac{9.8 \sin x_1 - \dfrac{m l x_2^2 \cos x_1 \sin x_1}{m_c + m}}{l\left(\dfrac{4}{3} - \dfrac{m \cos^2 x_1}{m_c + m}\right)},$$

$$g(x_1, x_2) = \frac{\dfrac{\cos x_1}{m_c + m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2 x_1}{m_c + m}\right)}$$

where m_c is the mass of the cart, m is the mass of the pole, $2l$ is the pole's length, and u is the applied force (control). The traditional fuzzy control rules (Wang, 1997), which are commonly applied to control the inverted pendulum, are obtained *heuristically*.

Assume that the model (3) is unknown. However, based on the physical intuition and the experience of balancing a pole, the perception-based information can be obtained as shown in Table 3. In the following, we will demonstrate that the fuzzy control rules can be derived from the perceptions by means of standard-fuzzy-arithmetic-based Lyapunov function in the framework of CW. Furthermore, the stability of the fuzzy controller can be guaranteed.

Table 3. Perceptions for balancing a pole.

	Perceptions	Remarks
S1	$\dot{x}_1 = x_2$	From the state description.
S2	$\ddot{\theta}$ is proportional to the control u	The angular acceleration is proportional to the force applied to the cart.
S3	u is inversely proportional to θ	As the pole is falling over to the right-hand side, one must move his/her finger to the right-hand side at once.
S4	u is inversely proportional to $\dot{\theta}$	From the knowledge of balancing a pole.

Remark 1. Note that the perceptions S3 and S4 in Table 3 can also be confirmed by the conditions of the asymptotic stability in (3). For example, by using Lyapunov's indirect method (Jenkins and Passino, 1999; Slotine and Li, 1991), from (3) we have

$$\bar{A} = \left.\begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} \end{bmatrix}\right|_{x=0}$$

$$= \left.\begin{bmatrix} 0 & 1 \\ \dfrac{9.8(m_c+m)}{\dfrac{l(4m_c+m)}{3}} + \dfrac{1}{\dfrac{(4m_c+m)}{3}}\dfrac{\partial u}{\partial x_1} & \dfrac{1}{\dfrac{(4m_c+m)}{3}}\dfrac{\partial u}{\partial x_2} \end{bmatrix}\right|_{x=0} \quad (4)$$

The eigenvalues of $\bar{A}$ are given by the determinant of $\lambda I - \bar{A}$. To ensure that the origin $x_e = 0$ is asymptotically stable, the eigenvalues λ_i of $\bar{A}$ must be in the left half of the complex plane. Hence, we can obtain the following conditions to ensure the asymptotic stability:

$$\frac{\partial u}{\partial x_1} < -9.8(m_c + m), \quad \frac{\partial u}{\partial x_2} < 0. \quad (5)$$

From (5), we can easily conclude that the force u is *inversely* proportional to the pendulum's angular velocity x_2, and inversely proportional to the pendulum's angle x_1. This is exactly reflected by the perceptions on balancing an inverted pendulum. For example, as the pole is falling over to the right-hand side, one must move his/her finger to the right-hand side at once.

Consider the Lyapunov function candidate $V(x_1, x_2) = 1/2(x_1^2 + x_2^2)$, which can be used to represent a measure of the distance of the pendulum's actual state (x_1, x_2) and the desired state $(x_1, x_2) = (0, 0)$. Differentiating V yields

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2. \quad (6)$$

Using the perception S2 in Table 3, $\dot{x}_2$ is proportional to the control u (the angular acceleration is proportional to the force applied to the cart). This can be further explained as $\dot{x}_2 = k_u u$ (assume $k_u = 1$ in this paper). Substituting $\dot{x}_2 = u$ into (6), we have

$$\dot{V} = x_1\dot{x}_1 + x_2 u = x_1 x_2 + x_2 u = x_2(x_1 + u). \quad (7)$$

Its linguistic description is given as

$$LV(\dot{V}(x)) = LVx_2(LVx_1 + LVu), \quad (8)$$

where $LV(\dot{V}(x))$, LVx_1, LVx_2 and LVu are linguistic values of $\dot{V}(x)$, x_1, x_2 and u, respectively.

