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DYNAMIC CONTACT PROBLEMS WITH VELOCITY CONDITIONS

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We consider dynamic problems which describe frictional contact between a body and a foundation. The constitutive law is viscoelastic or elastic and the frictional contact is modelled by a general subdifferential condition on the velocity, including the normal damped responses. We derive weak formulations for the models and prove existence and uniqueness results. The proofs are based on the theory of second-order evolution variational inequalities. We show that the solutions of the viscoelastic problems converge to the solution of the corresponding elastic problem as the viscosity tensor tends to zero and when the frictional potential function converges to the corresponding function in the elastic problem.

Keywords: viscoelastic, elastic, subdifferential boundary condition, dynamic process, nonlinear hyperbolic variational inequality, maximal monotone operator, weak solution

1. Introduction

Contact problems arise in many situations, for instance, in crack and impact mechanics, or in earthquake phenomena. Despite the importance of their practical applications and the considerable literature devoted to these topics, many problems involving contact phenomena still remain open.

A number of papers investigating quasistatic frictional contact problems with viscoelastic materials have recently been published (see, e.g., Awbi *et al.*, 2000; Chau *et al.*; 2001a; 2001b; Han and Sofonea, 2000; 2001). In (Chau *et al.*, 2001b) frictional contact was modelled by a general velocity-dependent dissipation functional, in (Chau *et al.*, 2001a) a bilateral contact with Tresca's friction law was analysed, while in (Han and Sofonea, 2001) frictional contact with normal compliance was studied, and in (Awbi *et al.*, 2000; Han and Sofonea, 2000) frictional contact with normal damped response was considered. Dynamic contact problems with normal compliance were considered in (Andrews *et al.*, 1997a; 1997b; Kuttler and Shillor, 1999; Martins and Oden, 1987).

This paper constitutes a contribution to the study of second-order evolution contact problems. Our aim is to give versions of the results obtained in (Chau *et al.*, 2001b) to a dynamic process. We investigate models for dynamic frictional contact between a body and an obstacle, in which Kelvin-Voigt viscoelastic or elastic constitutive laws are considered. The frictional contact is modelled by a general subdifferential boundary condition. Further examples and detailed explanations concerning the boundary conditions of this form can be found in the monograph by Panagiotopoulos (1985) and more recently in (Chau *et al.*, 2001b). Here, the study of viscoelastic or elastic materials in dynamic processes with subdifferential boundary conditions leads to a non-standard new mathematical model, implying nonlinear second-order evolution equations.

We prove the existence and uniqueness of weak solutions to the mechanical problems. We also show the continuous dependence of these solutions on the viscosity and frictional potential function, both of which may vary because of simultaneous changes in the viscosity of the body and in the roughness of the surface.

The outline of the paper is as follows. In Section 2 we introduce the notation and a preliminary material. In Section 3 we formulate the dynamic mechanical problems with a subdifferential frictional contact condition. Then, after specifying the assumptions on the data, we derive variational formulations for the problems, and we prove an existence and uniqueness result. The proof is based on second-order evolutionary inequalities with maximal monotone operators. In Section 4 we prove a convergence result which shows that the solutions to the viscoelastic problems converge to the solution to the elastic problem when the viscosity tends to zero and when the frictional potential function converges to the corresponding one in the elastic problem. Finally, in Section 5 we provide some examples of specific subdifferential conditions to which our results apply.

