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A GENERAL SOLUTION TO THE OUTPUT-ZEROING PROBLEM FOR MIMO LTI SYSTEMS

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The problem of zeroing the output in an arbitrary linear continuous-time system $S(A,B,C,D)$ with a nonvanishing transfer function is discussed and necessary conditions for output-zeroing inputs are formulated. All possible real-valued inputs and real initial conditions which produce the identically zero system response are characterized. Strictly proper and proper systems are discussed separately.

Keywords: linear multivariable systems, state-space methods, output-zeroing problem, invariant zeros

1. Introduction

As is known, the problem of zeroing the system output is strictly related to the notion of multivariable zeros. These zeros, however, are defined in many, not necessarily equivalent, ways (for a survey of these definitions see (MacFarlane and Karcanias, 1976; Schrader and Sain, 1989; Latawiec *et al*., 2000), where a new concept of the so-called "control zeros" was introduced and analysed). The most commonly used definition employs the Smith canonical form of the system (Rosenbrock) matrix and determines these zeros (which will be called in the sequel the Smith zeros) as the roots of diagonal (invariant) polynomials of the Smith form (Emami-Naeini and Van Dooren, 1982; Rosenbrock, 1970). Equivalently, the Smith zeros are defined as the points of the complex plane where the system matrix loses its normal rank. This definition treats zeros merely as complex numbers and for this reason it may create difficulties in their dynamical state-space interpretation. Most likely in order to overcome these difficulties, MacFarlane and Karcanias (1976) added to the notion of the Smith zeros the notions of state-zero and input-zero directions and gave certain dynamical (geometric) interpretation of these zeros. The output-zeroing problem in relationship with the Smith zeros was studied, under certain simplifying assumptions concerning the systems considered, in (Karcanias and Kouvaritakis, 1979; MacFarlane and Karcanias, 1976), and was interpreted geometrically in (Isidori, 1995, pp. 164, 296).

A more detailed analysis indicates, however, that for characterizing the output-zeroing problem the notion of Smith zeros may be too narrow. This observation can be motivated by a simple numerical example (see Example 4, Section 4) of a minimal (reachable and observable) and asymptotically stable system in which there are no Smith zeros and one could infer wrongly that there are no output-zeroing inputs which give nontrivial solutions of the state equation. However, extending in a natural way the concept of the Smith zeros, it is possible to show that there are infinitely many real-valued inputs for this system which give nontrivial solutions and the identically zero system response.

Such an extension is based on the definition of invariant zeros, see (Tokarzewski, 1998; 2000b) and (1a) below, which employs the system matrix and zero directions and treats the zeros as the triples (complex number, nonzero state-zero direction, input-zero direction). This definition enables us to extend in (Tokarzewski, 1998) the results of (El-Ghezawi *et al.*, 1982) (where square strictly proper systems of uniform rank are analysed) on nonsquare systems (by using the Moore-Penrose pseudoinverse and the singular value decomposition (SVD) of the first nonzero Markov parameter), as well as relate system zeros to the notions of reachability and observability (by using the Kalman canonical form and classical definitions of decoupling zeros). A crucial role in characterization of invariant and decoupling zeros is played in (Tokarzewski, 1998) by matrices $A - BD^{+}C$ and $K_k A$ (see Section 3 below, where these matrices appear in the characterization of the output-zeroing problem).

The invariant zeros defined in this way (see (1a) below) are invariant under similarity transformations of the state space and under constant state feedbacks. They do not change after introducing a nonsingular pre- or post-compensator to $S(A,B,C,D)$. Moreover, as is shown in (Tokarzewski, 2000b; Prop. 1), each Smith zero is also an invariant zero. The main differences between invariant and Smith zeros are as follows: The number of Smith zeros is always finite, while the number of invariant zeros may be infinite (then a system is called degenerate). On the other hand, each output decoupling zero is always an invariant zero, which is not the case when the Smith zeros are considered. In some cases the Smith zeros and invariant zeros coincide. It takes place, e.g., when the system matrix is of full column normal rank (Tokarzewski, 2000b; Cor. 1). This concerns in particular the classes of all systems diagonally decouplable by a static state feedback and of all systems of uniform rank (in particular, of all SISO systems with nonzero transfer function).

Because, as is noticed in (Tokarzewski, 2000b; Rem. 1) (see also Remarks 1 and 4 below), to each invariant zero we can assign a real initial condition and a real-valued input which produce the zero output, the invariant zeros can be easily interpreted (even in the degenerate case) in the context of the output-zeroing problem. Of course, since each Smith zero is an invariant zero, this interpretation remains valid also for Smith zeros.

Taking into account the above concept of invariant zeros, we can state the following question (cf. Tokarzewski, 2000a): Find a state-space characterization of the output-zeroing problem (at least in the form of necessary conditions for initial conditions and inputs zeroing the system output) which could convey in a compact form information about invariant zeros and their action in a system. More precisely, we want to characterize in a simple manner all the possible real-valued inputs and real initial conditions which produce the identically zero system response.

2. Preliminaries

Consider a system $S(A,B,C,D)$ with m inputs and r outputs

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$ and $A, B \neq 0$, $C \neq 0, D$ are real matrices of appropriate dimensions. By $\mathbb{U}$ we denote the set of admissible inputs which consists of all piecewise continuous real-valued functions of time $u(\cdot) : [0,\infty) \to \mathbb{R}^m$.

The point of departure for our discussion is the following formulation of the output-zeroing problem (in particular, of the notion of output-zeroing inputs) (see Isidori, 1995, p. 163): Find all pairs $(x^0, u_0(t))$, consisting of an initial state $x^0 \in \mathbb{R}^n$ and an admissible input $u_0(t)$, such that the corresponding output $y(t)$ of (1) is identically zero for all $t \geq 0$. Any nontrivial pair (i.e. such that $x^0 \neq 0$ or $u_0(t) \neq 0$) of this kind is called an output-zeroing input. Note that in each output-zeroing input $(x^0, u_0(t))$, $u_0(t)$ should be understood simply as an open-loop control signal which, when applied to (1) exactly at $x(0) = x^0$, yields $y(t) = 0$ for all $t \geq 0$.

Moreover, we consider the following definition of invariant zeros (Tokarzewski, 1998; 2000b): A complex number λ is an invariant zero of (1) if and only if (iff) there exist vectors $0 \neq x^0 \in \mathbb{C}^n$ (state-zero direction) and $g \in \mathbb{C}^m$ (input-zero direction) such that

$$P(\lambda)\begin{bmatrix} x^0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{1a}$$

where

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

denotes the system matrix. Transmission zeros of (1) are defined as invariant zeros of its minimal subsystem.

The same symbol x^0 is used to denote the *state-zero direction* in the definition of invariant zeros and the *initial state* in the definition of output-zeroing inputs. The *state-zero direction* x^0 must be a nonzero vector (real or complex). Otherwise, the definition of invariant zeros becomes senseless (for any system (1) each complex number may serve as an invariant zero). In other words, in the equation

$$P(\lambda)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the solutions of the form

$$\begin{bmatrix} 0 \\ u \end{bmatrix}$$

are not taken into account in the process of defining invariant zeros.

