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EXTERNALLY AND INTERNALLY POSITIVE SINGULAR DISCRETE-TIME LINEAR SYSTEMS

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Notions of externally and internally positive singular discrete-time linear systems are introduced. It is shown that a singular discrete-time linear system is externally positive if and only if its impulse response matrix is non-negative. Sufficient conditions are established under which a single-output singular discrete-time system with matrices in canonical forms is internally positive. It is shown that if a singular system is weakly positive (all matrices E, A, B, C are non-negative), then it is not internally positive.

Keywords: externally, internally, positive, singular, linear, system

1. Introduction

Singular (descriptor) discrete-time linear systems were considered in many papers and books (Cobb, 1984; Dai, 1989; Kaczorek, 1993; 1998b; Klamka, 1991; Lewis, 1984; 1986; Luenberger, 1977; 1978; Mertzios and Lewis, 1989; Ohta *et al.*, 1984). The properties of fundamental matrices of singular discrete-time linear systems were established and their solution was derived in (Lewis, 1986; Mertzios and Lewis, 1989). The reachability and controllability of singular and positive linear systems were considered in (Cobb, 1984; Dai, 1989; Fanti *et al.*, 1990; Kaczorek, 1993; Klamka, 1991; Ohta *et al.*, 1984). The notions of weakly positive discrete-time and continuous-time linear systems were introduced in (Kaczorek, 1997; 1998a; 1998b).

In the present paper a new class of externally and internally positive discrete-time linear systems will be introduced. Necessary and sufficient conditions will be established under which singular discrete-time linear systems are externally and internally positive. It will be shown that the singular weakly positive linear system is not internally positive.

2. Preliminaries

Let $\mathbb{Z}_+$ be the set of non-negative integers, $\mathbb{R}^{n\times m}$ be the set of $n\times m$ real matrices and $\mathbb{R}^m := \mathbb{R}^{m\times 1}$. The set of $m\times n$ real matrices with non-negative entries will be denoted by $\mathbb{R}_+^{m\times n}$ and $\mathbb{R}_+^m := \mathbb{R}_+^{m\times 1}$.

Consider the singular discrete-time linear system

$$Ex_{i+1} = Ax_i + Bu_i, \tag{1a}$$

$$y_i = Cx_i, \tag{1b}$$

where $i\in\mathbb{Z}_+$. Here $x_i\in\mathbb{R}^n$, $u_i\in\mathbb{R}^m$, $y_i\in\mathbb{R}^p$ are the state, input and output vectors, respectively, and $E, A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$. It is assumed that $\det E = 0$ and

$$\det[Ez - A] \neq 0 \tag{2}$$

for some $z\in\mathbb{C}$ (the field of complex numbers). If (2) holds, then (Kaczorek, 1993; Lewis, 1984)

$$[Ez - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)}, \tag{3}$$

where μ is the nilpotence index and the Φ_i's are the fundamental matrices satisfying the relations (Kaczorek, 1993; Lewis, 1984)

$$E\Phi_i - A\Phi_{i-1} = \Phi_i E - \Phi_{i-1}A = \begin{cases} I & \text{for } i = 0,\\ 0 & \text{for } i\neq 0,\end{cases} \tag{4}$$

and $E\Phi_{-\mu} = 0, \Phi_i = 0$ for $i < -\mu$, I and 0 being the identity and zero matrices, respectively.

The solution x_i to (1a) with admissible initial conditions is given by (Kaczorek, 1993; Lewis, 1984)

$$x_i = \Phi_i Ex_0 + \sum_{k=0}^{i+\mu-1} \Phi_{i-k-1}Bu_k \tag{5}$$

and the output y_i is determined by the formula

$$y_i = C\Phi_i E x_0 + \sum_{k=0}^{i+\mu-1} C\Phi_{i-k-1}Bu_k. \qquad (6)$$

Let $g_k \in \mathbb{R}^{p\times m}$, $k = 1-\mu, 2-\mu, \dots, 0, 1, \dots$ be the impulse response of the system (1). Applying the superposition principle and substituting

$$u_k = \begin{cases} 1 & \text{for } k=0, \\ 0 & \text{for } k>0 \end{cases}$$

and $x_0 = 0$ into (6), we obtain

$$g_i = C\Phi_{i-1}B \quad \text{for } i = 1-\mu, \dots, 0, 1, \dots . \qquad (7)$$

Using (7), we may write (6) in the form

$$y_i = C\Phi_i E x_0 + \sum_{k=0}^{i+\mu-1} g_{i-k}u_k. \qquad (8)$$

The transfer matrix of (1) is given by

$$T(z) = C[Ez-A]^{-1}B. \qquad (9)$$

From (3), (9) and (7) we obtain

$$T(z) = \sum_{i=-\mu}^{\infty} C\Phi_i B z^{-(i+1)} = \sum_{j=1-\mu}^{\infty} g_j z^{-j}. \qquad (10)$$

From (10) it follows that the impulse response matrix g_j can be found by expansion of $T(z)$.