Theorem 2. *If $V(x)$ is a Lyapunov function and the linguistic value $LV(\dot{V}(x)) = Negative$, where we have $\mathrm{Supp}\,(Negative) \subset (-\infty, 0]$, then the fuzzy controller designed by fuzzy Lyapunov synthesis is locally stable. Furthermore, if $\mathrm{Supp}\,(Negative) \subset (-\infty, 0)$, then the stability is asymptotic.*

A detailed explanation of Theorem 2 is given in (Zhou and Ruan, 2002). The theorem provides a guidance to design a *stable* fuzzy controller using perception-based information by the fuzzy-arithmetic-Lyapunov synthesis method.

3.1. Fuzzy Lyapunov Synthesis Approach

According to the linguistic version of Lyapunov synthesis (cf. (7)), the fuzzy control rules as shown in Table 4 can be obtained in a systematic manner in the domain of CW (Margaliot and Langholz, 1999; 2000).

For example, if $x_1 = Positive$ and $x_2 = Positive$, from our *heuristics*, u should be *Negative Big* to ensure $x_1 + u = Positive - NegativeBig = Negative$, and hence $x_2(x_1 + u) = Positive \cdot Negative = Negative$, that is $LV(\dot{V}(x)) = Negative$. From Theorem 2, if $\mathrm{Supp}\,(Negative) \subset (-\infty, 0]$, then the fuzzy controller designed by the fuzzy Lyapunov synthesis approach is locally stable. It can be seen that the fuzzy control rules are

Table 4. Fuzzy control rules derived from the Fuzzy Lyapunov Synthesis Approach.

x_1	x_2	u	x_1+u	$x_2(x_1+u)$
Positive	Positive	Negative Big	Negative	Negative
Positive	Negative	Zero	Positive	Negative
Negative	Positive	Zero	Negative	Negative
Negative	Negative	Positive Big	Positive	Negative

obtained in a systematic manner. However, the "words" are manipulated *heuristically*. On the other hand, only the sign (not magnitude) of the fuzzy linguistic values is utilized, and the number of fuzzy rules is hence limited.

3.2. Standard-Fuzzy-Arithmetic-Based Lyapunov Synthesis Approach

Assume that x_1, x_2 and u are all described by the fuzzy numbers as shown in Fig. 1. We also employ the standard fuzzy arithmetic operations defined in (1) and (2) in the following "words" manipulation for fuzzy Lyapunov synthesis.

Example 1. Consider $x_2 = \mathrm{PM}$ and choose $x_1 + u = \mathrm{NM}$. Then a set of fuzzy control rules as shown in Table 5 can be derived by using standard fuzzy arithmetic operations defined in (1) and (2). From (8), we have $LV(\dot{V}(x)) = \mathrm{PM}\cdot\mathrm{NM} = \mathit{Negative}$. This is illustrated in Fig. 4. It can be seen that $\mathrm{Supp}\,(\mathrm{PM}\cdot\mathrm{NM}) \subset [-9,-1] \subset (-\infty, 0]$. From Theorem 2, it can be seen that the fuzzy controller with fuzzy control rules as shown in Table 5 is stable.

Table 5. Fuzzy control rules ($x_2 = \mathrm{PM}$, $x_1 + u = \mathrm{NM}$).

x_1	$x_1+u=\mathrm{NM}$	u	Remarks
NM	$\mathrm{NM}+u=\mathrm{NM}$	ZE	$\mathrm{NM}+\mathrm{ZE}=\mathrm{NM}$
NS	$\mathrm{NS}+u=\mathrm{NM}$	NS	$\mathrm{NS}+\mathrm{NS}=\mathrm{NM}$
ZE	$\mathrm{ZE}+u=\mathrm{NM}$	NM	$\mathrm{ZE}+\mathrm{NM}=\mathrm{NM}$
PS	$\mathrm{PS}+u=\mathrm{NM}$	NB	$\mathrm{PS}+NB=\mathrm{NM}$
PM	$\mathrm{PM}+u=\mathrm{NM}$	NL	$\mathrm{PM}+NL=\mathrm{NM}$

