DYNAMIC CONTACT PROBLEMS WITH SLIP-DEPENDENT FRICTION IN VISCOELASTICITY

IOAN R. IONESCU*, QUOC-LAN NGUYEN**

* Laboratoire de Mathématiques, CNRS and Université de Savoie,
Campus Scientifique, 73376 Le Bourget-du-Lac Cedex, France,
e-mail: `ionescu@univ-savoie.fr`

** National Polytechnic Institute, Ho-Shi-Minh,
143/2D Xo Viet nghe Tinh (he m), P25, Q.BT, Ho-Shi-Minh, Vietnam,
e-mail: `quoc-lan.nguyen@hcm.fpt.vn`

The dynamic evolution with frictional contact of a viscoelastic body is considered. The assumptions on the functions used in modelling the contact are broad enough to include both the normal compliance and the Tresca models. The friction law uses a friction coefficient which is a non-monotone function of the slip. The existence and uniqueness of the solution are proved in the general three-dimensional case.

Keywords: slip-dependent friction, dynamic viscoelasticity, Tresca contact, normal compliance, existence and uniqueness

1. Introduction

Duvaut and Lions (1976) obtained the first existence and uniqueness results for contact problems with friction in elastodynamics. Some years later, the non-penetrability of mass was relaxed by Martins and Oden (1987) by considering the normal compliance model of contact with friction. In order to obtain existence and uniqueness results they considered only the viscous case (see also Kutler, 1997).

All the above results involve a fixed friction coefficient μ. In the study of many frictional processes (stick-slip motions, earthquakes modelling, etc.) the friction coefficient has to be considered variable during the slip. The simplest variation of μ is the discontinuous jump from a ‘static’ value μ_s down to a ‘dynamic’ or ‘kinetic’ value μ_d. Three current models of such a variation are considered in mechanics and geophysics. The first one, discussed latter, corresponds to a smooth dependence of the friction coefficient on the slip u_T, i.e. $\mu = \mu(|u_T|)$. The second one considers a slip rate dependence of the friction coefficient (Oden and Martins, 1985; Scholz, 1990), i.e. $\mu = \mu(|\dot{u}_T|)$. For this model the solution of the mathematical problem in dynamic elasticity is not uniquely determined and presents shocks (Ionescu and Paumier, 1993; 1994). However, the problem is well posed in dynamic viscoelasticity (Ionescu, 2001; Kuttler and Shillor, 1999). The third model, called the Dieterich and Ruina model, uses a rate- and state-dependent friction law (see, e.g., Dieterich, 1994; Perrin *et al.*, 1995; Rice and Ruina, 1983; Ruina, 1983). Though it tries to accommodate both slip and slip rate dependences, the qualitative behaviour of the solution is very close to the slip rate friction model (Favreau *et al.*, 1999b).

The physical model of slip-dependent friction was introduced by Rabinowicz (1951) in the geophysical context of earthquakes’ modelling to explain the stick-slip phenomenon. Generally speaking, the dependence of the friction forces upon the surface displacements is usually accepted when the slip is very small on laboratory scales (see, e.g., Ohnaka *et al.*, 1987; Scholz, 1990). Ohnaka *et al.* (1987) pointed out the good agreement of this model with experimental data. More recently, the slip weakening model (i.e. the decrease of the friction force with slip) was intensively used in the description of earthquake initiation (Campillo and Ionescu, 1997; Dascalu *et al.*, 2000; Favreau *et al.*, 1999a; Ionescu and Campillo; 1999). Indeed, since the model is rate independent, it can describe a large variation of the slip rate during the initiation phase.

The first mathematical results for the slip weakening model of friction in elastostatics were obtained by Ionescu and Paumier (1996). They proved the existence of a solution and gave sufficient conditions for uniqueness and stability. Moreover, they analyzed the bifurcation points between different branches of solutions. More recently, the quasi-static evolution of an elastic body with slip-dependent friction was studied in Corneschi *et al.* (2001). An existence result for a sufficiently small friction coefficient was proved. As far as we know, there is no existence and uniqueness result in dynamic elasticity involving slip-dependent friction.

The aim of this paper is to study the dynamic evolution of a viscoelastic body which is in frictional contact with a rigid foundation. The assumptions on the functions used in modelling the contact are general enough to include both the normal compliance and the Tresca models. For a constant normal stress (displacement) the friction force may exhibit a slip weakening behaviour. The main result is the existence and uniqueness of the solution in the general three-dimensional case. The proof, based on the Galerkin method, is constructive.

2. Problem Statement

Let $\Omega \subset \mathbb{R}^d \ (d = 2, 3)$ be a bounded domain, representing the interior of a viscoelastic body, with a smooth boundary $\Gamma = \partial\Omega$ divided into three disjoint parts $\Gamma = \bar{\Gamma}_d \cup \bar{\Gamma}_c \cup \bar{\Gamma}_f$ with $\operatorname{meas}(\Gamma_d) > 0$. The mechanical problem (MP) consists in finding the displacement field $u : [0, T] \times \Omega \longrightarrow \mathbb{R}^d$ such that

$$\sigma(t) = \mathcal{A}\epsilon\big(u(t)\big) + \eta\mathcal{C}\epsilon\big(\dot{u}(t)\big) \ \text{ in } \ \Omega, \tag{1}$$

$$\rho\ddot{u}(t) = \operatorname{div}\sigma(t) + r(t) \ \text{ in } \ \Omega \tag{2}$$

$$u(t) = 0 \ \text{ on } \ \Gamma_d, \tag{3}$$

$$\sigma(t)n = F(t) \ \text{ on } \ \Gamma_f, \tag{4}$$

$$\sigma_N(t) = -m_N(u_N^+(t)) \ \text{ on } \ \Gamma_c \tag{5}$$

$$\sigma_T(t) = -m_T\big(u_N^+(t)\big)\mu\big(|u_T(t)|\big)\frac{\dot{u}_T(t)}{|\dot{u}_T(t)|} \quad \text{if } \ \dot{u}_T(t) \neq 0 \ \text{ on } \ \Gamma_c, \tag{6}$$

$$\big|\sigma_T(t)\big| \leq m_T(u_N^+(t))\mu\big(|u_T(t)|\big) \quad \text{if } \ \dot{u}_T(t) = 0 \ \text{ on } \ \Gamma_c, \tag{7}$$

$$u(0) = u_0, \ \text{ in } \ \Omega, \tag{8}$$

$$\dot{u}(0) = u_1 \ \text{ in } \ \Omega, \tag{9}$$

where $\eta > 0$ is a viscosity coefficient, $\rho > 0$ is the density, $\mathcal{A}, \mathcal{C}$ are fourth-order tensors, σ is the stress tensor, $\epsilon(u) = (1/2)(\nabla u + \nabla^T u)$ is the small strain tensor, n is the unit outward normal vector on Γ, $\sigma_N = \sigma n \cdot n$ is the normal stress, $\sigma_T = \sigma n - \sigma_N n$ is the tangential stress, $u_N = u \cdot n$ is the normal displacement, u_N^+ is its positive part, and $u_T = u - u_N n$ is the tangential displacement. Here r represents given body forces and F is the load on Γ_f.

Equations (5)–(7) represent the contact with slip-dependent friction along a potential surface Γ_c with a rigid and fixed body. If there exists a normal gap g_{gap} between the viscoelastic body and the foundation, measured in the undeformed configuration, then u_N^+ has to be replaced by $(u_N - g_{\text{gap}})^+$.

In (5) the normal stress σ_N is a function of the penetration u_N^+. Two cases are often used in the literature. In the first one, the Tresca model, the normal stress is given, i.e. $m_N(s) = m_T(s) = S_N$, hence the contact surface is known. The second one is the normal compliance model often characterized by a power-law relationship, i.e. $m_N(s) = |s|^{h_n}, m_T(s) = |s|^{h_T}$.