Using (4) it can be shown that (Mertzios and Lewis, 1989)

$$\Phi_0 A \Phi_i = \begin{cases} \Phi_{i+1} & \text{for } i \geq 0, \\ 0 & \text{for } i < 0 \end{cases} \qquad (11a)$$

and

$$-\Phi_{-1} E \Phi_i = \begin{cases} 0 & \text{for } i \geq 0, \\ \Phi_{i-1} & \text{for } i < 0. \end{cases} \qquad (11b)$$

From (11a) we have $\Phi_1 = \Phi_0(A\Phi_0)$, $\Phi_2 = \Phi_0 A \Phi_1 = \Phi_0(A\Phi_0)^2$ and

$$\Phi_i = \Phi_0(A\Phi_0)^i \quad \text{for } i \geq 1. \qquad (12a)$$

Similarily, from (11b) we obtain $\Phi_{-2} = -\Phi_{-1}E\Phi_{-1}$, $\Phi_{-3} = \Phi_{-1}E\Phi_{-2} = (-\Phi_{-1}E)^2\Phi_{-1}$ and

$$\Phi_{-j} = (-\Phi_{-1}E)^{j-1}\Phi_{-1} \quad \text{for } j \geq 1. \qquad (12b)$$

3. Externally Positive Singular Systems

Definition 1. The singular system (1) is called *externally positive* if for any input sequence $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$ and the zero initial condition $x_0 = 0$ we have $y_i \in \mathbb{R}_+^p$ for $i \in \mathbb{Z}_+$.

Theorem 1. *The system (1) is externally positive if and only if*

$$g_i \in \mathbb{R}_+^{p\times m} \quad \textit{for } i = 1-\mu, \dots, 0, 1, \dots . \qquad (13)$$

Proof. The necessity follows immediately from Definition 1. To prove the sufficiency, note that for $x_0 = 0$ and $u_k \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$, from (8) we obtain

$$y_i = \sum_{k=0}^{i+\mu-1} g_{i-k}u_k \in \mathbb{R}_+^p$$

since (13) holds. ■

To simplify the notation, we shall assume that $m = p = 1$ and

$$E = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$A = \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline \multicolumn{2}{c}{a} \end{array}\right] \in \mathbb{R}^{n\times n},$$

$$a = [a_0 \ \ a_1 \ \ \cdots \ \ a_{r-1} \ \ -1 \ \ 0 \ \ \cdots \ \ 0], \qquad (14)$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n,$$

$$C = [b_0 \ \ b_1 \ \ \cdots \ \ b_{n-1}] \in \mathbb{R}^{1\times n}.$$

Theorem 2. *If the matrices E, A, B, C have the canonical form (14),*

and
$$\begin{aligned} a_i &\geq 0, \quad i = 0, 1, \dots, r-1 \\ b_j &\geq 0, \quad j = 0, 1, \dots, n-1, \end{aligned} \qquad (15)$$

then

$$\Phi_k B \in \mathbb{R}_+^n \quad \textit{for } k = -\mu, 1-\mu, \dots, \qquad (16)$$

$$\Phi_i \in \mathbb{R}_+^{n\times n} \quad \textit{for } i \in \mathbb{Z}_+, \qquad (17)$$

$$g_j \in \mathbb{R}_+^{p\times m} \quad \textit{for } j = 1-\mu, 2-\mu, \dots . \qquad (18)$$

Proof. If $E,\ A$ and B have the canonical form (14), then it is easy to show that

$$[Ez-A]_{\mathrm{ad}}B=\begin{bmatrix}1\\ z\\ \vdots\\ z^q\end{bmatrix}=H_qBz^q$$
$$+\cdots+H_1Bz+H_0B, \quad \text{(19a)}$$

where

$$H_qB=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix},\ \dots,\quad H_0B=\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix}. \quad \text{(19b)}$$

From (A4) (see the Appendix) and (19) it follows that $\Phi_kB\in\mathbb{R}_+^n$, $k=-\mu,1-\mu,\dots,r-1$ since $H_kB\in\mathbb{R}_+^n$, $k=-\mu,1-\mu,\dots,r-1$ and $q_k\geq 0$ for $k=1,2,\dots$.