Example 2. Consider $x_2 = \mathrm{NS}$ and choose $x_1+u = \mathrm{PS}$. Then a set of fuzzy control rules as shown in Table 6 can be derived. From (8), we have $LV(\dot{V}(x)) = \mathrm{NS}\cdot\mathrm{PS} = \mathit{Negative}$. This is illustrated in Fig. 5. It can be seen that $\mathrm{Supp}\,(\mathrm{NS}\cdot\mathrm{PS}) \subset [-4, 0] \subset (-\infty, 0]$. From Theorem 2, we can conclude that the fuzzy controller with the fuzzy rules as shown in Table 6 is stable.

Table 6. Fuzzy control rules ($x_2 = \mathrm{NS}$, $x_1 + u = \mathrm{PS}$).

x_1	$x_1+u=\mathrm{PS}$	u	Remarks
NM	$\mathrm{NM}+u=\mathrm{PS}$	PB	$\mathrm{NM}+\mathrm{PB}=\mathrm{PS}$
NS	$\mathrm{NS}+u=\mathrm{PS}$	PM	$\mathrm{NS}+\mathrm{PM}=\mathrm{PS}$
ZE	$\mathrm{ZE}+u=\mathrm{PS}$	PS	$\mathrm{ZE}+\mathrm{PS}=\mathrm{PS}$
PS	$\mathrm{PS}+u=\mathrm{PS}$	ZE	$\mathrm{PS}+\mathrm{ZE}=\mathrm{PS}$
PM	$\mathrm{PM}+u=\mathrm{PS}$	NS	$\mathrm{PM}+\mathrm{NS}=\mathrm{PS}$

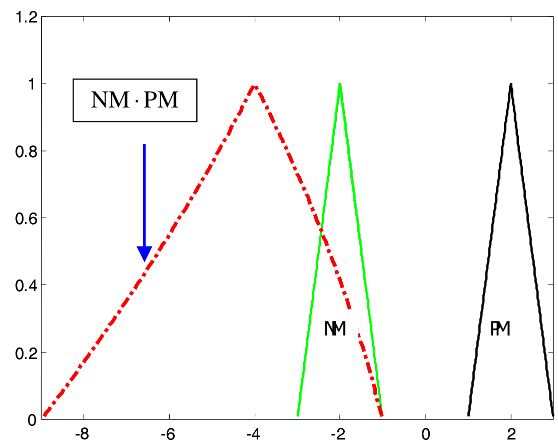

Fig. 4. Illustration of $LV(\dot{V}(x)) = \mathrm{PM}\cdot\mathrm{NM}$.

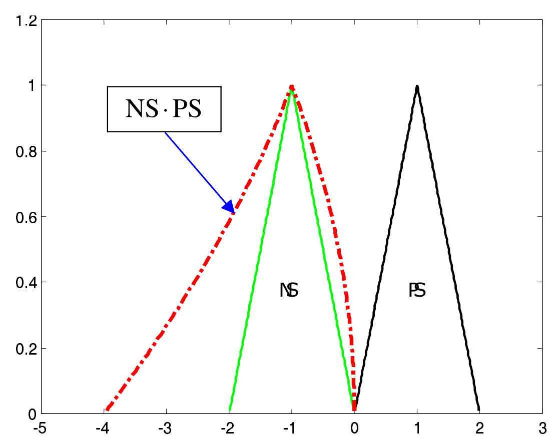

Fig. 5. Illustration of $LV(\dot{V}(x)) = \mathrm{NS}\cdot\mathrm{PS}$.