Equations (6) and (7) assert that if there is contact, the tangential (friction) stress is bounded by a function of the penetration u_N^+ multiplied by the value of the 'friction coefficient' $\mu(|u_T(t)|)$. If such a limit is not attained, sliding does not occur. Otherwise the friction stress is opposite to the slip rate and its absolute value depends on the slip. As a matter of fact, if we put $m_T(s) = R(m_N(s))m_N(s)$, then we get $\nu = |\sigma_T|/|\sigma_N| = R(|\sigma_N|)\mu(|u_T|)$, which corresponds to a generalization of Coulomb's friction law. Indeed, in this case the coefficient of friction ν is no more constant, and it accommodates the dependence on the normal stress and on the slip.

Since μ is a function of u_T, the friction model considered here is slip dependent. Indeed, for a constant normal stress (displacement) the friction force may have a slip-weakening behaviour. The physical model of slip-dependent friction was introduced in the geophysical context of earthquake modelling. In this context it is usual to suppose that the slip rate $\dot{u}_T$ (on the fault) has a single direction and a single sense during the slip, i.e. there exists a tangential vector T and a scalar $\dot{U}$, with $\dot{U} \geq 0$ (or $\dot{U} \leq 0$) such that $\dot{u}_T = \dot{U}T$. Even in this case, only the sequence 'stick-slip-stick' (i.e. $\dot{U} = 0$; $\dot{U} > 0$; $\dot{U} = 0$) has to be considered. Indeed, without an explicit loading/unloading criterion, the slip-dependent friction model (in the form used here) is more related to a surface potential than to a friction law, except for local monotonic loading.

3. Assumptions, Notation and Preliminaries

In the study of the problem (1)–(9) the following assumptions are used: $\mathcal{A}$ and $\mathcal{C}$ are symmetric and positive-definite fourth-order tensors, i.e.

$$\mathcal{A}_{ijkl}, \mathcal{C}_{ijkl} \in L^\infty(\Omega), \quad \mathcal{A}(x)\epsilon : \sigma = \mathcal{A}(x)\sigma : \epsilon, \quad \mathcal{C}(x)\epsilon : \sigma = \mathcal{C}(x)\sigma : \epsilon, \tag{10}$$

$$\exists\ \alpha > 0 \ \text{ such that } \quad \mathcal{A}(x)\epsilon : \epsilon \geq \alpha|\epsilon|^2, \quad \mathcal{C}(x)\epsilon : \epsilon \geq \alpha|\epsilon|^2, \tag{11}$$

a.e. $x \in \Omega$, $\forall\ i, j, k, l = \overline{1, d}$ and for all $\sigma, \epsilon \in \mathbb{R}_S^{d\times d}$.

Let us suppose that the friction coefficient μ: $\Gamma_c \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is differentiable with respect to the second variable, and there exist $M_1, \mu_0 > 0$ such that

$$0 \le \mu(x,u) \le \mu_0 \text{ a.e. } x \in \Gamma_f, \quad \forall\, u \in \mathbb{R}_+, \tag{12}$$

$$\big|\partial_u \mu(x,u)\big| \le M_1, \quad \forall\, u \in [0,+\infty[\text{ a.e. } x \in \Gamma_c, \tag{13}$$

and the functions $x \to \mu(x,u)$ and $x \to \partial_u\mu(x,u)$ are measurable for all $u \in \mathbb{R}_+$. As for the functions m_n and m_T, we suppose that

$$m_{_N}(x,u) \ge m_{_N}(x,0), \quad \forall\, u \in \mathbb{R}_+, \text{ a.e. } x \in \Gamma_{_c}, \tag{14}$$

$x \to m_i(x,u)$ is measurable for all $u \in \mathbb{R}_+$, $u \to m_i(x,u)$ is differentiable, and there exist $C_i, D_i, E_i \ge 0$ and $p_i \ge 1$ such that

$$\big|m_i(x,u)\big| \le C_i + D_i|u|^{p_i}, \tag{15}$$

$$\begin{aligned}&\big|m_i(x,u_1) - m_i(x,u_2)\big| \\ &\quad\le E_i\big(1 + |u_1|^{p_i-1} + |u_2|^{p_i-1}\big)|u_1-u_2|,\end{aligned} \tag{16}$$

a.e. $x \in \Gamma_c$, and for all $u, u_1, u_2 \in \mathbb{R}_+$, with $i = N$ or $i = T$. Set $q_N = P_N + 1, q_T = q_T + 1$, $q = \max\{q_N, q_T\}$ and suppose that

$$q < 3 \text{ if } d = 3. \tag{17}$$

We also suppose that the density $\rho \in L^\infty(\Omega)$ is positive, i.e. there exists ρ_0 such that $\rho(x) \ge \rho_0 > 0$. Finally, the load F and the body forces r are assumed to satisfy

$$F \in W^{1,2}\big(0,T,[L^2(\Gamma_f)]^N\big), \tag{18}$$

$$r \in W^{1,2}\big(0,T,[L^2(\Omega)]^N\big). \tag{19}$$

Set $H := [L^2(\Omega)]^d$ endowed with the inner product

$$(u,v) := \int_\Omega \rho u\cdot v\,\mathrm{d}x, \quad \forall\, u,v \in H,$$

which generates an equivalent norm denoted by $|\cdot|$. Denote by $|\cdot|_{_{q,\Gamma_c}}$ the norm in $L^q(\Gamma_{_c})$ and by $\|\cdot\|$ the norm in $[H^1(\Omega)]^d$. Let V_0 be the closed subspace of $[H^1(\Omega)]^N$ given by

$$V_0 := \big\{v \in [H^1(\Omega)]^d; \quad v = 0 \text{ on } \Gamma_d\big\},$$

and suppose that

$$u_0, u_1 \in V_0. \tag{20}$$

If we denote by a, c: $V_0 \times V_0 \to \mathbb{R}$ the following bilinear and symmetric applications:

$$a(u,v) := \int_\Omega \mathcal{A}\epsilon(u) : \epsilon(v),$$

$$c(u,v) := \int_\Omega \mathcal{C}\epsilon(u) : \epsilon(v), \quad \forall\, u,v \in V_0,$$

then from (10) we find $M > 0$ such that

$$\begin{aligned}|a(u,v)| &\le M\|u\|\,\|v\|,\\ |c(u,v)| &\le M\|u\|\,\|v\|, \quad \forall\, u,v \in V.\end{aligned} \tag{21}$$

From (11) and the Korn inequality, we deduce that there exists $D > 0$ such that

$$a(v,v) \ge D\|v\|^2, \quad c(v,v) \ge D\|v\|^2, \quad \forall\, v \in V. \tag{22}$$

Finally, we define M: $V_0 \to V_0'$, j: $V_0 \times V_0 \times V_0 \to \mathbb{R}$ and f: $V_0 \to V_0'$ as follows:

$$\big\langle M(w), v\big\rangle = \int_{\Gamma_c} m_{_N}\big(s,[w_{_N}]^+\big)v_{_N}\,\mathrm{d}s, \quad v \in V_0, \tag{23}$$

$$j(u,v,w) = \int_{\Gamma_c} m_{_T}\big(s,[u_{_N}]^+\big)\mu\big(s,|u_{_T}|\big)|w_{_T}|\,\mathrm{d}s, \quad u,v,w \in V_0, \tag{24}$$

$$\langle f(t), v\rangle = (r(t),v) + \int_{\Gamma_f} F(t)\cdot v\,\mathrm{d}s, \quad v \in V_0. \tag{25}$$

Using this notation, one can easily deduce that any solution of (1)–(8) satisfies the following variational problem:

(VP) Find $u : [0,T] \longrightarrow V_0$ such that

$$\begin{aligned}&\big\langle \ddot u(t), v - \dot u(t)\big\rangle + a\big(u(t), v - \dot u(t)\big) \\ &\quad + \eta c\big(\dot u(t), v - \dot u(t)\big) + \big\langle M(u(t)), v - \dot u(t)\big\rangle \\ &\quad + j\big(u(t), u(t), v\big) - j\big(u(t), u(t), \dot u(t)\big) \\ &\ge \big\langle f(t), v - \dot u(t)\big\rangle,\end{aligned} \tag{26}$$

$$u(0,x) = u_{_0}(x), \quad \dot u(0,x) = u_{_1}(x). \tag{27}$$

4. Existence and Uniqueness of the Solution

The main result of this section is the following:

Theorem 1. *There exists a unique solution of (VP) with the following regularity:*

$$u \in W^{1,\infty}(0,T,V) \cap W^{2,2}(0,T,H). \tag{28}$$

We recall here from (Ionescu, 2001) the following lemma, which will be useful in the proof of the theorem:

Lemma 1. *Let $\Omega \subset \mathbb{R}^d$ be as above and let $\alpha \in [2, 2(d-1)/(d-2)]$ if $d \ge 3$ and $\alpha \ge 2$ if $d = 2$. Then, for $\beta = [d(\alpha-2)+2]/2\alpha$ if $d \ge 3$ or if $d = 2$ and $\alpha = 2$,* *and for all* $\beta\in](\alpha-1)/\alpha,1[$ *if* $d=2$ *and* $\alpha>2$*, there exists a constant* $C=C(\beta)$ *such that*

$$\|v\|_{L^\alpha(\Gamma)}\le C\|v\|^{1-\beta}_{L^2(\Omega)}\|v\|^{\beta}_{H^1(\Omega)},\ \ \forall\, v\in H^1(\Omega). \tag{29}$$

Proof of Theorem 1. (Uniqueness) Let u_1 and u_2 be two solutions of (26)–(27) with regularity (28) and write $w=:u_1-u_2$. If we write the variational inequality (VP) successively for u_1 and u_2 taking $v=\dot u_2(t)$ in the first inequality and $v=\dot u_1(t)$ in the second one, and add the resulting inequalities, we obtain

$$\begin{aligned}&\big\langle \ddot w(t),\dot w(t)\big\rangle + a\big(\dot w(t),w(t)\big)+\eta c\big(\dot w(t),\dot w(t)\big)\\&\quad+\big\langle M(u_1(t))-M(u_2(t)),\dot w(t)\big\rangle\\&\quad+\int_{\Gamma_c}\big[m_T\big([u_{1N}(t)]^+\big)\mu\big(|u_{1T}(t)|\big)\\&\quad-m_N\big([u_{2N}(t)]^+\big)\mu\big(|u_{2T}(t)|\big)\big]\\&\qquad\times\big[|\dot u_{1T}(t)|-|\dot u_{2T}(t)|\big]\,\mathrm{d}s\le 0.\end{aligned}$$

Since the integrand of the last integral can be majorized by

$$\begin{aligned}&\big|m_T([u_{1N}(t)]^+)-m_T([u_{2N}(t)]^+)\big|\mu(|u_{1T}(t)|)\\&\quad+m_T([u_{2T}(t)]^+)\big|\,\mu(|u_{1T}(t)|)-\mu(|u_{2T}(t)|)\big|,\end{aligned}$$

we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[|\dot w(t)|^2+a\big(w(t),w(t)\big)\big]+\eta c\big(\dot w(t),\dot w(t)\big)\\&\le E_N\int_{\Gamma_c}\big(|u_{1N}(t)|^{p_n-1}+|u_{2N}(t)|^{p_n-1}\big)|w_N(t)|\,|\dot w_N(t)|\,\mathrm{d}s\\&\quad+\mu_o E_T\int_{\Gamma_c}\big(|u_{1N}(t)|^{p_T-1}+|u_{2N}(t)|^{p_T-1}\big)\\&\quad\times|w_N(t)|\,|\dot w_T(t)|\,\mathrm{d}s+\int_{\Gamma_c}(C_T+D_T|u_{1N}(t)|^{p_T})\\&\quad\times|w_T(t)|\,|\dot w_T(t)|\,\mathrm{d}s.\end{aligned}\tag{30}$$

The first two integrals on the right-hand side of (30) can be majorized using the Hölder inequality for $(q/(q-2);\ q;\ q)$ as follows:

$$\begin{aligned}&\int_{\Gamma_c}|u_{1N}(t)|^{p_n-1}|w_N(t)|\,|\dot w_N(t)|\,\mathrm{d}s\\&\qquad\le|u_1(t)|^{q-2}_{q,\Gamma_c}|w(t)|_{q,\Gamma_c}|\dot w(t)|_{q,\Gamma_c}\\&\qquad\le C_1\|u_1\|^{q-2}_{L^\infty(0,T;V_0)}\|w(t)\|\,\|\dot w(t)\|,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\Gamma_c}|u_{1N}(t)|^{p_n-1}|w_N(t)|\,|\dot w_N(t)|\,\mathrm{d}s\\&\qquad\le\frac{C_2}{\eta}\|w(t)\|^2+\frac{\eta}{6}\|\dot w(t)\|^2.\end{aligned}$$

In order to estimate the third integral, we use

$$C_T\int_{\Gamma_c}|w_T(t)|\,|\dot w_T(t)|\,\mathrm{d}s\le\frac{C_3}{\eta}\|w(t)\|^2+\frac{\eta}{6}\|\dot w(t)\|^2,$$

and the Hölder inequality for $((q+1)/(q-1);\ q+1;\ q+1)$

$$\begin{aligned}&\int_{\Gamma_c}|u_{1N}(t)|^{p_T}|w_T(t)|\,|\dot w_T(t)|\,\mathrm{d}s\\&\qquad\le|u_1(t)|^{q-1}_{q+1,\Gamma_c}\ |w(t)|_{q+1,\Gamma_c}|\dot w(t)|_{q+1,\Gamma_c}\\&\qquad\le C_4\|u_1\|^{q-1}_{L^\infty(0,T;V_0)}\|w(t)\|\,\|\dot w(t)\|\end{aligned}$$

to obtain

$$\begin{aligned}&D_T\int_{\Gamma_c}|u_{1N}(t)|^{p_T}|w_T(t)|\,|\dot w_T(t)|\,\mathrm{d}s\\&\qquad\le\frac{C_5}{\eta}\|w(t)\|^2+\frac{\eta}{6}\|\dot w(t)\|^2.\end{aligned}$$

From the above inequalities and (30), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ [|\dot w(t)|^2\ +a(w(t),w(t))]+\eta c(\dot w(t),\dot w(t))\\&\qquad\le C_6\|w(t)\|^2+\eta\|\dot w(t)\|^2.\end{aligned}$$

If we integrate this inequality from 0 to t, and use the coercivity of the bilinear applications $a(\cdot,\cdot)$ et $c(\cdot,\cdot)$ and the initial conditions $w(0)=\dot w(0)=0$, then

$$|w(t)|^2+\|w(t)\|^2\le C_6\int_0^t\big(|w(\tau)|^2+\|w(\tau)\|^2\big)\ \mathrm{d}\tau \tag{31}$$

By using the Gronwall lemma in (31), the uniqueness follows.