According to the formulation of the output-zeroing problem, the *initial state* x^0 must be a real vector (but not necessarily nonzero). If the *state-zero direction* x^0 is a complex vector, then it gives two *initial states* $\operatorname{Re} x^0$ and $\operatorname{Im} x^0$ (and, of course, at least one of these initial states must be a nonzero vector).

The differences mentioned above can be easily read out from the text (they are stressed in Remarks 1 and 4, and they are easily seen in Example 4, cf. Section 4).

We denote by M^+ the Moore-Penrose pseudoinverse of matrix M. Recall (Gantmacher, 1988) that for a given $r \times m$ real matrix M of rank p, a factorization $M = M_1 M_2$ with an $r \times p$ matrix M_1 and a $p \times m$ matrix M_2 is called the skeleton factorization of M. The skeleton factorization is not unique; however, in any such factorization M_1 has full column rank (i.e. is monic) and M_2 has full row rank (i.e. is epic). Then M^+ is uniquely determined (i.e. independently upon a particular choice of matrices M_1 and M_2 in the skeleton factorization of M) as $M^+ = M_2^+ M_1^+$, where $M_1^+ = (M_1^T M_1)^{-1} M_1^T$ and $M_2^+ = M_2^T (M_2 M_2^T)^{-1}$. From the definition of M^+ the relations $MM^+M = M$ and $M^+MM^+ = M^+$ follow. If M is square and nonsingular, then $M^+ = M^{-1}$.

Consider the equation $Mz = b$, where M is as above and $b \in \mathbb{R}^r$, and suppose that this equation is solvable (i.e. there exists at least one solution). Then any solution can be expressed in the form $z = z_0^* + z_h$, where $z_0^* = M^+ b$ and z_h is an arbitrary solution of the homogeneous equation $Mz = 0$.

3. Main Results

3.1. Proper Systems ($D \neq 0$)

Let $(x^0, u_0(t))$ be an output-zeroing input for a proper system (1) and let $x_0(t)$ denote the corresponding solution. Then for all $t \geq 0$ we have the equalities

$$\dot{x}_0(t) = Ax_0(t) + Bu_0(t), \quad 0 = Cx_0(t) + Du_0(t),$$
$$x_0(0) = x^0. \tag{2}$$

Consider the following equation:

$$Du(t) = -Cx_0(t) \tag{3}$$

with an unknown function $u(t) \in \mathbb{U}$. Since $u_0(t)$ satisfies (3), it can be written (see Section 2) as

$$u_0(t) = -D^+Cx_0(t) + u_h(t), \tag{4}$$

where $u_h(t)$ is some piecewise continuous function satisfying $Du_h(t) = 0$ for all $t \geq 0$. Because $(x^0, u_0(t))$ is assumed to be known, hence, by the uniqueness of solutions, $x_0(t)$ is known and $u_h(t)$ can be also treated as a known function uniquely determined by (4).

Introducing (4) into the second equality of (2), we get $DD^+Cx_0(t) = Cx_0(t)$, i.e. $x_0(t) \in \operatorname{Ker}(I_r - DD^+)C$ for all $t \geq 0$. Introducing (4) into the first equality of (2), we obtain

$$\begin{aligned} \dot{x}_0(t) &= (A - BD^+C)x_0(t) + Bu_h(t), \\ x_0(0) &= x^0, \quad t \geq 0 \end{aligned} \tag{5}$$

and, consequently,

$$x_0(t) = e^{t(A-BD^+C)}x^0 + \int_0^t e^{(t-\tau)(A-BD^+C)}Bu_h(\tau)\,\mathrm{d}\tau. \tag{6}$$

From (6) and (4) it follows that

$$\begin{aligned} u_0(t) = &-D^+Ce^{t(A-BD^+C)}x^0 \\ &-D^+C\int_0^t e^{(t-\tau)(A-BD^+C)}Bu_h(\tau)\,\mathrm{d}\tau + u_h(t). \end{aligned} \tag{7}$$

Thus we have the following characterization of output-zeroing inputs and the corresponding solutions:

Proposition 1. *Let $(x^0, u_0(t))$ be an output-zeroing input for a proper system (1) and let $x_0(t)$ denote the corresponding solution. Then $x^0 \in \operatorname{Ker}(I_r - DD^+)C$, and $u_0(t)$ is of the form (7) for some function $u_h(t) \in \mathbb{U}$ satisfying $Du_h(t) = 0$ for all $t \geq 0$, and $x_0(t)$ is of the form (6). Moreover, $x_0(t) \in \operatorname{Ker}(I_r - DD^+)C$ for all $t \geq 0$.*

Remark 1. Naturally, Proposition 1 does not tell us whether the output-zeroing inputs exist. However, if the set of invariant zeros is nonempty, for each such zero there exists an output-zeroing input (see (i) below) which in turn may be characterized as in Proposition 1 (see (ii) below). In order to discuss output-zeroing inputs corresponding to invariant zeros, it is convenient to treat system (1) as a complex one, i.e. admitting complex inputs, solutions and outputs, which are denoted respectively by $\tilde{u}$, $\tilde{x}$ and $\tilde{y}$.

(i) Suppose that $\lambda \in \mathbb{C}$ is an invariant zero of (1), i.e. a triple λ, $x^0 \neq 0$, g satisfies (1a). Then (1a) implies that the input $\tilde{u}_0(t) = ge^{\lambda t}$, $t \geq 0$, applied to system (1) (treated as a complex one) at the initial condition x^0 gives $\tilde{x}_0(t) = x^0 e^{\lambda t}$ and $\tilde{y}(t) = C\tilde{x}_0(t) + D\tilde{u}_0(t) \equiv 0$ (note that if the triples λ_1, $x_1^0 \neq 0$, g_1 and λ_2, $x_2^0 \neq 0$, g_2 satisfy (1a), then any linear combination of inputs $\tilde{u}_1(t) = g_1 e^{\lambda_1 t}$ and $\tilde{u}_2(t) = g_2 e^{\lambda_2 t}$, i.e. $\tilde{u}(t) = \alpha\tilde{u}_1(t) + \beta\tilde{u}_2(t)$ with $\alpha, \beta \in \mathbb{C}$, applied to (1) at the initial condition $\alpha x_1^0 + \beta x_2^0$, yields $\tilde{x}(t) = \alpha x_1^0 e^{\lambda_1 t} + \beta x_2^0 e^{\lambda_2 t}$ and $\tilde{y}(t) \equiv 0$).