From (A6) we have

$$\Phi_{r+k}B=\sum_{j=1}^{r}a_{r-j}\Phi_{r+k-j}B\in\mathbb{R}_+^n \quad \text{for } k=0,1,\dots \quad \text{(20)}$$

since by (15) we have $a_i\geq 0$ for $i=0,1,\dots,r-1$.

From (A4), (A8) and (A9) we get

$$\Phi_0=q_\mu H_q+q_{\mu-1}H_{q-1}+\cdots+q_0H_{r-1}$$
$$=\left[\begin{array}{c:c:c} I_r & & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \cdots\cdots & 0 & \\ W & & \begin{matrix}q_0\\ q_1\\ \vdots\\ q_{n-r}\end{matrix}\end{array}\right]\in\mathbb{R}_+^{n\times n}, \quad \text{(21)}$$

where $W=[w_{ij}]\in\mathbb{R}_+^{(n-r)\times r}$, $w_{ij}=\sum_{l=1}^{j}a_{j-l}q_{i-l}$ and

$$A\Phi_0=q_\mu AH_q+q_{\mu-1}AH_{q-1}+\cdots+q_0AH_{r-1}$$
$$=A\left(q_\mu H_{q-1}+q_{\mu-1}H_{q-2}+\cdots+q_0H_{r-2}\right)$$
$$=\left[\begin{array}{c:c:c:c} \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_{r-1} & & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \hdashline \multicolumn{2}{c:}{W} & 0 & \begin{matrix}q_0\\ \vdots\\ q_{n-r}\end{matrix}\\ \hdashline \multicolumn{2}{c:}{0\ \cdots\ 0} & & 0\end{array}\right]\in\mathbb{R}_+^{n\times n}. \quad \text{(22)}$$

From (12a) and (22) we have

$$\Phi_i=\Phi_0(A\Phi_0)^i\in\mathbb{R}_+^{n\times n} \quad \text{for } i=1,2,\dots. \quad \text{(23)}$$

Using (7) and (16), we obtain

$$g_j=C\Phi_{j-1}B\in\mathbb{R}_+^{p\times m} \quad \text{for } j=1-\mu,2-\mu,\dots. \quad \text{(24)}$$

■

4. Internally Positive Singular Systems

Definition 2. The system (1) is called *internally positive* if for any admissible initial conditions $x_0\in\mathbb{R}_+^n$ and all input sequences $u_i\in\mathbb{R}_+^m$, $i\in\mathbb{Z}_+$ we have $x_i\in\mathbb{R}_+^n$ and $y_i\in\mathbb{R}_+^p$ for $i\in\mathbb{Z}_+$.

From the comparison of Definitions 1 and 2 it follows that if the system (1) is internally positive, then it is always externally positive, but if the system (1) is externally positive, it may not be internally positive.

Theorem 3. *The system (1) with (14) is internally positive if relations (15) hold.*

Proof. By Theorem 2, if (15) hold, then $\Phi_i\in\mathbb{R}_+^{n\times n}$ for $i\in\mathbb{Z}_+$ and $\Phi_kB\in\mathbb{R}_+^n$ for $k=-\mu,1-\mu,\dots$. Hence, using (5), we obtain $x_i\in\mathbb{R}_+^n$ for $i\in\mathbb{Z}_+$ for any $x_0\in\mathbb{R}_+^n$ and all $u_i\in\mathbb{R}_+^m$. Similarly, taking into account that $g_j\in\mathbb{R}_+^{p\times m}$ for $j=1-\mu,2-\mu,\dots$, from (8) we obtain $y_i\in\mathbb{R}_+^p$ for $i\in\mathbb{Z}_+$. ■

Consider the system (1) with

$$E=\begin{bmatrix}I_{n-1} & 0\\ 0 & 0\end{bmatrix}\in\mathbb{R}^{n\times n},\quad A=\begin{bmatrix}A_1\\ A_2\end{bmatrix},\quad B=\begin{bmatrix}B_1\\ B_2\end{bmatrix}, \quad \text{(25)}$$

where $A_1 \in \mathbb{R}^{(n-1)\times n}$, $A_2 \in \mathbb{R}^{1\times n}$, $B_1 \in \mathbb{R}^{n-1}$, $B_2 \in \mathbb{R}$ and $C \in \mathbb{R}^{1\times n}$. From (1a) for $i = 0$ and (25) we have

$$0 = A_2 x_0 + B_2 u_0. \tag{26}$$

Equation (26) determines the set of admissible initial conditions for a given input sequence u_i, $i \in \mathbb{Z}_+$.

