ON THE DISCRETE TIME–VARYING JLQG PROBLEM

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In the present paper optimal time-invariant state feedback controllers are designed for a class of discrete time-varying control systems with Markov jumping parameter and quadratic performance index. We assume that the coefficients have limits as time tends to infinity and the boundary system is absolutely observable and stabilizable. Moreover, following the same line of reasoning, an adaptive controller is proposed in the case when system parameters are unknown but their strongly consistent estimators are available.

Keywords: jump linear systems, optimal control, time-varying systems, coupled Riccati equations

1. Introduction

Systems with jumping parameters have recently received great attention because of their potential applications in a number of technical problems, including flexible manufacturing control systems, fault tolerant systems design, analysis and synthesis of systems with abrupt changes in operating points or disturbances, and others.

Much effort has particularly been concentrated on different formulations of the JLQ problem in the case when the system state and the jumping parameters can be observed and consequently used for control, see, e.g., (Chizeck *et al.*, 1986; Costa and Fragoso, 1995; Ghosh, 1995; Griffiths and Loparo, 1985; Ji and Chizeck 1989; Mariton, 1987; Ghaoui, 1996; Sworder, 1969; Sworder and Robinson, 1973). Coupled Riccati equations related to this problem are studied among others by (Abou-Kandil *et al.*, 1994; 1995; Czornik, 2000; Ji and Chizeck, 1988). In (Pan and Bar-Shalom, 1996; Caines and Zhang, 1995; Dufour and Bertrand, 1994; Dufour and Elliott, 1998) the authors deal with a more complicated situation, where the system state or the jump parameter system cannot be directly observed and are consequently estimated.

This paper is devoted to the JLQG problem for a class of discrete time-varying systems on an infinite time interval with completely observable system state and jump parameters and addtitive white disturbances. It is well-known that for such a problem, in the case without jumps, the optimal control is not unique. So an interesting task is to find the simplest one. In the time-invariant case the situation is relatively easy, and the simplest control is the time-invariant feedback. But when we consider a time-varying system, the problem becomes much more complicated. It is interesting that when the coefficients of the system have limits as time tends to infinity, the set of optimal control strategies contains the control in time-invariant feedback form, see (Czornik, 1998; 1999). In the present paper we establish such a result for the discrete-time JLQG problem. We take into account the system with coefficients having limits as functions of time as time tends to infinity, and we show that the control minimizing the quadratic index can be realized in the form of a time-invariant feedback. The feedback matrix is equal to the one for the time-invariant system with coefficients equal to the limits of the time-varying system. To prove the results, the asymptotic behaviour of the solution of the time-varying coupled difference Riccati equation is studied. It enables us also to design an adaptive controller in the case of unknown system parameters.

2. Standard JLQG Problem

The system under study is described by the following state equation:

$$x_{k+1} = A_k(r_k)x_k + B_k(r_k)u_k + C_k(r_k)w_{k+1}, \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ stands for the control, and the disturbance $w_k \in \mathbb{R}^n$, $k = 0, 1 \dots$ is a second-order independent identically distributed sequence of random variables with $Ew_k = 0$ and $Ew_k w_k' = I$. Moreover, r_k is a strongly ergodic Markov chain with values in a finite set S and transition probabilities

$$P\big(r_{k+1} = j \mid r_k = i\big) = p_{ij}^{(k)}, \quad i, j \in S.$$

We also assume that for each $i, j \in S$, a limit p_{ij} of $p_{ij}^{(k)}$ as $k \to \infty$ exists and that the limit matrix $P = [p_{ij}]$ is a transition matrix of a Markov chain with a unique invariant distribution $\pi(i),\ i \in S$. The next assumption is that r_k is independent of $w_k,\ k = 0,\dots,N-1$. For each $i \in S,\ k = 0,\dots,N,\ A_k(i),\ B_k(i),\ C_k(i)$ there are given matrices of orders $n\times n,\ n\times m,\ n\times n$, respectively. The cost criterion to be minimized is

$$J(x_0,r_0,u) = \lim_{N\to\infty}\frac{1}{N}E\left[\sum_{k=0}^{N}\langle Q_k(r_k)x_k,x_k\rangle + \langle R_k(r_k)u_k,u_k\rangle\right], \tag{2}$$

where the matrices $Q_k(i)$ and $R_k(i)$ are positive-semidefinite and positive-definite, respectively, for each $i \in S$.