Write the triple λ, $x^0 \neq 0$, g under consideration as $\lambda = \sigma + j\omega$, $x^0 = \operatorname{Re} x^0 + j \operatorname{Im} x^0$, g. Then (1a) also holds for the triple $\bar{\lambda} = \sigma - j\omega$, $\bar{x}^0 = \operatorname{Re} x^0 - j\operatorname{Im} x^0$, $\bar{g}$ (i.e. $\bar{\lambda} = \sigma - j\,\omega$ is also an invariant zero). This means in turn that these triples generate two real initial conditions and two real-valued inputs which produce the identically zero system response. More precisely, the pair $(\operatorname{Re} x^0, \operatorname{Re}\tilde{u}_0(t))$, where $\operatorname{Re}\tilde{u}_0(t) = \frac{1}{2} g\, e^{\lambda\, t} + \frac{1}{2}\bar{g}\, e^{\bar{\lambda}\, t}$, is an output-zeroing input and yields the solution $x_0(t) = \operatorname{Re}\tilde{x}_0(t) = \frac{1}{2} x^0\, e^{\lambda\, t} + \frac{1}{2}\,\bar{x}^0\, e^{\bar{\lambda}\, t}$. Analogously, the pair $(\operatorname{Im} x^0, \operatorname{Im}\tilde{u}_0(t))$, where $\operatorname{Im}\tilde{u}_0(t) = -j\frac{1}{2}\, g\, e^{\lambda\, t} + j\frac{1}{2}\,\bar{g}\, e^{\bar{\lambda}\, t}$, constitutes an output-zeroing input which gives the solution $x_0(t) = \operatorname{Im}\tilde{x}_0(t) = -j\,\frac{1}{2} x^0\, e^{\lambda\, t} + j\,\frac{1}{2}\bar{x}^0\, e^{\bar{\lambda}\, t}$.

(ii) We show now that the inputs $\operatorname{Re}\tilde{u}_0(t)$ and $\operatorname{Im}\tilde{u}_0(t)$ may be expressed in the form (7). To this end, we use the following result (Tokarzewski, 1998, p. 1289, Prop. 5): If a triple λ, $x^0 \neq 0$, g satisfies (1a), then

$$g = g_1 + g_2, \quad g_1 \in \operatorname{Ker} D, \quad g_2 = -D^+Cx^0$$

and

$$\lambda x^0 - (A - BD^+C)x^0 = B\, g_1, \quad x^0 \in \operatorname{Ker}(I_r - DD^+)C.$$

Thus we can write $\tilde{u}_0(t)$ as

$$\begin{aligned}\tilde{u}_0(t) &= ge^{\lambda t} = g_2 e^{\lambda t} + g_1 e^{\lambda t} = -D^+ C x^0 e^{\lambda t} + g_1 e^{\lambda t} \\ &= -D^+ C \tilde{x}_0(t) + \tilde{u}_h(t), \end{aligned} \tag{8}$$

with $\tilde{u}_h(t) := g_1 e^{\lambda t}$. Since $\tilde{u}_0(t) = ge^{\lambda t}$ and $\tilde{x}_0(t) = x^0 e^{\lambda t}$ satisfy, at the initial condition x^0, the state equation of (1), i.e.

$$\dot{\tilde{x}}_0(t) = A\tilde{x}_0(t) + B\tilde{u}_0(t), \quad \tilde{x}_0(0) = x^0, \tag{9}$$

introducing the right-hand side of (8) into (9), we get

$$\dot{\tilde{x}}_0(t) = (A - BD^+C)\tilde{x}_0(t) + B\tilde{u}_h(t), \quad \tilde{x}_0(0) = x^0. \tag{10}$$

By virtue of the uniqueness of solutions, this yields

$$\begin{aligned}\tilde{x}_0(t) &= x^0 e^{\lambda t} = e^{t\,(A-BD^+C)} x^0 \\ &\quad + \int_0^t e^{(t-\tau)(A-BD^+C)}\, B\tilde{u}_h(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{11}$$

Introducing the right-hand side of (11) into the right-hand side of (8) and taking the real part, we obtain the desired result, i.e.

$$\begin{aligned}\mathrm{Re}\,\tilde{u}_0(t) &= -D^+C\, e^{t(A-BD^+C)} \mathrm{Re}\, x^0 \\ &\quad - D^+C \int_0^t e^{(t-\tau)(A-BD^+C)} \\ &\quad \times B\mathrm{Re}\,\tilde{u}_h(\tau)\,\mathrm{d}\tau + \mathrm{Re}\,\tilde{u}_h(t). \end{aligned} \tag{12}$$

As for the output-zeroing input $(\mathrm{Im}\, x^0,\ \mathrm{Im}\,\tilde{u}_0(t))$, we proceed similarly.

Corollary 1. *Let $(x^0, u_0(t))$ be an output-zeroing input for a proper system (1) and let $x_0(t)$ denote the corresponding solution. Then*

(i) If $B(I_m - D^+D) = 0$, then $x_0(t) = e^{t(A-BD^+C)}x^0$. Moreover, the pair $(x^0, u_0^*(t))$, where $u_0^*(t) = -D^+Ce^{t(A-BD^+C)}x^0$, is also output-zeroing and yields the solution $x_0(t) = e^{t(A-BD^+C)}x^0$.

(ii) If D has full column rank, then $u_0(t) = -D^+Ce^{t(A-BD^+C)}x^0$ and $x_0(t) = e^{t(A-BD^+C)}x^0$.

Proof. (i) To the state equation of (1) introduce the input

$$u_0^*(t) = -D^+Cx_0(t) \tag{13}$$

at the initial condition x^0. In other words, consider the Cauchy problem (i.e. the initial value problem, see (Sontag, 1990, Appendix C))

$$\dot{x}(t) = Ax(t) + Bu_0^*(t), \quad x(0) = x^0, \quad t \geq 0. \tag{14}$$

Introducing (13) into (14) and taking into account the first equality of (2), we can write

$$\begin{aligned}\dot{x}(t) - \dot{x}_0(t) &= A\big(x(t) - x_0(t)\big) \\ &\quad + (A - BD^+C)x_0(t) - \dot{x}_0(t). \end{aligned} \tag{15}$$

However, by virtue of (2), the last two terms on the right-hand side of (15) can be written as

$$\begin{aligned}&(A - BD^+C)x_0(t) - \dot{x}_0(t) \\ &\quad = Ax_0(t) - \dot{x}_0(t) - BD^+\big(-Du_0(t)\big) \\ &\quad = -B(I_m - D^+D)u_0(t). \end{aligned} \tag{16}$$

At $B(I_m - D^+D) = 0$, from (2) and (16) it follows that

$$x_0(t) = e^{t(A-BD^+C)}x^0. \tag{17}$$

This ends the proof of the first claim in (i). Moreover, from (13) and (17) we infer that

$$u_0^*(t) = -D^+C\, e^{t(A-BD^+C)}\, x^0. \tag{18}$$

Now, setting $z(t) = x(t) - x_0(t)$ and taking into account (16), we replace (15) by the Cauchy problem

$$\dot{z}(t) = Az(t) - B(I_m - D^+D)u_0(t), \quad z(0) = 0, \quad t \geq 0. \tag{19}$$

At $B(I_m - D^+D) = 0$ the unique solution of (19) is $z(t) \equiv 0$, which means that the unique solution $x(t)$ of (14) satisfies

$$x(t) = x_0(t) = e^{t(A-BD^+C)}x^0 \tag{20}$$

for all $t \geq 0$. In order to show that at $B(I_m - D^+D) = 0$ the pair $(x^0, u_0^*(t))$ is an output-zeroing input for (1), we use (2), (13) and the relations $x(t) = x_0(t)$ and $DD^+D = D$, and for all $t \geq 0$ we obtain

$$\begin{aligned}y(t) &= Cx(t) + Du_0^*(t) = Cx_0(t) - D\big(D^+Cx_0(t)\big) \\ &= Cx_0(t) + DD^+Du_0(t) = 0. \end{aligned} \tag{21}$$

This ends the proof of the second claim in (i).