Repeating a procedure similar to that shown in Examples 1 and 2, a complete set of fuzzy control rules as shown in Table 7 can be derived from the perception-based information (see Table 3) using the standard-fuzzy-arithmetic-based Lyapunov synthesis approach in the framework of CW. Note that the fuzzy control rules in Table 7 are the same as the conventional fuzzy control rules, which have been successfully used to control the inverted pendulum (Li and Shieh, 2000). But an important issue addressed here is that the fuzzy rules derived from the perception-based information are modelled on the standard-fuzzy-arithmetic-based Lyapunov synthesis approach in the context of CW. Therefore, the fuzzy controller is designed *systematically* rather than *heuristically*, and its stability can also be guaranteed.

Table 7. Fuzzy control rules derived from the perception-based information using the fuzzy-arithmetic-based Lyapunov synthesis approach.

u		x_1				
		NM	NS	ZE	PS	PM
x_2	NM	PL	PB	PM	PS	ZE
	NS	PB	PM	PS	ZE	PS
	ZE	PM	PS	ZE	PS	PM
	PS	PS	ZE	PS	PM	PB
	PM	ZE	PS	PM	PB	PL

4. Constrained-Fuzzy-Arithmetic-Based Lyapunov Synthesis

Consider the following fuzzy rule derived from the standard-fuzzy-arithmetic-based Lyapunov synthesis approach as shown in Table 7:

$$\text{If } x_1 \text{ is NS and } x_2 \text{ is PS Then } u \text{ is ZE.} \quad (9)$$

From (8), we have $LV(\dot{V}(x)) = \text{PS} \cdot (\text{NS} + \text{ZE})$. This is illustrated in Fig. 6. It can be seen that $\text{Supp}\,(\text{PS} \cdot (\text{NS} + \text{ZE})) = [-6, 2] \not\subset (-\infty, 0)$. The stability condition given in Theorem 2 is not satisfied. This is caused by the deficiency of the standard fuzzy arithmetic. The standard fuzzy arithmetic does not utilize some of the available information. Therefore, the obtained results may be more imprecise than necessary or, in some cases, even incorrect. To overcome this deficiency, a constrained fuzzy arithmetic (Klir, 1997) is needed to take all available information into account in terms of relevant requisite constraints.

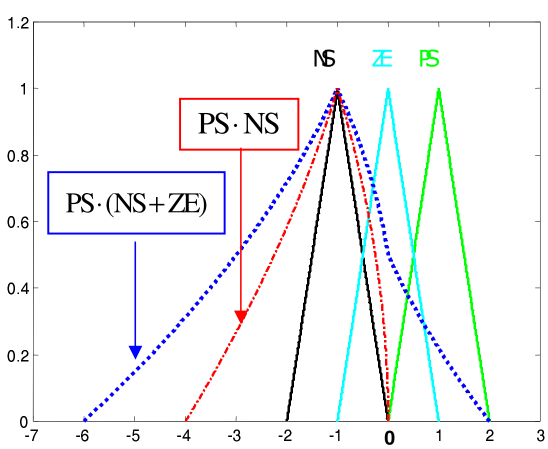

Fig. 6. Illustration of $LV(\dot{V}(x)) = \text{PS} \cdot (\text{NS} + \text{ZE})$.

4.1. Constrained Fuzzy Arithmetic

Results obtained by the standard fuzzy arithmetic suffer from imprecision greater than justifiable in all computations that involve the *requisite equality constraint* (Klir, 1997). However, the equality constraint is always satisfied in the classical arithmetic on real numbers. Because ignoring equality constraints will lead to results that are less precise than necessary, it is essential to include the constraints, when applicable, into the general definition of basic arithmetic operations on fuzzy numbers. In general, each constraint R on $A * B$ is a relation (crisp or fuzzy) on $A \times B$. For the extension principle of the fuzzy set theory, the constrained arithmetic operations $(A*B)_R$ are defined by the following equation:

$$\mu_{(A*B)_R}(z) = \sup_{z=x*y} \min\left(\mu_A(x), \mu_B(y), \mu_R(x,y)\right). \quad (10)$$

For the cut representation of the fuzzy intervals, we have

$$^{\alpha}(A * B)_R = \left\{x * y \mid \langle x, y\rangle \in ({}^{\alpha}A \times {}^{\alpha}B) \cap {}^{\alpha}R\right\}. \quad (11)$$