Note that the assumption (2) implies that A_2 is not a zero row and the singularity of the system implies that at least one entry of A_2 is zero.

From (26) for $u_0 = 0$ it follows that the equation $A_2 x_0 = 0$, $x_0 \in \mathbb{R}^n_+$, $x_0 \neq 0$ can be satisfied if A_2 contains at least one positive entry and at least one negative entry. Hence we have the following important corollaries:

Corollary 1. *The singular system (1) with (25) is not internally positive if* $A \in \mathbb{R}^{n\times n}_+$.

Corollary 2. *The singular weakly positive (Kaczorek, 1998a; 1998b) system (1) with (25) is not internally positive.*

5. Example

Consider the singular system (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -1 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = [b_0 b_1 b_2], \tag{27}$$

and $a \geq 0$, $b_i \geq 0$, $i = 0, 1, 2$. In this case $n = 3$, $r = 1$, $\mu = n - r = 2$ and

$$[Ez - A]^{-1} = \begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ -a & 1 & 0 \end{bmatrix}^{-1}$$
$$= \frac{1}{z-a} \begin{bmatrix} 1 & 0 & 1 \\ a & 0 & z \\ az & a-z & z^2 \end{bmatrix}$$
$$= \Phi_{-2} z + \Phi_{-1} + \Phi_0 z^{-1} + \Phi_1 z^{-2} + \cdots,$$

where

$$\Phi_{-2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Phi_{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ a & -1 & a \end{bmatrix},$$
$$\Phi_0 = \begin{bmatrix} 1 & 0 & 1 \\ a & 0 & a \\ a^2 & 0 & a^2 \end{bmatrix}, \qquad A\Phi_0 = \begin{bmatrix} a & 0 & a \\ a^2 & 0 & a^2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{28}$$
$$\Phi_i = \Phi_0 (A\Phi_0)^i, \quad i \geq 1.$$

Using (7), we obtain

$$\begin{aligned} g_{-1} &= C\Phi_{-2}B = b_2, \\ g_0 &= C\Phi_{-1}B = b_1 + b_2 a, \\ g_1 &= C\Phi_0 B = b_0 + b_1 a + b_2 a^2, \\ g_2 &= C\Phi_1 B = b_0 a + b_1 a^2 + b_2 a^3, \\ g_i &= a^{i-1} g_1, \quad i \geq 2. \end{aligned} \tag{29}$$

From (28) and (29) it follows that for the system (1) with (27), the conditions (16)–(18) are satisfied.

The transfer function of (1) with (27) has the form

$$T(z) = C[Ez - A]^{-1} B = \frac{b_2 z^2 + b_1^2 + b_0}{z - a}. \tag{30}$$

Expansion of (30) yields

$$T(z) = g_{-1} z + g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots,$$

where

$$g_{-1} = b_2, \quad g_0 = b_1 + b_2 a, \quad g_1 = b_0 + b_1 a + b_2 a^2$$
$$\text{and} \quad g_k = a^{k-1} g_1 \quad \text{for} \quad k \geq 2. \tag{31}$$

This result agrees with (29).

By Theorem 1, the system (1) with (27) is externally positive since $g_j \geq 0$ for $j = -1, 0, 1, \ldots$. By Theorem 3, the system (1) with (27) is also internally positive.

6. Concluding Remarks

The notions of externally and internally positive singular discrete-time linear systems have been introduced. It has been shown that:

1. The singular discrete-time linear system (1) is externally positive if and only if its impulse response matrix $g_i \in \mathbb{R}^{p\times m}_+$ for $i > -\mu$.
2. The singular system (1) with (14) is internally positive if the conditions (15) are satisfied.