(ii) If D is monic (i.e. $D^+D = I_m$), then (13) constitutes the unique solution of (3) and, consequently, we have $u_0(t) \equiv u_0^*(t)$ for $0 \leq t < \infty$. ■

Remark 2. Although the assumption $B(I_m - D^+D) = 0$ does not imply in general that $u_0^*(t) = u_0(t)$, it implies that $u_0(t)$ and $u_0^*(t)$ applied at the initial state x^0 affect the state equation of (1) in the same way. This follows immediately from the relations

$$\begin{aligned}Bu_0^*(t) - Bu_0(t) &= -B(D^+Cx_0(t)) - Bu_0(t) \\ &= BD^+Du_0(t) - Bu_0(t) \\ &= -B(I_m - D^+D)u_0(t) = 0. \end{aligned}$$

The relation $Du_0^*(t) - Du_0(t) = 0$ for all $t \geq 0$ is obvious (see (21)).

When D has full row rank, the necessary condition given by Proposition 1 becomes also sufficient.

Corollary 2. *In (1) let D have full row rank. Then $(x^0, u_0(t))$ is an output-zeroing input iff $u_0(t)$ has the form (7), where $x^0 \in \mathbb{R}^n$ and $u_h(t)$ is an element of $\mathbb{U}$ satisfying $Du_h(t) = 0$ for all $t \geq 0$. Moreover, the solution corresponding to $(x^0, u_0(t))$ has the form (6).*

Proof. The assumption implies $DD^+ = I_r$. We show first that (7) applied to (1) at the initial condition x^0 gives a solution of the form (6). To this end, in view of the uniqueness of solutions, it is enough to check that (6) and (7) satisfy the state equation. Next, introducing (7) and (6) to the output equation, we get $y(t) = Du_h(t)$. This proves that if $x^0 \in \mathbb{R}^n$ and $u_0(t)$ is as in (7) (at an arbitrary admissible $u_h(t) \in \operatorname{Ker} D$), then $(x^0, u_0(t))$ is an output-zeroing input. The converse implication follows immediately from Proposition 1. ■

A more detailed characterization of the output-zeroing problem than that obtained in Corollary 1(ii) is given by the following result.

Corollary 3. *In a proper system (1), let matrix D have full column rank. Then $(x^0,\ u_0(t))$ is an output-zeroing input if and only if*

(i) $$x^0 \in S_D^{cl} := \bigcap_{l=0}^{n-1} \operatorname{Ker}\{(I_r - DD^+)\, C\, (A - BD^+C)^l\}$$

and

(ii) $$u_0(t) = -D^+C\, e^{t(A-BD^+C)}x^0.$$

Moreover, the corresponding solution equals

(iii) $$x_0(t) = e^{t(A-BD^+C)}x^0$$

and is entirely contained in the subspace S_D^{cl}.

Proof. If $(x^0,\ u_0(t))$ is an output-zeroing input for the system, then, as is known from Corollary 1(ii), $u_0(t)$ has the form (ii) and $x_0(t)$ is as in (iii). So we need to show relation (i), and that $x_0(t) \in S_D^{cl}$ for all $t \geq 0$. However, by assumption, employing (ii) and (iii), we can write the following equality:

$$\begin{aligned} 0 \equiv y(t) &= Cx_0(t) + Du_0(t) \\ &= (I_r - DD^+)\, C\, e^{t(A-BD^+C)}x^0 \text{ for all } t \geq 0. \end{aligned}$$

Differentiating this equality $n-1$ times and using (iii), we get the desired relation

(iv) $$\begin{cases} (I_r - DD^+)Cx_0(t) = 0, \\ \vdots \\ (I_r - DD^+)C(A - BD^+C)^{n-1}x_0(t) = 0, \end{cases}$$

for all $t \geq 0$, i.e. $x_0(t) \in S_D^{cl}$ for all $t \geq 0$. Substituting $t = 0$ in (iv), one gets (i).

In order to prove the converse implication, we have to show that any pair $(x^0,\ u_0(t))$ such that $x^0 \in S_D^{cl}$ and $u_0(t)$ has the form (ii) constitutes an output-zeroing input and produces a solution of the state equation of the form (iii). To this end, we check first that functions (ii) and (iii) satisfy the state equation of (1). Then we observe that the system response corresponding to the input (ii), when applied to the system at the initial condition x^0, is equal to

(v) $$y(t) = (I_r - DD^+)\, C\, e^{t(A-BD^+C)}x^0.$$

Now, expanding the term $e^{t(A-BD^+C)}$ in a finite series $\sum_{l=0}^{n-1} \alpha_l(t)\,(A - BD^+C)^l$ and taking into account that $x^0 \in S_D^{cl}$, from (v) we obtain

(vi) $$\begin{aligned} y(t) &= \sum_{l=0}^{n-1} \alpha_l(t)\big[(I_r - DD^+)C(A - BD^+C)^l\big]x^0 \\ &= 0 \quad \text{for all} \quad t \geq 0. \end{aligned}$$

This means that $(x^0,\ u_0(t))$ is an output-zeroing input.

Finally, in order to prove that $x_0(t) \in S_D^{cl}$ for all $t \geq 0$, we proceed analogously as in the first part of the proof (cf. (iv)). ■

Remark 3. Any proper system (1) can be transformed, by introducing an appropriate precompensator, into a proper system in which the first nonzero Markov parameter has full column rank. In fact, suppose that in (1) D is not monic, i.e. $\operatorname{rank} D = p < m$. Let $D = D_1D_2$, with $D_1 \in \mathbb{R}^{r\times p}$ monic and $D_2 \in \mathbb{R}^{p\times m}$ epic, be a skeleton factorization of D. Introduce the precompensator D_2^T to (1) i.e. consider the p-input, r-output system

(i) $$\dot{x}(t) = Ax(t) + B'v(t), \quad y(t) = Cx(t) + D'v(t),$$

where $B' = BD_2^T$, $D' = DD_2^T$ and $v \in \mathbb{R}^p$. Since $D_2D_2^T$ is nonsingular, we have $\operatorname{rank} D' = \operatorname{rank} D_1 = p$, i.e. D' has full column rank.