Consider the noise-free system

$$x_{k+1} = A_k(r_k)x_k + B_k(r_k)u_k, \tag{3}$$

$$y_k = \sqrt{Q_k(r_k)}x_k. \tag{4}$$

Definition 1. If for any initial form r_0 and initial x-states $x_1^{(0)}, x_2^{(0)}$ the minimum time N is finite, such that equivalent outputs $y(x_0 = x_1^{(0)}) = y(x_0 = x_2^{(0)})$ and known inputs in the interval $0 \le k \le N$ imply that $x_1^{(0)} = x_2^{(0)}$, then the system $\left\{A_k(i), \sqrt{Q_k(i)}, i \in S\right\}$ is called *absolutely observable*.

The algebraic conditions equivalent to the absolute observability for time-invariant systems are given in (Ji and Chizeck, 1988).

Definition 2. The system

$$x_{k+1} = A_k(r_k)x_k \tag{5}$$

is *stochastically stable* if

$$\lim_{N\to\infty}E\left(\|x_N\|^2 \Big| r_0 = i\right) = 0, \quad i \in S$$

for any initial state x_0.

It can be shown (Ji and Chizeck, 1988) that (5) is stochastically stable if and only if

$$\lim_{N\to\infty}E\left(\sum_{k=0}^{N}\|x_k\|^2 \,\big|\, r_0 = i\right) < \infty \tag{6}$$

for any initial state x_0.

Definition 3. The system $\{A_k(i), B_k(i), i \in S\}$ is called *stochastically stabilizable* if there exists a feedback control $u_k = L_k(r_k)x_k$ such that the resulting closed-loop system $x_{k+1} = (A_k(r_k) + B_k(r_k)L_k(r_k))x_k$ is stochastically stable.

With these definitions we can formulate a solution of the control problem for the time-invariant case:

$$A_k(r_k) = A(r_k), \quad B_k(r_k) = B(r_k), \quad C_k(r_k) = C(r_k), \tag{7}$$

$$Q_k(r_k) = Q(r_k), \quad R_k(r_k) = R(r_k), \quad p_{ij}^{(k)} = p_{ij}. \tag{8}$$

Theorem 1. *If the system $\{A(i), B(i), i \in S\}$ is stochastically stabilizable and the system $\{A(i), \sqrt{Q(i)}, i \in S\}$ is absolutely observable, then the coupled algebraic Riccati equation*

$$P(i) = A'(i)F(i)\big(A(i) - B(i)L(i)\big) + Q(i), \tag{9}$$

where

$$L(i) = \Big(R(i) + B'(i)F(i)B(i)\Big)^{-1}B'(i)F(i)A(i), \tag{10}$$

$$F(i) = \sum_{j\in S}p_{ij}P(j), \tag{11}$$

has a unique positive definite solution P_i and the optimal control law is given by

$$\widetilde{u}_k = -L(r_k)x_k, \quad i \in S. \tag{12}$$

The value of the optimal cost is given by

$$J(x_0,r_0,\widetilde{u}) = \sum_{i\in S}\sum_{j\in S}\pi(i)p_{ij}\,\mathrm{tr}\,\Big(C'(i)P^0(j)C(i)\Big). \tag{13}$$

3. Asymptotic Behaviour of the Coupled Difference Riccati Equation

In this section we shall investigate properties of the time-varying coupled Riccati equation.

The next theorem, which describes the asymptotic behaviour of the coupled difference Riccati equation, is proved in (Czornik and Świerniak, 2001).