(Existence) In order to prove the existence of the solution u to (VP), we shall use the Faedo-Galerkin method. For this let us consider $\phi_i\in V$ as a sequence of linearly independent functions such that $V=\overline{\bigcup_{m=1}^{\infty}V_m}$, where $V_m=\mathrm{Span}\,\{\phi_1,\phi_2,\dots,\phi_m\}$. Since $u_0,\ v_0\in V$, let $u_0^m,\ v_0^m\in V_m$ be such that

$$u_0^m\longrightarrow u_0,\quad u_1^m\longrightarrow u_1\ \text{ strongly in } V. \tag{32}$$

If we consider the family of convex and differentiable functions $\Psi_\varepsilon\colon\mathbb{R}^d\to\mathbb{R}$ given by

$$\Psi_\varepsilon(v)=\sqrt{|v|^2+\varepsilon^2}-\varepsilon,\quad v\in\mathbb{R}^d$$

for all positive ε, then we have

$$0\le\Psi_\varepsilon(v)\le|v|,\quad\forall\, v\in\mathbb{R}^d, \tag{33}$$

$$|\Psi'_\varepsilon(v)(w)|\le|w|,\quad\forall\,(v,w)\in\mathbb{R}^d\times\mathbb{R}^d, \tag{34}$$

$$|\Psi_\varepsilon(v)-|v||\le\varepsilon,\quad\forall\, v\in\mathbb{R}^d. \tag{35}$$

Next we define $j_\varepsilon\colon V_0\times V_0\times V_0\to\mathbb{R}$, a family of regularized frictional functionals depending on $\epsilon>0$,

$$j_\varepsilon(u,v,w)=\int_{\Gamma_c} m_{\scriptscriptstyle T}(s,[u_{\scriptscriptstyle N}]^+)\mu(s,|v_{\scriptscriptstyle T}|)\Psi_\varepsilon(w_{\scriptscriptstyle T})\,\mathrm{d}s,\quad \forall u,v,w\in V_0.$$

The functional j_ε is Gâteaux-differentiable with respect to the third argument and represents an approximation of j, i.e. there exists a constant C such that

$$|j_\varepsilon(u,v,w)-j(u,v,w)| \le C\varepsilon\left(1+\|u\|^{q-1}\right),\quad \forall u,v,w\in V_0. \tag{36}$$

We denote by $J_\varepsilon\colon V_0\times V_0\times V_0\to V_0'$ the derivative of j_ε with respect to the third variable given by

$$\langle J_\varepsilon(u,v,w),z\rangle =\int_{\Gamma_c} m_{\scriptscriptstyle T}(s,[u_{\scriptscriptstyle N}]^+)\mu(s,|v_{\scriptscriptstyle T}|)\Psi'_\varepsilon(w_{\scriptscriptstyle T})(z_{\scriptscriptstyle T})\,\mathrm{d}s,\quad u,v,w,z\in V_0.$$

We can introduce now the following variational problem with regularized friction in the finite-dimensional space V_m:

$$(\mathrm{VP}_\epsilon^m):\quad \text{Find } u_\epsilon^m:\ [0,T]\longrightarrow V_m$$

such that

$$\begin{aligned}\langle \ddot u_\varepsilon^m(t),v\rangle &+ a(u_\varepsilon^m(t),v)+\eta c(\dot u_\varepsilon^m(t),v)\\ &+\langle M(u_\varepsilon^m(t),v\rangle\\ &+\langle J_\varepsilon(u_\varepsilon^m(t),u_\varepsilon^m(t),\dot u_\varepsilon^m(t)),v\rangle\\ &=\langle f(t),v\rangle,\end{aligned} \tag{37}$$

$$u_\varepsilon^m(0)=u_0^m,\quad \dot u_\varepsilon^m(0)=u_1^m. \tag{38}$$

Since $(u;v)\to J_\varepsilon(u,v,v)$ is a locally Lipschitz continuous function on $V_m\times V_m$, we deduce that (37)–(38) has a unique maximal solution $u_\varepsilon^m\in\mathcal{C}^2([0,T_\varepsilon^m];V_m)$.

The continuation of the proof is divided into three parts. We begin by proving that each problem has a unique solution u_ϵ^m for all $\epsilon>0$ and all $m\in\mathbb{N}$. To do this, we need some *a priori* estimates, which will be deduced in the first two parts of the proof. Only after that shall we prove that when $\epsilon\to 0$ and $m\longrightarrow+\infty$, the limit of u_ϵ^m, in an appropriate sense, is the solution to (VP).

In order to simplify the notation, we shall omit the indices ε and m in the first two parts of the proof.

(i) A priori estimates I

Since $\langle J_\varepsilon(u,v,w),w\rangle\ge 0$ for all $u,v,w\in V_0$, setting $v=\dot u(t)$ in (37) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac12|\dot u(t)|^2+\frac12 a(u(t),u(t))\right]+\eta\, c(\dot u(t),\dot u(t)) +\langle M(u(t)),\dot u(t)\rangle\le\langle f(t),\dot u(t)\rangle. \tag{39}$$

Let us introduce the following notation:

$$\tilde m_{\scriptscriptstyle N}(s,u)=m_{\scriptscriptstyle N}(s,u)-m_{\scriptscriptstyle N}(s,0),\quad \forall u\in\mathbb{R}_+,$$

$$P(s,u)=\int_0^u\tilde m_{\scriptscriptstyle N}(s,v)\,\mathrm{d}v,$$

$$\hat m(u)=\int_{\Gamma_c}P(s,u(s))\,\mathrm{d}s,\quad \forall u\in L^2(\Gamma_c)$$

a.e. $s\in\Gamma_c$. From (14) we find that the energy associated with the normal compliance $\hat m([u_{\scriptscriptstyle N}]^+)$ is positive, i.e.

$$\hat m([u_{\scriptscriptstyle N}(t)]^+)\ge 0,\quad \forall u\in L^2(0,T;V_0).$$

For all $v\in L^2(0,T,V_0)$, we have

$$\langle M(v(t)),\dot v(t)\rangle-\int_{\Gamma_c}m_{\scriptscriptstyle N}(s,0)\dot v_{\scriptscriptstyle N}(t)\,\mathrm{d}x =\int_{\Gamma_c}\tilde m_{\scriptscriptstyle N}(s,[v_{\scriptscriptstyle N}(t)]^+)\dot v_{\scriptscriptstyle N}(t)\,\mathrm{d}x,$$

and after differentiation of the associated energy $\hat m(v)$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat m([v_{\scriptscriptstyle N}(t)]^+)&=\int_{\Gamma_c}\frac{\mathrm{d}}{\mathrm{d}t}P(s,[v_{\scriptscriptstyle N}(t)]^+)\,\mathrm{d}x\\ &=\int_{\Gamma_c}\tilde m_{\scriptscriptstyle N}(s,[v_{\scriptscriptstyle N}(t)]^+)H(v_{\scriptscriptstyle N}(t))\dot v_{\scriptscriptstyle N}(t)\,\mathrm{d}x,\end{aligned}$$

where $H(x)$ is the Heaviside function. Since $\tilde m_{\scriptscriptstyle N}(s,0)=0$, we obtain

$$\tilde m_{\scriptscriptstyle N}([v_{\scriptscriptstyle N}(t)]^+)H(v_{\scriptscriptstyle N}(t))\dot v_{\scriptscriptstyle N}(t)=\tilde m_{\scriptscriptstyle N}([v_{\scriptscriptstyle N}(t)]^+)\dot v_{\scriptscriptstyle N}(t),$$

and the following equality follows:

$$\langle M(v(t)),\dot v(t)\rangle =\frac{\mathrm{d}}{\mathrm{d}t}\hat m([v_{\scriptscriptstyle N}(t)]^+)+\int_{\Gamma_c}m_{\scriptscriptstyle N}(s,0)\dot v_{\scriptscriptstyle N}(t,s)\,\mathrm{d}s,\quad v\in L^2(0,T;V_0).$$

Bearing in mind that $\hat m([v_{\scriptscriptstyle N}(t)]^+)\ge 0$, we integrate this equation to deduce that

$$\begin{aligned}\int_0^t\langle M(v(\tau)),\dot v(\tau)\rangle\,\mathrm{d}\tau &\ge\int_{\Gamma_c}m_{\scriptscriptstyle N}(s,0)v_{\scriptscriptstyle N}(t,s)\,\mathrm{d}s\\ &\quad-\int_{\Gamma_c}m_{\scriptscriptstyle N}(s,0)v_{\scriptscriptstyle N}(0,s)\,\mathrm{d}s-\hat m(v_{\scriptscriptstyle N}(0)),\end{aligned}$$

for all $v \in L^2(0,T;V_0)$. If we integrate (39) over $(0,t)$ and use the last inequality, then we obtain

$$|\dot{u}(t)|^2 + D\|u(t)\|^2 + 2\eta \int_0^t \|\dot{u}(\tau)\|^2 \,\mathrm{d}\tau \le |u_1|^2 + a(u_0,u_0) + \hat{m}([u_{0N}]^+) + 2\int_{\Gamma_c} m_N(s,0)u_{0N}\,\mathrm{d}s - 2\int_{\Gamma_c} m_N(s,0)u_N(t)\,\mathrm{d}s + 2\langle f(t),u(t)\rangle - 2\langle f(0),u_0\rangle - 2\int_0^t \langle \dot{f}(\tau),u(\tau)\rangle\,\mathrm{d}\tau. \tag{40}$$