Any operations $A * B$ or $B * A$ are unconstrained, even though $A = B$, while operations $A * B$ and $B * A$ are subject to the equality constraint. These constrained operations, for example, on A, may conveniently be expressed as follows, where E denotes the relation R representing the equality constraint:

$$^{\alpha}(A + A)_E = \{x + x \mid x \in^{\alpha} A\} =^{\alpha} [2\underline{a}, 2\bar{a}], \quad (12)$$

$$^{\alpha}(A - A)_E = \{x - x \mid x \in^{\alpha} A\} = 0, \quad (13)$$

$$^{\alpha}(A \cdot A)_E = \{x \cdot x \mid x \in^{\alpha} A\}, \quad (14)$$

$$^{\alpha}(A/A)_E = \{x/x \mid x \in^{\alpha} A, 0 \notin^{\alpha} A\} = 1. \quad (15)$$

Under the equality constraint for X, where $A, B, X \in \mathbb{R}$, we obtain

$$A + X = B \Leftrightarrow X = B - A, \quad (16)$$

$$A \cdot X = B \Leftrightarrow X = B/A \;\; (0 \notin^{\alpha} A). \quad (17)$$

But in general, these are not solutions in the standard fuzzy arithmetic (Klir, 1997).

4.2. Constrained-Fuzzy-Arithmetic-Based Lyapunov Synthesis Approach

In the following, we will demonstrate how to use the constrained-fuzzy-arithmetic-based Lyapunov synthesis approach to derive fuzzy control rules from the perception-based information given in Table 3.

Example 3. Consider $x_2 = \text{PM} = \langle 1, 2, 3\rangle$ and choose $x_1 + u = \text{NM}$. Under the equality constraint for u, from (16) we have $u = \mathrm{NM} - x_1$. If $x_1 = \mathrm{NM} = \langle -3, -2, -1 \rangle$ as shown in Fig. 1, under the equality constraint, then ${}^{\alpha}(LVu) = {}^{\alpha}(\mathrm{NS} - \mathrm{NM})_E$. Considering ${}^{\alpha}(\mathrm{NS}) = [(-2+\alpha), -\alpha)]$ and ${}^{\alpha}(\mathrm{NM}) = [(-3+\alpha), -1-\alpha)]$, we have ${}^{\alpha}(LVu) = [(-2+\alpha) - (-3+\alpha), (-\alpha) - (-1-\alpha)] = [1, 1]$. This leads to $u = 1$. Hence the following fuzzy control rule can be derived:

$$\text{If } x_1 \text{ is NM and } x_2 \text{ is PS Then } u = 1. \tag{18}$$

It is a fuzzy rule with a singleton consequent, i.e., a singleton fuzzy rule (Sugeno, 1999). The rest of fuzzy rules for the condition $x_2 = \mathrm{PM}$ are illustrated in Table 8.

Table 8. Singleton fuzzy control rules $(x_2 = \mathrm{PM})$.

x_1	${}^{\alpha}(LVu) = {}^{\alpha}(\mathrm{NM} - x_1)_E$	u
NM	$\big[(-3+\alpha)-(-3+\alpha), (-1-\alpha)-(-1-\alpha)\big] = [0, 0]$	0
NS	$\big[(-3+\alpha)-(-2+\alpha), (-1-\alpha)-(-\alpha)\big] = [-1, -1]$	-1
ZE	$\big[(-3+\alpha)-(-1+\alpha), (-1-\alpha)-(1-\alpha)\big] = [-2, -2]$	-2
PS	$\big[(-3+\alpha)-\alpha, (-1-\alpha)-(2-\alpha)\big] = [-3, -3]$	-3
PM	$\big[(-3+\alpha)-(1+\alpha), (-1-\alpha)-(-3-\alpha)\big] = [0, 0]$	-4

Example 4. Consider $x_2 = \mathrm{PS} = \langle 0, 1, 2 \rangle$ and choose $x_1 + u = \mathrm{NS}$. Under the equality constraint for u, from (16), we have $u = \mathrm{NS} - x_1$. Following the same procedure as shown in Example 3, a set of singleton fuzzy rules as shown in Table 9 can be derived.