Theorem 2. *Assume that the sequence $(A_N(j), B_N(j), Q_N(j), R_N(j), p_{ij}(N))$ with $A_N(j) \in \mathbb{R}^{n\times n}$, $B_N(j) \in \mathbb{R}^{n\times m}$, $C_N(j) \in \mathbb{R}^{n\times n}$, $Q_N(j) \in \mathbb{R}^{n\times n}$, $R_N(j) \in \mathbb{R}^{m\times m}$, $Q_N(j) \ge 0$, $R_N(j) > 0, i,\ j \in S$, is such that the limits of $A_N(j)$, $B_N(j)$, $Q_N(j)$, $R_N(j)$ as $N \to \infty$ exist for each $j \in S$ and $R(j) > 0$, $Q(j) \ge 0$, $(A(i), B(i), i \in S)$ is stochastically stabilizable, $(A(i), \sqrt{Q(i)}, i \in S)$ is absolutely observable, where*

$$A(j) = \lim_{N\to\infty}A_N(j), \quad B(j) = \lim_{N\to\infty}B_N(j), \quad Q(j) = \lim_{N\to\infty}Q_N(j), \tag{14}$$

$$R(j)=\lim_{N\to\infty} R_N(j),\quad p_{ij}=\lim_{N\to\infty} p_{ij}(N),\quad i,j\in S. \tag{15}$$

Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N} P_k^{(N)}\big(i,K(i)\big)=P(i), \tag{16}$$

for any initial condition $\{K(i):K(i)\ge 0, i\in S\}$*, where* $P_k^{(N)}(i,K(i))$ *is given by*

$$P_k^{(N)}\big(i,K(i)\big)=A'_{N-k}(i)F_{k-1}^{(N)}(i)\Big[A_{N-k}(i) - B_{N-k}(i)L_k^{(N)}(i)\Big]+Q_k(i), \tag{17}$$

$$P_0^{(N)}\big(i,K(i)\big)=K(i)$$

with

$$F_{k-1}^{(N)}(i)=\sum_{j\in S} p_{ij}(N)P_{k-1}^{(N)}\big(j,K(j)\big),$$

$$L_k^{(N)}(i)=\Big(R_{N-k}(i)+B'_{N-k}(i)F_{k-1}^{(N)}(i)B_{N-k}(i)\Big)^{-1}\times B'_{N-k}(i)F_{k-1}^{(N)}(i)A_{N-k}(i) \tag{18}$$

for $k=1,\dots,N$ *and* $P(i)$ *being the unique solution of (9).*

Yet another characterization of the asymptotic behaviour of the coupled Riccati equation is given in the next theorem. The proof can be obtained in much the same way as in (Czornik, 2000) for its continuous time counterpart.

Theorem 3. *Under the assumptions of the previous theorem, there exists* N_0 *such that for all* $N\ge N_0$ *the coupled algebraic Riccati equation*

$$P_N(i)=A'_N(i)F_N(i)\big(A_N(i)-B_N(i)L_N(i)\big)+Q_N(i),$$

where

$$F_N(j)=\sum_{i\in S} p_{ji}^{(N)}P_N(i),$$

and

$$L_N(i)=\Big(R_N(i)+B'_N(i)F_N(i)B_N(i)\Big)^{-1}\times B'_N(i)F_N(i)A_N(i),$$

has a solution $P_N(j)$, $j\in S$*, and*

$$\lim_{N\to\infty}P_N(j)=P(j),\quad j\in S,$$

where $P(j), j\in S$ *are the solutions of (9).*

4. Optimal Control in the Time–Varying Case

The main result of this section is based on the following theorem:

Theorem 4. *Suppose that the assumptions of Theorem 2 hold. Then the optimal control law for the time-varying control problem (1), (2) is given by*

$$\widetilde{u}_k=-L\big(r(k)\big)x_k,\quad i\in S, \tag{19}$$

where

$$L(i)=\Big(R(i)+B'(i)F(i)B(i)\Big)^{-1}B'(i)F(i)A(i), \tag{20}$$

$$F(i)=\sum_{j\in S}p_{ij}P(j) \tag{21}$$

and $P(i)$ *is the unique solution of (9).*

This theorem was proved in (Czornik and Świerniak, 2001) using Theorem 2.