In order to estimate the normal displacement, we have

$$\Big|\int_{\Gamma_c} m_N(s,0)u_N(t)\,\mathrm{d}s\Big| \le C_T \text{ mes}(\Gamma_c)|u(t)|_{2,\Gamma_c} \le \frac{C}{D} + \frac{D}{8}\|u(t)\|^2.$$

By using the last inequality in (40), we deduce that

$$|\dot{u}(t)|^2 + \|u(t)\|^2 + \eta \int_0^t \|\dot{u}(\tau)\|^2\,\mathrm{d}\tau \le C + C\int_0^t \left(|\dot{u}(\tau)|^2 + \|u(\tau)\|^2\right)\,\mathrm{d}\tau. \tag{41}$$

From the Gronwall lemma we obtain that the solution $t \to (u_\varepsilon^m(t);\dot{u}_\varepsilon^m(t))$ of (37)–(38) is bounded on its interval of existence, and hence $t \to (u_\varepsilon^m(t);\dot{u}_\varepsilon^m(t))$ is a global solution, i.e. $T_\varepsilon^m = T$. Moreover, we have

$$\{u_\varepsilon^m\}_{m,\varepsilon} \text{ is bounded in } L^\infty(0,T;V_0), \tag{42}$$

$$\{\dot{u}_\varepsilon^m\}_{m,\varepsilon} \text{ is bounded in } L^\infty(0,T;H)\cap L^2(0,T;V_0). \tag{43}$$

(ii) A priori estimates II

If we let $v = \ddot{u}(t)$ in (37) and notice that

$$\langle J_\varepsilon(u(t),u(t),\dot{u}(t)),\ddot{u}(t)\rangle = \int_{\Gamma_c} m_T([u_N(t)]^+)\mu(|u_T(t)|)\Psi'_\varepsilon(\dot{u}_T(t))(\ddot{u}_T(t))\,\mathrm{d}s = \int_{\Gamma_c} m_T([u_N(t)]^+)\mu(|u_T(t)|)\frac{\mathrm{d}}{\mathrm{d}t}\{\Psi_\varepsilon(\dot{u}_T(t))\}\,\mathrm{d}s,$$

then we get

$$|\ddot{u}(t)|^2 + a(u(t),\ddot{u}(t)) + \langle M(u(t)),\ddot{u}(t)\rangle + \frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{c(\dot{u}(t),\dot{u}(t))\} + \int_{\Gamma_c} m_T([u_N(t)]^+)\mu(|u_T(t)|)\frac{\mathrm{d}}{\mathrm{d}t}\{\Psi_\varepsilon(\dot{u}_T(t))\}\,\mathrm{d}s = \langle f(t),\ddot{u}(t)\rangle.$$

After integration from 0 to t, we obtain

$$\int_0^t |\ddot{u}(\tau)|^2\,\mathrm{d}\tau + \frac{\eta}{2}\|\dot{u}(t)\|^2 \le C + \int_0^t a(\dot{u}(\tau),\dot{u}(\tau))\,\mathrm{d}\tau - a(u(t),\dot{u}(t)) - \int_0^t \langle M(u(t)),\ddot{u}(t)\rangle + \langle f(t),\dot{u}(t)\rangle - \int_0^t \langle \dot{f}(\tau),\dot{u}(\tau)\rangle\,\mathrm{d}\tau - \int_0^t\int_{\Gamma_c} m_T([u_N(\tau)]^+)\mu(|u_T(\tau)|)\frac{\mathrm{d}}{\mathrm{d}\tau}\{\Psi_\varepsilon(\dot{u}_T(\tau)\}\,\mathrm{d}s\,\mathrm{d}\tau. \tag{44}$$

The virtual power of the normal displacement can be written as

$$\int_0^t \langle M(u(t)),\ddot{u}(t)\rangle = \int_0^t\int_{\Gamma_c} m_N(s,[u_N(\tau)]^+)\ddot{u}_N(\tau)\,\mathrm{d}s\,\mathrm{d}\tau = \langle M(u(t)),\dot{u}(t)\rangle - \langle M(u_0),u_1\rangle - \int_0^t\int_{\Gamma_c} \frac{\partial m_N}{\partial u}([u_N(\tau)]^+)H(u_N(\tau))(\dot{u}_N(\tau))^2\,\mathrm{d}s\,\mathrm{d}\tau.$$

Using the above inequalities and

$$|a(u(\tau),\dot{u}(\tau))| \le C\|u(\tau)\|\,\|\dot{u}(\tau)\| \le 2\frac{C^2}{\eta}\|u(\tau)\|^2 + \frac{\eta}{8}\|\dot{u}(\tau)\|^2,$$

$$|\langle M(u(t)),\dot{u}(t)\rangle| \le \int_{\Gamma_c}(C_N + D_N|u(t)|^{p_n})|\dot{u}(t)|\,\mathrm{d}s \le \frac{C(1+\|u(t)\|^{q-1})}{\eta} + \frac{\eta}{8}\|\dot{u}(t)\|^2,$$

we deduce from (44) that

$$\eta\|\dot{u}(t)\|^2 + \int_0^t |\ddot{u}(\tau)|^2\,\mathrm{d}\tau \le C + C\int_0^t \|\dot{u}(\tau)\|^2\,\mathrm{d}\tau + \Big|\int_0^t\int_{\Gamma_c} |u_N(\tau)|^{p_n-1}\dot{u}_N^2(\tau)\,\mathrm{d}s\,\mathrm{d}\tau\Big| + \Big|\int_0^t\int_{\Gamma_c} m_T([u_N(\tau)]^+)\mu(|u_T(\tau)|)\frac{\mathrm{d}}{\mathrm{d}\tau} \times \{\Psi_\varepsilon(\dot{u}_T(\tau))\}\,\mathrm{d}s\,\mathrm{d}\tau\Big|. \tag{45}$$

The second integral on the right-hand side of (45) can be estimated as follows:

$$\int_{\Gamma_c} |u_N(\tau)|^{p_n-1}\dot{u}_N^2(\tau)\,\mathrm{d}s \le |u(\tau)|_{q,\Gamma_c}^{q-2}|\dot{u}(\tau)|_{q,\Gamma_c}^2 \le \|u(\tau)\|^{q-2}\|\dot{u}(\tau)\|^2 \le C\|\dot{u}(\tau)\|^2.$$

In order to estimate the last integral of (45), we replace the non-differential term $|u_T(\tau)|$ with $\Psi_r(u_T(\tau))$ $(r>0)$ to get

$$\begin{aligned}&\int_0^t\!\!\int_{\Gamma_c} m_T([u_N(\tau)]^+)\mu(|u_T(\tau)|)\frac{\mathrm{d}}{\mathrm{d}\tau}\\&\quad\times\{\Psi_\varepsilon(\dot u_T(\tau))\}\,\mathrm{d}s\,\mathrm{d}\tau\\&=\int_0^t\!\!\int_{\Gamma_c} m_T([u_N(\tau)]^+)\mu(\Psi_r(u_T(\tau)))\frac{\mathrm{d}}{\mathrm{d}\tau}\\&\quad\times\{\Psi_\varepsilon(\dot u_T(\tau))\}\,\mathrm{d}s\,\mathrm{d}\tau\\&\quad+\int_0^t\!\!\int_{\Gamma_c} m_T([u_N(\tau)]^+)\big[\mu(|u_T(\tau)|)\\&\quad-\mu(\Psi_r(u_T(\tau)))\big]\frac{\mathrm{d}}{\mathrm{d}\tau}\{\Psi_\varepsilon(\dot u_T(\tau))\}\,\mathrm{d}s\,\mathrm{d}\tau,\end{aligned}$$