3. If the singular system (1) with (25) is weakly positive, then it is not internally positive.

The consideration presented for single-input single-output discrete-time linear systems can be easily extended to multi-input multi-output singular discrete-time linear systems.

An extension to singular continuous-time linear systems is also possible. A generalization of this approach to singular two-dimensional linear systems (Kaczorek, 1993) will be considered in a separate paper.

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Appendix

Lemma 1. *Let*

$$\begin{aligned} p(z) &:= \det[Ez - A] \\ &= z^r - a_{r-1}z^{r-1} - \cdots - a_1 z - a_0, \end{aligned} \tag{A1}$$

$$[Ez - A]_{\mathrm{ad}} = H_q z^q + \cdots + H_1 z + H_0, \tag{A2}$$

and

$$[Ez - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)}. \tag{A3}$$

Then

$$\begin{bmatrix} \Phi_{-\mu} \\ \Phi_{1-\mu} \\ \Phi_{2-\mu} \\ \vdots \\ \Phi_{r-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ q_1 & 1 & 0 & \cdots & 0 & 0 \\ q_2 & q_1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1} & q_{n-2} & q_{n-3} & \cdots & q_1 & 1 \end{bmatrix} \begin{bmatrix} H_q \\ H_{q-1} \\ H_{q-2} \\ \vdots \\ H_0 \end{bmatrix}, \tag{A4}$$

where $n = r + \mu, \ \ q = n - 1$,

$$q_k := \sum_{i=1}^{k} a_{r-i} q_{k-i} \quad \textit{for} \quad k = 1, 2, \ldots \quad (q_0 := 1), \tag{A5}$$

and

$$\Phi_{r+k} = \sum_{j=1}^{r} a_{r-j} \Phi_{r+k-j} \quad \textit{for} \quad k = 0, 1, \ldots. \tag{A6}$$

Proof. Using the well-known equality $[Ez - A]_{\mathrm{ad}} = (\det[Ez - A])\,[Ez - A]^{-1}$, and (A1), (A2) with (A3), we can write

$$\begin{aligned} &\left(H_q z^q + H_{q-1} z^{q-1} + \cdots + H_1 z + H_0\right) \\ &\quad = \left(z^r - a_{r-1} z^{r-1} - \cdots - a_1 z - a_0\right) \\ &\qquad \times \left(\Phi_{-\mu} z^{\mu-1} + \Phi_{1-\mu} z^{\mu-2} + \cdots \right. \\ &\qquad \left. + \Phi_{-1} + \Phi_0 z^{-1} + \Phi_1 z^{-2} + \cdots \right). \end{aligned} \tag{A7}$$

The comparison of the coefficients at the same powers of z^k for $k = q, q-1, \ldots, 0$ of (A7) yields

$$\begin{aligned} \Phi_{-\mu} &= H_q, \quad H_{q-1} = \Phi_{1-\mu} - a_{r-1}\Phi_{-\mu}, \\ \Phi_{1-\mu} &= H_{q-1} + a_{r-1} H_q, \\ H_{q-2} &= \Phi_{2-\mu} - a_{r-1}\Phi_{1-\mu} - a_{r-2}\Phi_{-\mu}, \\ \Phi_{2-\mu} &= H_{q-2} + a_{r-1}\Phi_{1-\mu} + a_{r-2}\Phi_{-\mu} \\ &= H_{q-2} + a_{r-1} H_{q-1} + \left(a_{r-1}^2 + a_{r-2}\right) H_q \\ &= H_{q-2} + q_1 H_{q-1} + q_2 H_q \end{aligned}$$

and (A4), where q_k is defined by (A5).

Comparing the coefficients of (A7) at $z^{-1}, z^{-2}, \ldots,$ we obtain

$$\Phi_r = a_{r-1}\Phi_{r-1} + a_{r-2}\Phi_{r-2} + \cdots + a_0\Phi_0,$$

$$\Phi_{r+1} = a_{r-1}\Phi_r + a_{r-2}\Phi_{r-1} + \cdots + a_0\Phi_1,$$

and the formula (A6). ■

Lemma 2. *Let $H_k,\ k = 0, 1, \ldots, q$ be defined by (A2) and let the matrices E,A have the canonical form (14). Then*

$$AH_k = \begin{cases} EH_{k-1} + a_k I_n & \text{for } k = 1, \ldots, r-1, \\ EH_{r-1} - I_n & \text{for } k = r, \\ EH_{k-1} & \text{for } k = r+1, \ldots, q, \end{cases} \tag{A8}$$