After simple matrix manipulations, we obtain $A - B'(D')^+C = A - BD^+C$, i.e. the matrix characterizing output-zeroing inputs in system (i) is exactly the same as in the original system (1). Each invariant zero of (i) is also an invariant zero of (1) (although the converse implication is false; for instance, system (i) is never degenerate, even if such is system (1)). Of course, if $(x^0, v_0(t))$ is an output-zeroing input for (i), then $(x^0, u_0(t))$, with $u_0(t) = D_2^Tv_0(t)$, is an output-zeroing input for (1). Finally, by introducing a precompensator D_2^T, the controllability of (1) may be lost.

3.2. Strictly Proper Systems ($D=0$)

If $D=0$, then the first nonzero Markov parameter of (1) is denoted by CA^kB, where $0 \le k \le n-1$ (i.e. $CB = \cdots = CA^{k-1}B = 0$ and $CA^kB \neq 0$). In (1) let $\operatorname{rank} CA^kB = p$. Define the matrix

$$K_k := I - B(CA^kB)^+CA^k \tag{22}$$

and let H_1H_2, with $H_1 \in \mathbb{R}^{r\times p}$ and $H_2 \in \mathbb{R}^{p\times m}$, denote a skeleton factorization of CA^kB. The following lemma characterizes some useful algebraic properties of K_k.

Lemma 1. (Tokarzewski, 1998, p. 1287) *The matrix (22) has the following properties:*

(i) $K_k^2 = K_k$,

(ii) $C^n(\mathbb{R}^n) = \Sigma_k \oplus \Omega_k$, *with* $\Sigma_k := \{x : K_kx = x\} = \operatorname{Ker}(H_1^TCA^k)$, $\Omega_k := \{x : K_kx = 0\} = \operatorname{Im}(BH_2^T)$ *and* $\dim\Sigma_k = n-p$, $\dim\Omega_k = p$.

Moreover,

(iii) $K_kBH_2^T = 0, \quad H_1^TCA^kK_k = 0,$
$C(K_kA)^l = CA^l$ *for* $0 \le l \le k$,

(iv) $$H_1^TC(K_kA)^l = \begin{cases} H_1^TCA^l & \text{for } 0 \le l \le k, \\ 0 & \text{for } l \ge k+1. \end{cases}$$

Since K_k is determined uniquely, its properties listed in Lemma 1 do not depend upon a particular choice of matrices H_1 and H_2 in the skeleton factorization of CA^kB. In the sequel, only property (iv) of the lemma will be used (see the proof of Corollary 5).

Suppose now that $(x^0, u_0(t))$ is an output-zeroing input for the strictly proper system (1) and denote by $x_0(t)$ the corresponding solution. Thus for all $t \ge 0$ we have the equalities

$$\dot{x}_0(t) = Ax_0(t) + Bu_0(t), \quad x_0(0) = x^0 \tag{23}$$

and

$$y(t) = Cx_0(t) = Ce^{tA}x^0 + C\int_0^t e^{(t-\tau)A}Bu_0(\tau)\,\mathrm{d}\tau \equiv 0. \tag{24}$$

Differentiating (24) successively k times according to the well-known rule of differentiation (Chen, 1984)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^t g(t-\tau)u(\tau)\,\mathrm{d}\tau = g(t-\tau)u(\tau)\,|_{\tau=t} + \int_{t_0}^t \frac{\mathrm{d}}{\mathrm{d}t} g(t-\tau)u(\tau)\,\mathrm{d}\tau,$$

and using the relations

$$CB = \cdots = CA^{k-1}B = 0$$

and

$$x_0(t) = e^{tA}x^0 + \int_0^t e^{(t-\tau)A}Bu_0(\tau)\,\mathrm{d}\tau,$$

we obtain at the first step $y^{(1)}(t) = CAx_0(t) + CBu_0(t) \equiv 0$. Since $CB=0$, we have $CAx_0(t) \equiv 0$. For the i-th derivative of $y(t)$, $1 < i \le k$, we obtain $y^{(i)}(t) = CA^ix_0(t) + CA^{i-1}Bu_0(t) \equiv 0$, which, in view of $CA^{i-1}B = 0$, yields $CA^ix_0(t) \equiv 0$. Thus $x_0(t)$ is entirely contained in the subspace

$$S_k := \bigcap_{l=0}^{k} \operatorname{Ker} CA^l, \tag{25}$$

i.e. $x_0(t) \in S_k$ for all $t \ge 0$.

The $(k+1)$-th derivative of (24) yields $y^{(k+1)}(t) = CA^{k+1}x_0(t) + CA^kBu_0(t) \equiv 0$, which can be written as

$$CA^{k+1}x_0(t) = -CA^kBu_0(t) \quad \text{for all } t \ge 0. \tag{26}$$

Note that premultiplying (23) by CA^k and using (26), we obtain $CA^k\dot{x}_0(t) \equiv 0$ and, consequently, by virtue of (22), we get the following relation:

$$K_k\dot{x}_0(t) = \dot{x}_0(t). \tag{27}$$

Consider the equation

$$CA^{k+1}x_0(t) = -CA^kBu(t) \tag{28}$$

with an unknown function $u(t) \in \mathbb{U}$. Because, by assumption, $u_0(t)$ satisfies (28), $u_0(t)$ (see Section 2) can be written as

$$u_0(t) = -(CA^kB)^+CA^{k+1}x_0(t) + u_h(t), \tag{29}$$

where $u_h(t) \in \mathbb{U}$ is some function which satisfies $CA^kBu_h(t) = 0$ for all $t \ge 0$. Because $(x^0, u_0(t))$ and, consequently, $x_0(t)$ are assumed to be known, $u_h(t)$ is treated as a known function which is uniquely determined by (29). Introducing (29) to (23) and employing (22), we can write (23) as

$$\dot{x}_0(t) = K_kAx_0(t) + Bu_h(t), \quad x_0(0) = x^0. \tag{30}$$

Thus we have

$$x_0(t) = e^{tK_kA}x^0 + \int_0^t e^{(t-\tau)K_kA}Bu_h(\tau)\,\mathrm{d}\tau \tag{31}$$

and

$$u_0(t) = -(CA^kB)^+CA^{k+1}e^{tK_kA}x^0 - (CA^kB)^+CA^{k+1}\int_0^t e^{(t-\tau)K_kA}Bu_h(\tau)\,\mathrm{d}\tau + u_h(t). \tag{32}$$

The above discussion leads to the following characterization of output-zeroing inputs.

Proposition 2. *Let* $(x^0, u_0(t))$ *be an output-zeroing input for a strictly proper system (1) and let* $x_0(t)$ *denote the corresponding solution. Then* $x^0 \in S_k$*, cf. (25), and* $u_0(t)$ *has the form (32), for some* $u_h(t) \in \mathbb{U}$ *satisfying* $CA^kBu_h(t) = 0$ *for all* $t \geq 0$*, and* $x_0(t)$ *is as in (31). Moreover,* $x_0(t) \in S_k$ *for all* $t \geq 0$.