and, after integration by parts, we have

$$\begin{aligned}&\int_0^t\!\!\int_{\Gamma_c} m_T([u_N(\tau)]^+)\mu(\Psi_r(u_T(\tau)))\frac{\mathrm{d}}{\mathrm{d}\tau}\{\Psi_\varepsilon(\dot u_T(\tau))\}\,\mathrm{d}s\,\mathrm{d}\tau\\&=\int_{\Gamma_c} m_T([u_N(t)]^+)\mu(\Psi_r(u_T(t)))\Psi_\varepsilon(\dot u_T(t))\,\mathrm{d}s\\&\quad-\int_{\Gamma_c} m_T([u_{0N}]^+)\mu(\Psi_r(u_{0T}))\Psi_\varepsilon(u_{1T})\,\mathrm{d}s\\&\quad-\int_0^t\!\!\int_{\Gamma_c}\Big[\frac{\partial m_T}{\partial u}([u_N(\tau)]^+)H(u_N(\tau))\dot u_N(\tau)\\&\quad\times\mu(\Psi_r(u_T(\tau)))\Psi_\varepsilon(\dot u_T(\tau))+m_T([u_N(\tau)]^+)\frac{\partial\mu}{\partial u}\\&\quad\times(\Psi_r(u_T(\tau)))\Psi_r'(u_T(\tau))(\dot u_T(\tau))\Psi_\varepsilon(\dot u_T)\Big]\,\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}$$

We now use the estimates: (12) for μ, (33) for Ψ_ε, (34) for Ψ_r', and (15) for m_T, and obtain

$$\begin{aligned}&\Big|\int_0^t\!\!\int_{\Gamma_c} m_T([u_N(\tau)]^+)\mu(\Psi_r(u_T(\tau)))\frac{\mathrm{d}}{\mathrm{d}\tau}\\&\quad\times\{\Psi_\varepsilon(\dot u_T(\tau))\}\,\mathrm{d}s\,\mathrm{d}\tau\Big|\\&\le C++C\int_{\Gamma_c}(C_T+D_T|u(t)|^{p_T})|\dot u_T(t)|\,\mathrm{d}s\\&\quad+C\int_0^t\!\!\int_{\Gamma_c}|u(\tau)|^{p_T-1}|\dot u_T(\tau)|^2\mathrm{d}s\,\mathrm{d}\tau\\&\quad+C\int_0^t\!\!\int_{\Gamma_c}(C_T+D_T|u(\tau)|^{p_T})|\dot u_T(\tau)|^2\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}$$

Using (35) to estimate the difference between $\Psi_r(u_T(\tau))$ and $|\dot u_T(\tau)|$, we have

$$\begin{aligned}&\Big|\int_0^t\!\!\int_{\Gamma_c} m_T([u_N(\tau)]^+)[\mu(|u_T(\tau)|)-\mu(\Psi_r(u_T(\tau)))]\frac{\mathrm{d}}{\mathrm{d}\tau}\\&\quad\{\Psi_\varepsilon(\dot u_T(\tau))\}\,\mathrm{d}s\,\mathrm{d}\tau\Big|\le C(\varepsilon,m_T,\mu)r,\end{aligned}$$

where $C(\varepsilon,m_T,\mu)$ is a constant independent of r. We pass to the limit $r\to 0$ to obtain

$$\begin{aligned}&\Big|\int_0^t\!\!\int_{\Gamma_c} m_T([u_N(\tau)]^+)\mu(|u_T(\tau)|)\frac{\mathrm{d}}{\mathrm{d}\tau}\{\Psi_\varepsilon(\dot u_T(\tau))\}\,\mathrm{d}s\,\mathrm{d}\tau\Big|\\&\le C+C\int_{\Gamma_c}(C_T+D_T|u(t)|^{p_T})|\dot u_T(t)|\,\mathrm{d}s\\&\quad+C\int_0^t\!\!\int_{\Gamma_c}|u(\tau)|^{p_T-1}|\dot u_T(\tau)|^2\mathrm{d}s\,\mathrm{d}\tau\\&\quad+C\int_0^t\!\!\int_{\Gamma_c}(C_T+D_T|u(\tau)|^{p_T})|\dot u_T(\tau)|^2\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}\tag{46}$$

If we use the Hölder inequality in the second part of (46), then the following estimates are obtained:

$$\begin{aligned}\int_{\Gamma_c}|u(t)|^{p_T}|\dot u_T(t)|\,\mathrm{d}s&\le|u(t)|^{q-1}_{q,\Gamma_c}|\dot u_T(t)|_{q,\Gamma_c}\\&\le\|u(t)\|^{q-1}\|\dot u(t)\|,\\\int_{\Gamma_c}|u(\tau)|^{p_n-1}|\dot u(\tau)|^2\mathrm{d}s&\le|u(\tau)|^{q-2}_{q,\Gamma_c}|\dot u(\tau)|^2_{q,\Gamma_c}\\&\le\|u(\tau)\|^{q-2}\|\dot u(\tau)\|^2,\\\int_{\Gamma_c}|u(\tau)|^{p_T}|\dot u_T(\tau)|^2\mathrm{d}s&\le|u(\tau)|^{q-1}_{q,\Gamma_c}|\dot u_T(\tau)|^2_{q,\Gamma_c}\\&\le\|u(\tau)\|^{q-1}\|\dot u(\tau)\|^2.\end{aligned}$$

Since the functions u and $\dot u$ are bounded in $L^\infty(0,T;V)$ and $L^2(0,T;V)$), respectively, from the above estimates and (45) we deduce that

$$\|\dot u(t)\|^2+\int_0^t|\ddot u(\tau)|^2\mathrm{d}\tau\le C+C\int_0^t\|\dot u(\tau)\|^2\mathrm{d}\tau.\tag{47}$$

Using the Gronwall lemma, we conclude that

$$\{\dot u_\varepsilon^m\}_{m,\varepsilon}\ \text{ is bounded in }\ L^\infty(0,T;V_0),\tag{48}$$

$$\{\ddot u_\varepsilon^m\}_{m,\varepsilon}\ \text{ is bounded in }\ L^2(0,T;H).\tag{49}$$

(iii) Passage to the limit in m and ε

From (42), (43), (48), (49) we deduce that there exists a subsequence of $\{u_\varepsilon^m\}_{m,\varepsilon}$ (again denoted by $\{u_\varepsilon^m\}_{m,\varepsilon}$), such that

$$u_\varepsilon^m\overset{*}{\rightharpoonup}u\ \text{ weak* in }\ L^\infty(0,T;V_0),\tag{50}$$

$$\dot u_\varepsilon^m\overset{*}{\rightharpoonup}\dot u\ \text{ weak* in }\ L^\infty(0,T;V_0),\tag{51}$$

$$\ddot u_\varepsilon^m\overset{*}{\rightharpoonup}\ddot u\ \text{ weak* in }\ L^\infty(0,T;H),\tag{52}$$

as $\varepsilon \to 0$ and $m \to +\infty$. If we write

$$\Omega_T = \Omega \times]0, T[,$$

then

$$\{u^m_\varepsilon\}_{m,\varepsilon}, \ \{\dot{u}^m_\varepsilon\}_{m,\varepsilon} \quad \text{are bounded in } H^1(\Omega_T).$$

Since the embedding of $H^1(\Omega_T)$ in $L^2(\Omega_T) = L^2(0,T;H)$ is compact, we find that there exists a subsequence of $\{u^m_\varepsilon\}$ (again denoted by $\{u^m_\varepsilon\}$), such that

$$u^m_\varepsilon \to u \ \text{ strongly in } L^2(0,T;H), \tag{53}$$

$$\dot{u}^m_\varepsilon \to \dot{u} \ \text{ strongly in } L^2(0,T;H). \tag{54}$$