2. Notation and Preliminaries

Let S_d be the space of second-order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), and denote the inner product and the Euclidean norm on $\mathbb{R}^d$ and S_d by '$\cdot$' and $|\cdot|$, respectively. Thus,

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \qquad |\boldsymbol{v}| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2}, \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d.$$

Here and below, the indices i and j run between 1 and d, the summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ. We shall use the notation

$$H = L^2(\Omega)^d = \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\},$$

$$\mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

$$H_1 = \{\boldsymbol{u} = (u_i) \mid u_i \in H^1(\Omega)\},$$

$$\mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \sigma_{ij,j} \in L^2(\Omega)\}.$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \,\mathrm{d}x,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \,\mathrm{d}x,$$

$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau})_H,$$

respectively, where $\boldsymbol{\varepsilon} : H_1 \to \mathcal{H}$ and $\mathrm{Div} : \mathcal{H}_1 \to H$ are the *deformation* and the *divergence* operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

$$\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

We denote the norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$, $\gamma : H_1 \to H_\Gamma$ be the trace map and $\boldsymbol{\nu}$ be the outer unit normal on Γ. For every element $\boldsymbol{v} \in H_1$ we still write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ, and we denote by v_ν and $\boldsymbol{v}_\tau$ the *normal* and *tangential* components of $\boldsymbol{v}$ on the boundary Γ given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}.$$

Let H'_Γ be the dual of H_Γ and let $\langle\cdot,\cdot\rangle$ stand for the pairing between H'_Γ and H_Γ. For every $\boldsymbol{\sigma} \in \mathcal{H}_1$, there exists an element, denoted by $\boldsymbol{\sigma\nu} \in H'_\Gamma$, such that

$$\langle \boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}\rangle = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \boldsymbol{v})_H, \quad \forall \boldsymbol{v} \in H_1. \tag{1}$$

In addition, if $\boldsymbol{\sigma}$ is regular enough (e.g. of class C^1), we have

$$\langle \boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}\rangle = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,\mathrm{d}a, \quad \forall \boldsymbol{v} \in H_1. \tag{2}$$

Relations (1) and (2) imply the following Green formula:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,\mathrm{d}a, \quad \forall \boldsymbol{v} \in H_1. \tag{3}$$

In a similar manner, the *normal* and *tangential* components of $\boldsymbol{\sigma}$ are defined by

$$\sigma_\nu = (\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}.$$

Finally, for every real Hilbert space X we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \le p \le +\infty$, $k = 1, 2, \dots$, and we denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, respectively. We recall now an existence and uniqueness result concerning evolution problems, taken from (Barbu, 1976, p. 268).

Theorem 1. *Let V and H be two real Hilbert spaces such that $V \subset H$ and the inclusion mapping of V into H is continuous and densely defined. We suppose that V is endowed with the norm $\|\cdot\|$ induced by the inner product $(\cdot,\cdot)$ and H is endowed with the norm $|\cdot|$. We denote by V' the dual space of V, by $\langle\cdot,\cdot\rangle_{V'\times V}$ the duality pairing between an element of V and an element of V', and H is identified with its own dual H'. We assume that M is a maximal monotone set in $V \times V'$ and A is a linear, continuous and symmetric operator from V to V' satisfying the following coerciveness condition:*

$$\langle A\boldsymbol{u}, \boldsymbol{u}\rangle_{V'\times V} + \alpha|\boldsymbol{u}|^2 \ge \omega\|\boldsymbol{u}\|^2, \quad \forall\, \boldsymbol{u} \in V, \tag{4}$$

where $\alpha \in \mathbb{R}$ and $\omega > 0$. Let $\boldsymbol{g}$ be given in $W^{1,1}(0,T;H)$ and $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ be given with

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in D(M), \quad \{A\boldsymbol{u}_0 + M\boldsymbol{v}_0\} \cap H \neq \emptyset. \tag{5}$$

Then there exists a unique solution $\boldsymbol{u}$ to the following problem:

$$\begin{cases} \dfrac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2} + A\boldsymbol{u} + M\left(\dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right) \ni \boldsymbol{g}(t) \ \ \text{a.e. on } (0,T) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(0) = \boldsymbol{v}_0, \end{cases}$$

which satisfies

$$u \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H).$$

We use Theorem 1 in Section 3 to prove the existence and the uniqueness of the solution to the variational problem associated with our mechanical model.