Note that on the assumptions of Proposition 2 the input (32) applied to (1) at an arbitrary initial condition $x(0) \in \mathbb{R}^n$ yields the solution of the state equation of the form $x(t) = e^{tA}(x(0) - x^0) + x_0(t)$, where $x_0(t)$ is as in (31), and the system output equals $y(t) = Ce^{tA}(x(0) - x^0)$.

Remark 4. Suppose that $\lambda \in \mathbb{C}$ is an invariant zero of a strictly proper system (1), i.e. the triple $\lambda, \quad x^0 \neq 0, \quad g$ satisfies (1a). Then $\tilde{u}_0(t) = ge^{\lambda t}$, $t \geq 0$, applied to system (1) (treated as a complex one) at x^0 yields $\tilde{x}_0(t) = x^0e^{\lambda t}$ and $\tilde{y}(t) = C\tilde{x}_0(t) \equiv 0$. We show now that the output-zeroing inputs $(\operatorname{Re} x^0, \operatorname{Re} \tilde{u}_0(t))$ and $(\operatorname{Im} x^0, \operatorname{Im} \tilde{u}_0(t))$ corresponding to λ can be written as in (32). To this end, it is enough to use the following result (Tokarzewski, 1998, p. 1287, Prop. 2): If a triple $\lambda, x^0 \neq 0, g$ satisfies (1a), then

$$\begin{aligned} &g = g_1 + g_2, \quad g_1 \in \operatorname{Ker} CA^kB, \\ &g_2 = -(CA^kB)^+CA^{k+1}x^0, \end{aligned} \tag{33}$$

where g_1, g_2 are uniquely determined by g, and

$$\begin{aligned} &\lambda x^0 - K_kAx^0 = Bg_1, \quad K_kAx^0 - Ax^0 = Bg_2, \\ &x^0 \in \bigcap_{l=0}^{k} \operatorname{Ker} CA^l. \end{aligned} \tag{34}$$

Now, using (33), we can write $\tilde{u}_0(t)$ as

$$\begin{aligned} \tilde{u}_0(t) &= ge^{\lambda t} = g_2e^{\lambda t} + g_1e^{\lambda t} \\ &= -(CA^kB)^+CA^{k+1}x^0e^{\lambda t} + g_1e^{\lambda t} \\ &= -(CA^kB)^+CA^{k+1}\tilde{x}_0(t) + \tilde{u}_h(t), \end{aligned} \tag{35}$$

where $\tilde{u}_h(t) := g_1e^{\lambda t}$.

For $\tilde{u}_0(t) = ge^{\lambda t}$ and $\tilde{x}_0(t) = x^0e^{\lambda t}$ we can write equalities of the form (23). Then, employing (35), from (23) we get equalities of the form (30). By virtue of the uniqueness of solutions, this means that

$$\tilde{x}_0(t) = x^0e^{\lambda t} = e^{tK_kA}x^0 + \int_0^t e^{(t-\tau)K_kA}B\tilde{u}_h(\tau)\,\mathrm{d}\tau. \tag{36}$$

Finally, introducing the right-hand side of (36) into the right-hand side of (35) and taking the real part of the resultant form of $\tilde{u}_0(t)$, we obtain the desired result. We proceed similarly with $(\operatorname{Im} x^0,\ \operatorname{Im} \tilde{u}_0(t))$.

Corollary 4. *Let* $(x^0, u_0(t))$ *be an output-zeroing input for a strictly proper system (1) and let* $x_0(t)$*,* $t \in [0, +\infty)$ *denote the corresponding solution. Then*

(i) If $K_kB = 0$*, then* $x_0(t) = e^{tK_kA}x^0$*. Moreover, at* $K_kB = 0$ *the pair* $(x^0, u_0^*(t))$*, where* $u_0^*(t) = -(CA^kB)^+CA^{k+1}e^{tK_kA}x^0$*, is also output-zeroing and yields the solution* $x_0(t) = e^{tK_kA}x^0$.

(ii) If CA^kB *has full column rank, then* $u_0(t) = -(CA^kB)^+CA^{k+1}e^{tK_kA}x^0$ *and* $x_0(t) = e^{tK_kA}x^0$.

Proof. (i) Premultiplying both the sides of the first equality in (23) by K_k and using (27), we obtain

$$K_kAx_0(t) - \dot{x}_0(t) = -K_kBu_0(t), \quad x_0(0) = x^0. \tag{37}$$

At $K_kB = 0$, from (37) it follows that

$$x_0(t) = e^{tK_kA}x^0. \tag{38}$$

This ends the proof of the first claim in (i). For the proof of the second claim, let us introduce to the state equation of (1) the input

$$u_0^*(t) = -(CA^kB)^+CA^{k+1}x_0(t) \tag{39}$$

at the initial condition x^0. That is, consider the Cauchy problem

$$\dot{x}(t) = Ax(t) + Bu_0^*(t), \quad x(0) = x^0. \tag{40}$$

After using (39) and (22), eqn. (40) can be rewritten as

$$\begin{aligned} \dot{x}(t) - \dot{x}_0(t) &= A\big(x(t) - x_0(t)\big) \\ &\quad + \big(K_kAx_0(t) - \dot{x}_0(t)\big). \end{aligned} \tag{41}$$

Now, setting $z(t) = x(t) - x_0(t)$ and taking into account (37), the problem (41) can be replaced by

$$\dot{z}(t) = Az(t) - K_kBu_0(t), \quad z(0) = 0. \tag{42}$$

At $K_kB = 0$ the unique solution of (42) is $z(t) \equiv 0$, which means in turn that the unique solution $x(t)$ of (40) satisfies $x(t) = x_0(t) = e^{tK_kA}x^0$ for all $t \geq 0$. Consequently, since from Proposition 2 we have $x_0(t) \in S_k \subset \operatorname{Ker} C$, the pair $(x^0, u_0^*(t))$, where in view of (38) and (39)

$$u_0^*(t) = -(CA^kB)^+CA^{k+1}e^{tK_kA}x^0, \tag{43}$$

is an output-zeroing input and gives the same solution of (1) as $(x^0, u_0(t))$. This proves the second claim of (i).

(ii) By virtue of (22), we get $K_kB = B(I_m - (CA^kB)^+(CA^kB))$. If CA^kB is monic, then $I_m - (CA^kB)^+(CA^kB) = 0$, i.e. $K_kB = 0$. Moreover, in this case the unique solution of (28) has the form (39). Hence $u_0(t) \equiv u_0^*(t)$, $t \in [0, +\infty)$, where $u_0^*(t)$ is as in (43). ■

Remark 5. The assumption $K_kB = 0$ does not imply in general the equality $u_0^*(t) = u_0(t)$ for all $t \geq 0$, although it implies $x(t) \equiv x_0(t)$. The reason behind this becomes clear if we consider the relations $Bu_0^*(t) - Bu_0(t) = (K_k - I)Ax_0(t) - Bu_0(t) = K_kAx_0(t) - \dot{x}_0(t) = -K_kBu_0(t)$. Thus, at $K_kB = 0$, although in general $u_0(t) \neq u_0^*(t)$ (cf. Example 1), both these inputs applied at the initial condition x^0 affect the state equation of (1) in exactly the same way (since we have then $B\,u_0^*(t) - B\,u_0(t) = 0$).