$$H_0 = \left[\begin{array}{c:c} -a^{(0)} & \\ \cdots\cdots & e_1 \\ a_0 I_q & \end{array}\right],$$

$$a^{(0)} := [a_1\ a_2\ \cdots\ a_{r-1}\ -1\ 0\ \cdots\ 0],$$

$$H_i = \left[\begin{array}{c:c} -a^{(1)} & \\ \vdots & \\ -a^{(i+1)} & e_{i+1} \\ \bar{a}^{(1)} & \\ \vdots & \\ \bar{a}^{(q-i)} & \end{array}\right] \quad \text{for } i = 1, \ldots, r-1,$$

$$a^{(i)} = [\overbrace{0\ \cdots\ 0}^{i-1}\ a_{i+1}\ \cdots\ a_{r-1}\ -1\ 0\ \cdots\ 0],$$

$$\bar{a}^{(j)} := [\overbrace{0\ \cdots\ 0}^{j-1}\ a_0\ \cdots\ a_i\ 0\ \cdots\ 0], \quad j = 1, \ldots, q-i,$$

$$H_i = \left[\begin{array}{c} \left.\begin{array}{cccc} 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{array}\right\} n-i+1 \\ \cdots\cdots\cdots \\ \hat{a}^{(1)} \\ \vdots \\ \hat{a}^{(i-1)} \end{array}\right] \quad \text{for } i = r, \ldots, n-2,$$

$$\hat{a}^{(j)} := [\overbrace{0\ \cdots\ 0}^{j-1}\ a_0\ a_1\ \cdots\ a_{r-1}\ -1\ 0\ \cdots\ 0],$$

$$j = 1, \ldots, i-1,$$

$$H_{n-1} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{A9}$$

Here e_i is the i-th column of the identity matrix I_n and $a_i,\ i = 0, 1, \ldots, r-1$ are the coefficients of the polynomial (A1).

Proof. Using the equality $[Ez - A][Ez - A]_{\mathrm{ad}} = I_n \det[Ez - A]$ and (A1), (A2), we may write

$$\begin{aligned} &[Ez - A]\left[H_q z^q + H_{q-1} z^{q-1} + \cdots + H_1 z + H_0\right] \\ &\quad = I_n \left(z^r - a_{r-1} z^{r-1} - \cdots a_1 z - a_0\right). \end{aligned} \tag{A10}$$

The comparison of the coefficients at the same powers of z of (A10) yields

$$AH_0 = I_n a_0, \quad AH_1 = EH_0 + a_1 I_n, \quad \cdots,$$

$$AH_{r-1} = EH_{r-2} + a_{r-1} I_n, \quad AH_r = EH_{r-1} - I,$$

$$AH_{r+1} = EH_r, \quad \cdots, \quad AH_q = EH_{q-1}, \quad EH_q = 0.$$

It is easy to check that it satisfies the equality $AH_0 = I_n a_0$.

Using the canonical form of E and A, it is easy to show that

$$[Ez - A]_{\mathrm{ad}}$$

$$= \begin{bmatrix} m_{11} & m_{12} & \cdots & 0 & 0 & 1 \\ a_0 & m_{22} & \cdots & 0 & 0 & z \\ a_0 z & a_1 z + a_0 & \cdots & 0 & 0 & z^2 \\ a_0 z^2 & z(a_1 z + a_0) & \cdots & 0 & 0 & z^3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_0 z^{n-3} & z^{n-4}(a_1 z + a_0) & \cdots & -p(z) & 0 & z^{n-2} \\ a_0 z^{n-2} & z^{n-3}(a_1 z + a_0) & \cdots & -zp(z) & -p(z) & z^{n-1} \end{bmatrix}$$

$$= H_q z^q + H_{q-1} z^{q-1} + \cdots + H_1 z + H_0, \tag{A11}$$

where $m_{11} = z^{r-1} - a_{r-1} z^{r-2} - \cdots - a_1$, $m_{12} = z^{r-2} - a_{r-1} z^{r-3} - \cdots - a_2$, $m_{22} = z(z^{r-2} - a_{r-1} z^{r-3} - \cdots - a_2)$, $p(z)$ being defined by (A1).

The comparison of the coefficients at the same powers of z^k for $k = 0, 1, \ldots, q$ of (A11) yields (A9). ■

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