Repeating the same procedure as shown in Examples 3 and 4, a singleton fuzzy controller as shown in Table 10 can be devised by using the constrained-fuzzy-arithmetic-based Lyapunov synthesis approach in the framework of CW.

Table 9. Singleton fuzzy control rules $(x_2 = \mathrm{PS})$.

x_1	${}^{\alpha}(LVu) = {}^{\alpha}(\mathrm{NM} - x_1)_E$	u
NM	$\big[(-2+\alpha)-(-3+\alpha), (-\alpha)-(-1-\alpha)\big] = [1, 1]$	1
NS	$\big[(-2+\alpha)-(-2+\alpha), (-\alpha)-(-\alpha)\big] = [0, 0]$	0
ZE	$\big[(-2+\alpha)-(-1+\alpha), (-\alpha)-(1-\alpha)\big] = [-1, -1]$	-1
PS	$\big[(-2+\alpha)-\alpha, (-\alpha)-(2-\alpha)\big] = [-2, -2]$	-2
PM	$\big[(-2+\alpha)-(1+\alpha), (-\alpha)-(3-\alpha)\big] = [-3, -3]$	-3

Remark 2. To investigate the stability of the above fuzzy control rules with a singleton consequent, let us consider the same condition as that of the fuzzy control rule (9). The corresponding rule in Table 10 is given as follows:

$$\text{If } x_1 \text{ is NS and } x_2 \text{ is PS Then } u \text{ is } 0. \tag{19}$$

Under the equality constraint, we have $(LVx_1 + LVu)_E = \mathrm{NS}$. From (8), we get $\mathrm{Supp}\,(LV(\dot{V}(x))) = LVx_2(LVx_1 + LVu) = \mathrm{PS} \cdot \mathrm{NS}$. In Fig. 6, we can observe that $\mathrm{Supp}\,(\mathrm{PS} \cdot \mathrm{NS}) = [-4, 0] \subset (-\infty, 0]$. From Theorem 2, the fuzzy controller with the singleton fuzzy control rule (19) is stable. Comparing this with the fuzzy control rule (9), where $\mathrm{Supp}\,(LV(\dot{V}(x))) = \mathrm{Supp}\,(\mathrm{PS} \cdot (\mathrm{NS} + \mathrm{ZE})) = [-6, 2] \not\subset (-\infty, 0]$ (see Fig. 6), it can be seen that the deficiency of the fuzzy Lyapunov synthesis with the standard fuzzy arithmetic can be overcome by the constrained fuzzy arithmetic.

Table 10. Singleton fuzzy rules derived by the constrained-fuzzy-arithmetic-based Lyapunov synthesis approach.

u		x_1				
		NM	NS	ZE	PS	PM
x_2	NM	4	3	2	1	0
	NS	3	2	1	0	-1
	ZE	2	1	0	-1	-2
	PS	1	0	-1	-2	-3
	PM	0	-1	-2	-3	-4

Remark 3. By using the equality constrained fuzzy arithmetic, we can easily prove that $\mathrm{Supp}\,(LV(\dot{V}(x))) \subset (-\infty, 0]$, i.e., $\dot{V}(x) \le 0$ for all the singleton fuzzy control rules with $x_2 = \mathrm{NM}, \mathrm{NS}, \mathrm{PS}$ and PM in Table 5. However, for $x_2 = \mathrm{ZE}$, $\mathrm{Supp}\,(LV(\dot{V}(x))) = \mathrm{Supp}\,(\mathrm{ZE} \cdot \mathrm{ZE})_E$. Note that under the equality constraint, $\mathrm{Supp}\,(\mathrm{ZE} \cdot \mathrm{ZE})_E = [0, a_0^2]$ (see Fig. 7). This means that once $a_0 \to 0$, we have $\mathrm{Supp}\,(\mathrm{ZE} \cdot \mathrm{ZE})_E \to [0, 0]$, or $\dot{V}(x) \to 0$. This confirms the intuition that more rules result in more powerful fuzzy control systems. We should also notice that the linguistic terms like PM, PS, NS and NM need not be very specific. On the contrary, the description of the area close to zero should be defined in greater detail to make the control actions more specific and assure enough sensitivity in the generated control actions (Pedrycz, 1994).