Corollary 5. *In a strictly proper system (1) let CA^kB have full row rank. Then $(x^0, u_0(t))$ is an output-zeroing input iff $x^0 \in S_k$ and $u_0(t)$ is as in (32) with $u_h(t) \in \mathbb{U}$ satisfying $u_h(t) \in \operatorname{Ker} CA^kB$. Moreover, the corresponding solution $x_0(t)$ has the form (31) and is entirely contained in S_k.*

Proof. We write the skeleton factorization of CA^kB as H_1H_2, where $H_1 = I_r, H_2 = CA^kB$. We show first that the input $u_0(t)$ in (32), with an arbitrarily fixed admissible $u_h(t) \in \operatorname{Ker} CA^kB$ and $x^0 \in S_k$, applied to the system at the initial condition x^0, produces a solution of the form (31). To this end, it is enough to verify that (32) and (31) satisfy the state equation of (1). The corresponding output equals $y(t) = Ce^{tK_kA}x^0 + \int_0^t Ce^{(t-\tau)K_kA}Bu_h(\tau)\,\mathrm{d}\tau$. Now, using Lemma 1(iv) (at $H_1 = I_r$) and the assumption $x^0 \in S_k$, for the power series expansion of $Ce^{tK_kA}x^0$ we can write $Ce^{tK_kA}x^0 = \sum_{l=0}^{k}(t^l/l!)CA^lx^0 = 0$.

Analogously,

$$Ce^{(t-\tau)K_kA}Bu_h(\tau) = \frac{(t-\tau)^k}{k!}CA^kBu_h(\tau) = 0.$$

This yields $y(t) = 0$, i.e. $(x^0, u_0(t))$ is output-zeroing. The converse implication is an immediate consequence of Proposition 2. ■

Corollary 6. *In a strictly proper system (1) let CA^kB have full column rank. Then a pair $(x^0,\ u_0(t))$ is an output-zeroing input if and only if*

(i) $$x^0 \in S_k^{cl} := \bigcap_{l=0}^{n-1} \operatorname{Ker} C(K_kA)^l$$

and $u_0(t)$ has the form

(ii) $$u_0(t) = -(CA^kB)^+CA^{k+1}\,e^{t\,K_kA}\,x^0.$$

Moreover, the solution of the state equation corresponding to $(x^0,\ u_0(t))$ has the form

(iii) $$x_0(t) = e^{t\,K_kA}\,x^0$$

and is entirely contained in S_k^{cl}, i.e. $x_0(t) \in S_k^{cl}$ for all $t \geq 0$.

Proof. Suppose first that $(x^0,\ u_0(t))$ is an output-zeroing input. Then, as we know from Corollary 4(ii), $u_0(t)$ has the form (ii) and the corresponding solution is as in (iii). Moreover, by assumption, we have

(iv) $$y(t) = C\,x_0(t) = C\,e^{t\,K_kA}\,x^0 \equiv 0 \quad \text{for} \quad t \in [0,\,+\infty).$$

Differentiating the identity (iv) $n-1$ times, we can write

(v) $$\begin{cases} Cx_0(t) = 0, \\ C(K_kA)x_0(t) = 0, \\ \vdots \\ C(K_kA)^{n-1}x_0(t) = 0 \end{cases}$$

for all $t \geq 0$. This means that $x_0(t) \in S_k^{cl}$ for all $t \geq 0$. In particular, taking $t = 0$, we get the relation $x^0 \in S_k^{cl}$.

In order to prove the converse implication, we should show that any pair $(x^0,\ u_0(t))$ such that $x^0 \in S_k^{cl}$ and $u_0(t)$ has the form (ii) constitutes an output-zeroing input. To this end, we verify first that functions (ii) and (iii) satisfy the state equation of system (1). This means that the input function (ii) applied to the system at the initial condition x^0 yields the solution of the form (iii). Furthermore, the system response is equal to

(vi) $$y(t) = C\,x_0(t) = C\,e^{t\,K_kA}\,x^0.$$

Now, expanding the term $e^{t\,K_kA}$ in a finite power series $\sum_{l=0}^{n-1}\alpha_l(t)(K_kA)^l$ and making use of the assumption $x^0 \in S_k^{cl}$, we can evaluate the system output (vi) as follows:

$$y(t) = C\,e^{t\,K_kA}\,x^0 = \sum_{l=0}^{n-1}\alpha_l(t)\,C\,(K_kA)^l\,x^0 \equiv 0$$

for all $t \geq 0$. ■

Remark 6. Any strictly proper system (1) can be transformed, by introducing an appropriate precompensator, into a strictly proper system in which the first nonzero Markov parameter has full column rank. In fact, assume that in (1), CA^kB is not monic, i.e. $\operatorname{rank} CA^kB = p < m$. Let $CA^kB = H_1H_2$, with $H_1 \in \mathbb{R}^{r\times p}$ monic and $H_2 \in \mathbb{R}^{p\times m}$ epic, be a skeleton factorization. Introduce to (1) a precompensator H_2^T, i.e. consider the p-input, r-output system

(i) $$\dot{x}(t) = Ax(t) + B'v(t), \quad y(t) = Cx(t),$$

where we have $B' = BH_2^T$ and $v \in \mathbb{R}^p$. The first nonzero Markov parameter CA^kB' of (i) has full column rank. It follows from the skeleton factorization $CA^kB' = H_1H_2'$, where $H_2' = H_2H_2^T$ is nonsingular. The output-zeroing inputs for system (i) are characterized by matrix K_kA of system (1). For system (i) we form matrix $K_k' := I - B'(CA^kB')^+CA^k$, where $(CA^kB')^+ = (H_2')^+H_1^+ = (H_2H_2^T)^{-1}(H_1^TH_1)^{-1}H_1^T$ and, consequently, $K_k' = I - BH_2^T[(H_2H_2^T)^{-1}(H_1^TH_1)^{-1}H_1^T]CA^k = I - BH_2^+H_1^+CA^k = K_k$, i.e. $K_k'A = K_kA$, as claimed.

If $(x^0, v_0(t))$ is output-zeroing for (i), then $(x^0, u_0(t))$, with $u_0(t) = H_2^Tv_0(t)$, is an output-zeroing input for (1). Since H_2^T is monic, the converse implication does not hold in general. Of course, (i) is never degenerate, even if the original system (1) is. By introducing the precompensator H_2^T, the controllability of (1) may be lost.