Consider now the following situation: Let the assumptions of Theorem 2 be satisfied, but neither the values $A_N(i)$, $B_N(i)$ nor their limits $A(i)$, $B(i)$ are known for the control purposes. Instead, we know the sequences $\overline{A}_N(i)$, $\overline{B}_N(i)$ of their estimators and we know that the estimators are strongly consistent, i.e.

$$\lim_{N\to\infty}\big\|A_N(j)-\overline{A}_N(j)\big\|=0,$$

$$\lim_{N\to\infty}\big\|B_N(j)-\overline{B}_N(j)\big\|=0,\quad j\in S$$

or, equivalently,

$$\begin{aligned}\lim_{N\to\infty}\overline{A}_N(j)&=A(j),\\ \lim_{N\to\infty}\overline{B}_N(j)&=B(j),\quad j\in S.\end{aligned} \tag{22}$$

It appears that under this assumption we are still able to solve the optimal control problem (1), (2). For that purpose, we will use the following theorem, which is shown in (Czornik, 2002):

Theorem 5. *Suppose that the matrices* $L(i)$ *are such that for the control*

$$u_k=-L\big(r(k)\big)x_k$$

the cost functional (2) takes a value of J*. Suppose now that the matrices* $L_N(i)$ *are such that*

$$\lim_{N\to\infty}L_N(i)=L(i),\quad i\in S.$$

Then if the control

$$u_k=-L_N\big(r(k)\big)x_k$$

is applied, the value of the cost functional (2) is J.

Now we can propose a solution to the adaptive control problem formulated above.

Theorem 6. *Suppose that neither the values* $A_N(i)$, $B_N(i)$ *nor their limits* $A(i)$, $B(i)$ *are known, and that the assumptions of Theorem 2 are satisfied. Moreover, let the sequences* $\overline{A}_N(i)$, $\overline{B}_N(i)$ *of known matrices be such that (22) holds. Then the control*

$$\widetilde{u}_k = -\overline{L}_N(i)x_k, \quad i \in S, \tag{23}$$

where

$$\overline{L}_N(i) = \Big(R_N(i) + \overline{B}_N^{'}(i)F_N(i)\overline{B}_N(i)\Big)^{-1} \times \overline{B}_N^{'}(i)F_N(i)\overline{A}_N(i),$$

$$F_N(i) = \sum_{j \in S} p_{ij} P_N(j),$$

and $P_N(i)$ *is the unique solution of*

$$P_N(i) = \overline{A}_N^{'}(i)F_N(i)\big(\overline{A}_N(i) - \overline{B}_N(i)\overline{L}_N(i)\big) + Q_N(i), \tag{24}$$

when it exists and zero otherwise, is optimal for the cost functional (2).

Proof. From Theorem 3 we conclude that there exists N_0 such that for all $N \geq N_0$ the coupled algebraic Riccati equation (24) exists and that

$$\lim_{N\to\infty} \overline{L}_N(i) = L(i),$$

where $L(i)$ is given by (20). Then by Theorem 4, $L(i)$ is the optimal feedback, and therefore by Theorem 5 we conclude that control $\widetilde{u}_k$ is optimal for the control problem (1), (2). ■

5. Conclusion

In this paper the discrete time-varying JLQG problem has been revisited. It was shown that for a system with coefficients having limits as time tends to infinity the optimal control can be realized in the form of a time-invariant feedback with the feedback matrix equal to the one for the time invariant system with coefficients equal to the limits of the time-varying system. Based on this fact, a solution to the adaptive control problem was proposed under the assumption that strongly consistent estimators of unknown parameters are available.