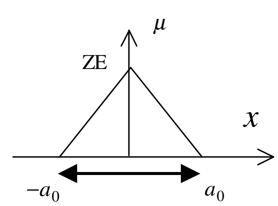

Fig. 7. Partition in the *zero* region.

Remark 4. The singleton fuzzy rules in Table 10 are derived based on the assumption S2 of Table 3, i.e., $\dot{x}_2 = u$. For a more general case, if we assume $u = k\dot{x}_2$, where k is a real number, then under the equality constrained fuzzy arithmetic, (17) can be rewritten as

$$LV\big(\dot{V}(x)\big) = LVx_2\big(LVx_1 + LV(u/k)\big). \tag{20}$$

If $x_2 = \mathrm{PM}$, then by choosing $x_2 + u = \mathrm{NM}$, under the equality constraint for u, from (16) we have $u = k(\mathrm{NM} - x_1)_E$ or ${}^{\alpha}(LVu) = [k, k]$, i.e., $u = k$. By repeating the same procedure, we can have a singleton fuzzy controller as shown in Fig. 8. This means that we could improve the stability of the singleton fuzzy controller by tuning k.

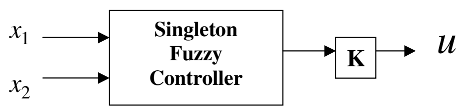

Fig. 8. Configuration of a singleton fuzzy controller with a general assumption of S2 in Table 3.

5. Experimental Results

To demonstrate the effectiveness of the proposed fuzzy controller design method, a real-time experiment of the fuzzy control of an autonomous pole-balancing mobile robot with an onboard TMS 320C32 DSP processor was conducted (see Fig. 9). This project aims to design and fabricate an autonomous mobile robot to participate in the Singapore Robotic Games (SRG). The mobile robot is able to balance a free-falling pole by means of horizontal movements. While balancing the pole, it would also travel with a pre-designed slope profile. The mobile robot with the highest number of successful cycles in a single untouched attempt within a predefined time slot will be considered the winning entry.

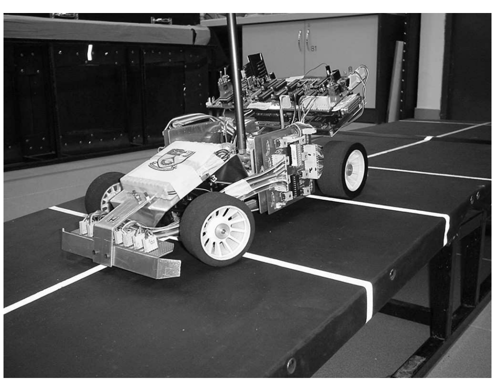

Fig. 9. An autonomous pole-balancing mobile robot.

The parameters of the physical robot are given as follows: the pole's length is $2l = 1$ m, the mass of the pole is $m = 0.1$ kg, and the mass of the cart is $m_c = 2.5$ kg. Figure 10 shows the trajectory of the pole angle and the velocity tracking results using the conventional fuzzy control rules (Table 7) derived from perception-based information using the standard-fuzzy-arithmetic-based Lyapunov synthesis approach. It can be seen that the pole never falls down as the mobile robot can always track the desired trajectory though the pole swings very much occasionally. This may be due to the limited perception-based information. A similar experiment is also conducted using the singleton fuzzy control rules (Table 10) derived by the constrained-fuzzy-arithmetic-based Lyapunov synthesis approach. The results are similar to those presented in Fig. 10. From Fig. 11, it can be found that the pole angle is sometimes greater than 0.2 rad. However, for the fuzzy control rules in Table 7, the pole angle is always less than 0.2 rad.