Moreover, since the trace map from $H^1(\Omega_T)$ to $L^2(\partial\Omega_T)$ is a compact operator and $\partial\Omega_T = \partial\Omega\times]0,T[\bigcup \Omega\times\{0\}\bigcup\Omega\times\{T\}$, we deduce that

$$u^m_\varepsilon(T) \to u(T) \ \text{ strongly in } H, \tag{55}$$

$$\dot{u}^m_\varepsilon(T) \to \dot{u}(T) \ \text{ strongly in } H, \tag{56}$$

and from (50)–(52) we have

$$u^m_\varepsilon(T) \rightharpoonup u(T) \ \text{ weakly in } V.$$

We only have to verify that u is the solution of (26) and (27). Let $w \in L^2(0,T;V_0)$ be fixed and let $w^m \in L^2(0,T;V_m)$ be a sequence such that

$$w^m \to w \ \text{ strongly in } L^2(0,T;V_0).$$

If we let $v = w^m(t) - \dot{u}^m_\varepsilon(t)$ in (37) and use the inequality

$$j_\varepsilon(u,v,w) - j_\varepsilon(u,v,z) \geq \langle J_\varepsilon(u,v,z), w-z\rangle, \quad \forall\, u,v,w,z \in V_0,$$

after integration of (37) from 0 to T, we have

$$\begin{aligned}
&\int_0^T \{(\ddot{u}^m_\varepsilon(t), w^m(t) - \dot{u}^m_\varepsilon(t)) + a(u^m_\varepsilon(t), w^m(t) - \dot{u}^m_\varepsilon(t)) \\
&\quad + \eta c(\dot{u}^m_\varepsilon(t), w^m(t) - \dot{u}^m_\varepsilon(t)) \\
&\quad + \langle M(u^m_\varepsilon(t)), w^m(t) - \dot{u}^m_\varepsilon(t)\rangle\}\,\mathrm{d}t \\
&\quad + \int_0^T \big[j_\varepsilon(u^m_\varepsilon(t), u^m_\varepsilon(t), w^m(t)) \\
&\quad - j_\varepsilon(u^m_\varepsilon(t), u^m_\varepsilon(t), \dot{u}^m_\varepsilon(t))\big]\mathrm{d}t \\
&\geq \int_0^T \langle f(t), w^m(t) - \dot{u}^m_\varepsilon(t)\rangle\,\mathrm{d}t.
\end{aligned}$$

If we use the estimate (36), after some algebra we obtain

$$\begin{aligned}
&C\varepsilon(1 + \|u^m_\varepsilon\|_{L^2(0,T;V)}) + \int_0^T (\ddot{u}^m_\varepsilon(t), w^m)\,\mathrm{d}t \\
&\quad + \frac{1}{2}|u_1|^2 - \frac{1}{2}|\dot{u}^m_\varepsilon(T)|^2 + \int_0^T a(u^m_\varepsilon(t), w^m)\,\mathrm{d}t \\
&\quad + a(u_0, u_0) + \eta\int_0^T c(\dot{u}^m_\varepsilon(t), w^m)\,\mathrm{d}t \\
&\quad + \int_0^T \big[j(u^m_\varepsilon(t), u^m_\varepsilon(t), w^m) \\
&\quad - j(u^m_\varepsilon(t), u^m_\varepsilon(t), \dot{u}^m_\varepsilon(t))\big]\mathrm{d}t \\
&\quad + \int_0^T \langle M(u^m_\varepsilon(t)), w^m\rangle\,\mathrm{d}t \\
&\quad + \hat{m}([u^m_{\varepsilon N}(T)]^+) - \hat{m}([u_{0N}]^+) \\
&\quad - \int_{\Gamma_c} m_N(s,0)\,\big[u^m_{\varepsilon N}(T) - u_{0N}\big]\,\mathrm{d}s \\
&\geq a(u^m_\varepsilon(T), u^m_\varepsilon(T)) + \eta\int_0^T c(\dot{u}^m_\varepsilon(t), \dot{u}^m_\varepsilon(t))\,\mathrm{d}t \\
&\quad + \int_0^T \langle f(t), w^m(t) - \dot{u}^m_\varepsilon(t)\rangle\,\mathrm{d}t.
\end{aligned} \tag{57}$$

Now let us verify the convergence of the terms of the left-hand side of (57). First, we prove that for $m \to +\infty$ and $\varepsilon \to 0$ we have

$$\int_0^T j(u^m_\varepsilon(t), u^m_\varepsilon(t), \dot{u}^m_\varepsilon(t))\,\mathrm{d}t \to \int_0^T j(u(t), u(t), \dot{u}(t))\,\mathrm{d}t. \tag{58}$$

Indeed, after some algebra, we get

$$\begin{aligned}
&|j(u^m_\varepsilon(t), u^m_\varepsilon(t), \dot{u}^m_\varepsilon(t)) - j(u(t), u(t), \dot{u}(t))| \\
&\leq C\Big[\Big(|u^m_\varepsilon(t)|^{q-2}_{q,\Gamma_c} + |u(t)|^{q-2}_{q,\Gamma_c}\Big)\,|u^m_\varepsilon(t) - u(t)|_{q,\Gamma_c} \\
&\quad \times |\dot{u}^m_\varepsilon(t)|_{q,\Gamma_c} + \Big(1 + |u(t)|^{q-1}_{q+1,\Gamma_c}\Big) \\
&\quad \times |u^m_\varepsilon(t) - u(t)|_{q+1,\Gamma_c}\,|\dot{u}^m_\varepsilon(t)|_{q+1,\Gamma_c} \\
&\quad + \Big(1 + |u(t)|^{q-1}_{q,\Gamma_c}\Big)\,|\dot{u}^m_\varepsilon(t) - \dot{u}(t)|_{q,\Gamma_c}\Big].
\end{aligned}$$

If we use now Lemma 1 for

$$\beta = \frac{3(q-1)+2}{2(q+1)} < 1 \ \text{ if } d = 3$$

and for

$$\beta \in]\frac{q}{q+1}, 1[\quad \text{if} \quad d = 2,$$

we obtain

$$\begin{aligned} &|u_\varepsilon^m(t) - u(t)|_{q+1,\Gamma_c} \\ &\quad \le C|u_\varepsilon^m(t) - u(t)|^{1-\beta} \\ &\qquad \times (\|u_\varepsilon^m\|_{L^\infty(0,T;V_0)} + \|u\|_{L^\infty(0,T;V_0)})^\beta, \\ &|\dot u_\varepsilon^m(t) - \dot u(t)|_{q+1,\Gamma_c} \\ &\quad \le C|\dot u_\varepsilon^m(t) - \dot u(t)|^{1-\beta} \\ &\qquad \times (\|\dot u_\varepsilon^m\|_{L^\infty(0,T;V_0)} + \|\dot u\|_{L^\infty(0,T;V_0)})^\beta. \end{aligned}$$

From the last three inequalities we deduce that

$$\begin{aligned} &\int_0^T \big|j(u_\varepsilon^m(t), u_\varepsilon^m(t), \dot u_\varepsilon^m(t)) - j(u(t), u(t), \dot u(t))\big| \, \mathrm{d}t \\ &\qquad \le C \left[\|u_\varepsilon^m - u\|^{1-\beta}_{L^2(0,T;H)} + \|\dot u_\varepsilon^m - \dot u\|^{1-\beta}_{L^2(0,T;H)} \right], \end{aligned}$$

and, by using (53) as $m \to +\infty$ and $\varepsilon \to 0$, we obtain (58). In a similar way, we conclude that

$$\int_0^T \langle M(u_\varepsilon^m(t)), w^m(t)\rangle \, \mathrm{d}t \to \int_0^T \langle M(u(t)), w(t)\rangle \, \mathrm{d}t,$$

and therefore

$$\hat m([u^m_{\varepsilon N}(T)]^+) \to \hat m([u_N(T)]^+) \quad \text{as } m \to +\infty \text{ and } \varepsilon \to 0.$$