Remark 7. If in a strictly proper system (1) matrix B is not of full column rank, i.e. $\operatorname{Ker} B \neq \{0\}$, then any pair $(x^0 = 0, u_h(t))$, where $u_h(t)$ is an arbitrary nonzero admissible input satisfying $u_h(t) \in \operatorname{Ker} B$, forms an output-zeroing input. It is clear that each input of this kind affects the state equation in exactly the same way as the pair $(x^0 = 0, u_0(t) \equiv 0)$ (i.e. it gives the trivial solution $x(t) \equiv 0$). We do not associate the set of all pairs $(x^0 = 0, u_h(t) \in \operatorname{Ker} B)$ with invariant zeros since it may exist independently upon these zeros (as in the system

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is minimal and has no invariant zeros).

Finally, one can note that if $(x^0, u_0(t))$ is an output-zeroing input for (1), then any pair $(x^0, u_0(t) + u_h(t))$, with $u_h(t) \in \operatorname{Ker} B$, is also output-zeroing and gives the same solution as $(x^0, u_0(t))$ (cf. Example 1).

It is now clear that the operation of introducing the precompensator H_2^T to (1) (see Remark 6) removes from (1) all the output-zeroing inputs of the form $(x^0 = 0, u_h(t) \in \operatorname{Ker} B)$ (since in (i) the matrix $B' = BH_2^T$ has full column rank). On the other hand, this operation removes from (1) all those invariant zeros which occur outside $\sigma(K_kA)$ ($\sigma(\cdot)$ stands for the spectrum of a matrix) (because all the invariant zeros of system (i) in Remark 6 remain in $\sigma(K_kA)$, see (Tokarzewski, 1998, p. 1288, Cor. 2)) and, consequently, all the output-zeroing inputs corresponding to such zeros.

4. Examples

Example 1. (Tokarzewski, 1998). In (1) let

$$A = \begin{bmatrix} -1 & 0 & -3 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The system is minimal and nondegenerate (in the sense of (1a)). We have $k = 0$ and

$$(CB)^+ = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The triple

$$\lambda = 0, \quad x^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad g = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

satisfies (1a) ($\lambda = 0$ is the only invariant zero), and g can be written as $g = g_1 + g_2$ with

$$g_1 = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} \in \operatorname{Ker} CB$$

and

$$g_2 = -(CB)^+CAx^0 = \begin{bmatrix} 3/2 \\ 0 \\ 3/2 \end{bmatrix}.$$

Any pair $(x^0, u_0(t))$, where

$$u_0(t) = g_2 + g_1 + \begin{bmatrix} f(t) \\ 0 \\ -f(t) \end{bmatrix}$$

and $f : [0, \infty) \to \mathbb{R}^1$ is an arbitrary piecewise continuous function, is output-zeroing and gives the solution $x_0(t) = x^0$. ◆

Example 2. Consider a square strictly proper system (1) of uniform rank (i.e. such that $(CA^kB)^{-1}$ exists). Then $(x^0, u_0(t))$ is output-zeroing iff $x^0 \in S_k$ and $u_0(t) = -(CA^kB)^{-1}CA^{k+1}e^{tK_kA}x^0$, where $K_k := I - B(CA^kB)^{-1}CA^k$. Moreover, the zero dynamics (Isidori, 1995) are governed by the equation $\dot{x}(t) = K_kAx(t)$ and initial conditions $x(0) \in S_k$. ◆

Example 3. Consider a square proper system (1) of uniform rank (i.e. such that D^{-1} exists). Then $(x^0, u_0(t))$ is output-zeroing iff $u_0(t) = -D^{-1}Ce^{t(A-BD^{-1}C)}x^0$,

$x^0 \in \mathbb{R}^n$. Moreover, the zero dynamics are governed by the equation $\dot{x}(t) = (A - BD^{-1}C)x(t)$ and initial conditions $x(0) \in \mathbb{R}^n$. ♦

Example 4. (Tokarzewski, 2000b). In (1) let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The system is degenerate (in the sense of (1a)) and it has no Smith zeros.

In particular,

$$\lambda = j\omega, \quad x^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad g = \begin{bmatrix} j\omega + 1 \\ -1 \end{bmatrix}$$

satisfy (1a) for any $\omega \neq 0$. To $\lambda = j\omega$ we assign

$$\left(\operatorname{Re} x^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \operatorname{Re} \tilde{u}_0(t) = \begin{bmatrix} \cos \omega t - \omega \sin \omega t \\ -\cos \omega t \end{bmatrix}\right)$$

and

$$\left(\operatorname{Im} x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \operatorname{Im} \tilde{u}_0(t) = \begin{bmatrix} \sin \omega t + \omega \cos \omega t \\ -\sin \omega t \end{bmatrix}\right)$$

as output-zeroing inputs (see Remark 4). The corresponding solutions are respectively

$$x_0(t) = \begin{bmatrix} 0 \\ 0 \\ \cos \omega t \end{bmatrix} \quad \text{and} \quad x_0(t) = \begin{bmatrix} 0 \\ 0 \\ \sin \omega t \end{bmatrix}.$$

According to the notation of Remark 4,

$$\tilde{u}_h(t) = \begin{bmatrix} j\omega + 1 \\ 0 \end{bmatrix} e^{j\omega t}.$$

Due to the stability of (1), each of the inputs $\operatorname{Re} \tilde{u}_0(t)$ and $\operatorname{Im} \tilde{u}_0(t)$ applied to the system at any initial state yields an asymptotically vanishing system response. ♦

5. Concluding Remarks

In this paper we derived necessary conditions concerning the form of output-zeroing inputs and the corresponding solutions (Propositions 1 and 2) for a general class of linear time-invariant systems (with nonzero transfer functions) described by a state-space model $S(A, B, C, D)$. By showing how to assign to each invariant zero an appropriate output-zeroing input (Remarks 1 and 4), we studied dynamic properties of these zeros (the question of the algebraic characterization and calculation of zeros is discussed in (Tokarzewski, 1998; 2000b)). It is shown that if the first Markov parameter has full row rank, the necessary conditions become also sufficient (Corollaries 2 and 4 and Examples 2 and 3). Necessary and sufficient conditions for output-zeroing inputs for systems with the first nonzero Markov parameter of full column rank are given in Corollaries 3 and 6.

Finally, some remarks concerning systems with identically zero transfer functions should be made. As is noticed in (Tokarzewski, 2000b; Remark 3), if $G(s) \equiv 0$ in (1), then the system is degenerate (in the sense of (1a)). More precisely, any $\lambda \notin \sigma(A)$ is its invariant zero. Furthermore, using the Kalman canonical form of (1), it is possible to show that the trajectories of the solutions corresponding to such zeros are contained in the subspace of all controllable and unobservable states. Note that $G(s) \equiv 0$ can be the desired property of a system. It takes place, e.g., when the disturbance decoupling problem is analysed, cf. (Sontag, 1990; p. 146).

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