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References

Abou-Kandil H., Freiling G. and Jank G. (1994): *Solution and asymptotic behavior of coupled Riccati equations in jump linear systems*. — IEEE Trans. Automat. Contr., Vol. 39, No. 8, pp. 1631–1636.

Abou-Kandil H., Freiling G. and Jank G. (1995): *On the solution of discrete-time Markovian jump linear quadratic control problems*. — Automatica, Vol. 31, No. 5, pp. 765–768.

Caines P.E. and Zhang J.F. (1995): *On adaptive control of jump parameter systems via nonlinear filtrating*. — SIAM J. Contr. Optim., Vol. 33, No. 6, pp. 1758–1777.

Costa O.L.V. and Fragoso M.D. (1995): *Discrete-time LQ-optimal control problems for infinite Markov jump parameter systems*. — IEEE Trans. Automat. Contr., Vol. 40, No. 12, pp. 2076–2088.

Chizeck H.J., Willsky A.S. and Castanon D. (1986): *Discrete-time Markovian-jump linear quadratic optimal control*. — Int. J. Contr., Vol. 43, No. 1, pp. 213–231.

Czornik A. (1998): *On time-varying LQG*. — Prep. IFAC Conf. *System Structure and Control*, France, Nantes, pp. 427–432.

Czornik A. (1999): *On discrete-time linear quadratic-control*. — Syst. Contr. Lett., Vol. 36, No. 2, pp. 101–107.

Czornik A. (2000): *Continuity of the solution of the Riccati equations for continuous time JLQP*. — IEEE Trans. Automat. Contr., Vol. 45, No. 5, pp. 934–937.

Czornik A. (2002): *Adaptive control for jump linear system with quadratic cost*. — Int. J. Adapt. Contr. Signal Process., (submitted).

Czornik A. and Świerniak A. (2001): *On the discrete JLQ and JLQG problems*. — Nonlin. Anal., Vol. 47, No. 1, pp. 423–434.

Dufour F. and Bertrand P. (1994): *Stabilizing control law for hybrid models*. — IEEE Trans. Automat. Contr., Vol. 38, No. 11, pp. 2354–2357.

Dufor F. and Elliott R. (1998): *Adaptive control for linear systems with markov perturbations*. — IEEE Trans. Automat. Contr., Vol. 43, No. 3, pp. 351–372.

Ghosh M.K. (1995): *A note on an LQG regulator with Markovian switching and pathwise average cost*. — IEEE Trans. Automat. Contr., Vol. 40, No. 2, pp. 1919–1921.

Ghaoui L. (1996): *LMI optimization for nonstandard Riccati equations arising in stochastic control*. — IEEE Trans. Automat. Contr., Vol. 41, No. 11, pp. 1666–1671.

Griffiths B.E. and Loparo K. (1985): *Optimal control of jump-linear gaussian systems*. — Int. J. Contr., Vol. 42, No. 4, pp. 791–819.

Ji Y. and Chizeck H.J. (1988): *Controllability, observability and discrete-time markovian jump linear quadratic control*. — Int. J. Contr., Vol. 48, No. 2, pp. 481–498.

Ji Y. and Chizeck H.J. (1989): *Optimal quadratic control of jump linear systems with separately controlled transition probabilities*. — Int. J. Contr., Vol. 49, No. 2, pp. 481–491.

Mariton M. (1987): *Jump linear quadratic control with random state discontinuities*. — Automatica, Vol. 23, No. 2, pp. 237–240.

Pan G. and Bar-Shalom Y. (1996): *Stabilization of jump linear gaussian systems without mode observations*. — Int. J. Contr., Vol. 64, No. 4, pp. 631–666.

Sworder D.D. (1969): *Feedback control of a class of linear systems with jump parameters.* — IEEE Trans. Automat. Contr., Vol. 14, No. 1, pp. 9–14.

Sworder D.D. and Robinson V.G. (1973): *Feedback regulators for jump parameter systems with state and control dependent transition rates*. — IEEE Trans. Automat. Contr. Vol. 18, No. 4, pp. 355–359.

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