Indeed, we have the estimate

$$\begin{aligned} &\big|\hat m([u^m_{\varepsilon N}(T)]^+) - \hat m([u_N(T)]^+)\big| \\ &\quad \le \int_{\Gamma_c} \big|P(s, [u^m_{\varepsilon N}(T)]^+) - P(s, [u_N(T)]^+)\big| \, \mathrm{d}s \\ &\quad \le \int_{\Gamma_c} \big(|u_\varepsilon^m(T)|^{q-1} + |u(T)|^{q-1}\big) \, |u_\varepsilon^m(T) - u(T)| \, \mathrm{d}s. \end{aligned}$$

Using the Hölder inequality and Lemma 1, we deduce that

$$\begin{aligned} &\int_{\Gamma_c} |u_\varepsilon^m(T)|^{q-1} |u_\varepsilon^m(T) - u(T)| \, \mathrm{d}s \\ &\quad \le \|u_\varepsilon^m\|^{q-1}_{L^\infty(0,T;V)} |u_\varepsilon^m(T) - u(T)|_{q,\Gamma_c} \\ &\quad \le C\|u_\varepsilon^m\|^{q-1}_{L^\infty(0,T;V)} |u_\varepsilon^m(T) - u(T)|^{1-\beta} \\ &\qquad \times (\|u_\varepsilon^m\|_{L^\infty(0,T;V)} + \|u\|_{L^\infty(0,T;V)})^\beta, \end{aligned}$$

and from (55) we get the strong convergence of the associated energy.

If we pass to the limit in (57) as $m \to +\infty$ and $\varepsilon \to 0$, and we bear in mind the strong convergence proved above, we obtain

$$\begin{aligned} &\int_0^T (\ddot u(t), w(t)) \, \mathrm{d}t + \frac{1}{2}|u_1|^2 - \frac{1}{2}|\dot u(T)|^2 + a(u_0, u_0) \\ &\quad + \int_0^T a(u(t), w(t)) \, \mathrm{d}t + \eta \int_0^T c(\dot u(t), w(t)) \, \mathrm{d}t \\ &\quad + \int_0^T [j(u(t), u(t), w(t)) - j(u(t), u(t), \dot u(t))] \, \mathrm{d}t \\ &\quad + \int_0^T \langle M(u(t)), w(t)\rangle \, \mathrm{d}t + \hat m([u_N(T)]^+) \\ &\quad - \hat m([u_{0N}]^+) \, \mathrm{d}t - \int_{\Gamma_c} m_N(s, 0) \, [u_N(T) - u_{0N}] \, \mathrm{d}s \\ &\ge \liminf_{m \to +\infty, \varepsilon \to 0} \Big[a(u_\varepsilon^m(T), u_\varepsilon^m(T)) \\ &\quad + \eta \int_0^T c(\dot u_\varepsilon^m(t), \dot u_\varepsilon^m(t)) \, \mathrm{d}t \Big] + \int_0^T \langle f(t), w(t) - \dot u(t)\rangle \, \mathrm{d}t \\ &\ge a(u(T), u(T)) + \eta \int_0^T c(\dot u(t), \dot u(t)) \, \mathrm{d}t \\ &\quad + \int_0^T \langle f(t), w(t) - \dot u(t)\rangle \, \mathrm{d}t. \end{aligned}$$

Finally, for all $w \in L^2(0, T; V_0)$ we have

$$\begin{aligned} &\int_0^T [\, (\ddot u(t), w(t) - \dot u(t)) + a(u(t), w(t) - \dot u(t)) \\ &\qquad + \eta c(\dot u(t), v - \dot u(t)) + \langle M(u(t)), v - \dot u(t)\rangle \\ &\qquad + j(u(t), u(t), w(t)) - j(u(t), u(t), \dot u(t))] \, \mathrm{d}t \\ &\quad \ge \int_0^T \langle f(t), w(t) - \dot u(t)\rangle \, \mathrm{d}t, \end{aligned}$$

and the pointwise inequality (26) follows. ■

References

Duvaut G. and Lions J.L. (1976): *Inequalities in Mechanics and Physics*. — Berlin: Springer.

Campillo M. and Ionescu I.R. (1997): *Initiation of antiplane shear instability under slip dependent friction*. — J. Geophys. Res., Vol. 122, No. B9, pp. 20363–20371.

Corneschi C., Hoarau-Martel T.-H. and Sofonea M. (2002): *A quasistatic contact problem with slip dependent coefficient of friction for elastic materials*. — J. Appl. Anal. (to appear).

Dascalu C., Ionescu I.R. and Campillo M. (2000): *Fault finiteness and initiation of dynamic shear instability*. — Earth Planet. Sci. Lett., No. 177, pp. 163–176.

Dieterich J.H. (1994): *A constitutive law for rate of earthquake production and its application to earthquake clustering*. — J. Geophys. Res., Vol. 99, No. B2, pp. 2601–2618.

Favreau P., Campillo M. and Ionescu I.R. (1991): *Initiation of inplane shear instability under slip dependent friction*. — Bull. Seism. Soc. Amer., Vol. 89, No. 5, pp. 1280–1295.

Favreau P., Ionescu I.R. and Campillo M. (1991b): *On the dynamic sliding with rate and state dependent friction laws*. — Geophys. J., Vol. 139, pp. 671–678.

Ionescu I.R. (2001): *Viscosity solutions for dynamic problems with slip-rate dependent friction*. — Quart. Appl. Math. (to appear).

Ionescu I.R. and Campillo M. (1999): *The influence of the shape of the friction law and fault finiteness on the duration of initiation*. — J. Geophys. Res., Vol. 104, No. B2, pp. 3013–3024.

Ionescu I.R. and Paumier J.C. (1993): *Dynamic stick-slip motions with sliding velocity-dependent friction*. — Comptes Rendus de l'Acad. Sci. Paris, S. I, Vol. 316, pp. 121–125.

Ionescu I.R. and Paumier J.C. (1994): *On the contact problem with slip rate dependent friction in elastodynamics*. — Europ. J. Mech., A-Solids, Vol. 13, No. 4, pp. 556–568.

Ionescu I.R. and Paumier J.C. (1996): *On the contact problem with slip displacement dependent friction in elastostatics*. — Int. J. Eng. Sci., Vol. 34, No. 4, pp. 471–491.

Kuttler K.L. (1997): *Dynamic friction contact problems for general normal and friction laws*. — Nonlin. Anal., Vol. 28, No. 3, pp. 559–575.

Kuttler K.L. and Shillor K.L. (1999): *Set-valued pseudomonotone maps and degenerate evolution inclusions*. — Commun. Contemp. Math., Vol. 1, pp. 87–123.

Martins J.A.C. and Oden J.T. (1987): *Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface law*. — Nonlin. Anal., Vol. 11, No. 3, pp. 407–428.

Oden J.T. and Martins J.A.C. (1985): *Models and computational methods for dynamic friction phenomena*. — Comp. Math. Appl. Mech. Eng., Vol. 52, pp. 527–634.

Ohnaka Y., Kuwahara Y. and Yamamoto K. (1987): *Constitutive relations between dynamic physical parameters near a tip of the propagation slip during stick-slip shear failure*. — Tectonophysics, Vol. 144, pp. 109–125.

Perrin G., Rice J.R. and Zheng G. (1995): *Self-healing slip pulse on a frictional surface*. — J. Mech. Phys. Solids, Vol. 43, No. 9, pp. 1461–1495.

Rabinowicz E. (1951): *The nature of the static and kinetic coefficients of friction*. — J. Appl. Phys., Vol. 22. No. 11, pp. 13730-1379.

Rice J.R. and Ruina A.L. (1983): *Stability of steady frictional slipping*. — J. Appl. Mech., Vol. 50, pp. 343–349.

Ruina A.L. (1983): *Slip instabilities and state variable friction laws*. — J. Geophys. Res., Vol. 88, No. B12, pp. 10359–10370.

Scholz C.H. (1990): *The Mechanics of Earthquakes and Faulting*. — Cambridge: Cambridge University Press.