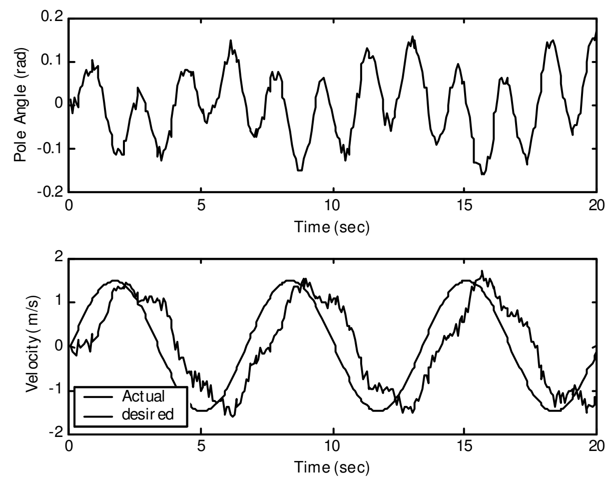

Fig. 10. Balancing and tracking results using the fuzzy control rules derived from the perception-based information by means of the standard-fuzzy-arithmetic-based Lyapunov function.

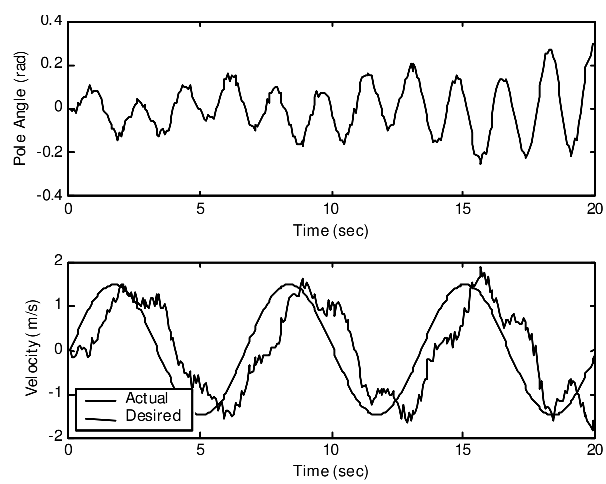

Fig. 11. Balancing and tracking results using the fuzzy control rules derived from the perception-based information by means of the constrained-fuzzy-arithmetic-based Lyapunov function.

From the results shown in Figs. 10 and 11, we can conclude that a stable fuzzy controller for the autonomous pole balancing mobile robot can be devised systematically rather than heuristically from perception-based information by means of standard- and constrained-fuzzy-arithmetic-based Lyapunov synthesis approaches. To improve the pole-balancing performance, further learning is necessary, e.g., by fuzzy reinforcement learning methods (Zhou, 2001; Zhou *et al.*, 2001).

6. Concluding Remarks

A novel approach to design fuzzy controllers using the fuzzy-arithmetic-based Lyapunov synthesis approach that gives a linguistic description of the plant and the control objective is presented. It is found that by using the standard-fuzzy-arithmetic-based Lyapunov synthesis approach, a set of conventional fuzzy control rules can be produced, while by using the constrained-fuzzy-arithmetic-based Lyapunov synthesis approach, the fuzzy control rules with a singleton consequent can be derived. We also demonstrate that the constrained fuzzy arithmetic can be utilised to overcome some deficiencies in the standard fuzzy arithmetic for the fuzzy controller design. In a real-time experiment of the fuzzy control of the autonomous pole-balancing mobile robot, we found that the pole does not fall down as the robot tracks the desired trajectory even without further tuning the fuzzy controller proposed in this paper, though it swings very much occasionally.

The perception-based information is very limited in relation to designing a controller. How to integrate both measurement-based information and perception-based information to design an intelligent controller using CW will be a new challenge. We will also try to incorporate some other techniques in the fuzzy controller design approach presented in this paper. Some CW versions of conventional control theory, such as a CW version of the fuzzy slide mode control, will be studied in the future.

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