3. Problem Statement. Existence and Uniqueness Result

In this section we describe the mechanical contact problem, derive its variational formulation and prove an existence and uniqueness result.

The physical setting is the following: We consider a body that occupies a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz continuous boundary divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that the measure of Γ_1, denoted by $|\Gamma_1|$, is positive. Let $T > 0$ and $[0,T]$ be the time interval of interest. Let $\rho : \Omega \to \mathbb{R}_+$ be the mass density of the body and $\boldsymbol{f}_0 : \Omega \times (0,T) \to \mathbb{R}^d$ the volume force density acting in $\Omega \times (0,T)$. The body is clamped on $\Gamma_1 \times (0,T)$ and therefore the displacement field vanishes there. A surface traction of density $\boldsymbol{f}_2 : \Gamma_2 \to \mathbb{R}^d$ assumed to be time-independent acts on Γ_2. On $\Gamma_3 \times (0,T)$ the body may come in contact with an obstacle, the so-called foundation, and we suppose that the contact condition may be described by a subdifferential-type inequality.

We denote by $\boldsymbol{u} = (u_i) : \Omega \times [0,T] \to \mathbb{R}^d$ the displacement field, by $\boldsymbol{\sigma} = (\sigma_{ij}) : \Omega \times [0,T] \to S_d$ the stress field, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ represents the linearized strain tensor. Moreover, dots above a function will represent the derivative with respect to the time variable, i.e. $\dot{\boldsymbol{u}} = \mathrm{d}\boldsymbol{u}/\mathrm{d}t$ or $\ddot{\boldsymbol{u}} = \mathrm{d}^2\boldsymbol{u}/\mathrm{d}t^2$.

We now describe the mechanical model for the process of frictional contact between the body and the obstacle. We use a Kelvin-Voigt constitutive law of the form

$$\boldsymbol{\sigma} = c\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}),$$

where $\mathcal{A}$ is the viscosity operator, $\mathcal{G} \equiv (g_{ijkh})$, the elasticity tensor, $c \geq 0$ is the viscosity coefficient. When c is positive, the body exhibits a viscoelastic behavior, while for $c = 0$ the body is elastic. We model the frictional contact with a subdifferential boundary condition on Γ_3 of the form

$$\boldsymbol{u} \in U, \quad \varphi(\boldsymbol{v}) - \varphi(\dot{\boldsymbol{u}}) \geq -\boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}), \quad \forall \boldsymbol{v} \in U,$$

where $U \subset H_1$ represents the set of contact admissible test functions, $\boldsymbol{\sigma\nu}$ denotes the Cauchy stress vector on the contact boundary and $\varphi : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ is a given function. The initial displacement field $\boldsymbol{u}_0$ and the initial velocity field $\boldsymbol{v}_0$ are given.

To summarize, the frictional mechanical problem can be formulated as follows.

Problem $\boldsymbol{P^c}$: *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$ *such that*

$$\rho\ddot{\boldsymbol{u}} = \operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in } \Omega \times (0,T), \tag{6}$$

$$\boldsymbol{\sigma} = c\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega \times (0,T), \tag{7}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{8}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{9}$$

$$\boldsymbol{u} \in U, \quad \varphi(\boldsymbol{v}) - \varphi(\dot{\boldsymbol{u}}) \geq -\boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}), \quad \forall \boldsymbol{v} \in U \quad \text{on } \Gamma_3 \times (0,T), \tag{10}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \quad \text{in } \Omega. \tag{11}$$

To obtain the variational formulation of Problem P^c, we consider the set

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = 0 \ \text{ on } \ \Gamma_1\} \cap U. \tag{12}$$

Let us define the functional $j : V \to \mathbb{R} \cup \{+\infty\}$ by

$$j(\boldsymbol{v}) = \begin{cases} \displaystyle\int_{\Gamma_3} \varphi(\boldsymbol{v})\,\mathrm{d}a & \text{if} \quad \varphi(\boldsymbol{v}) \in L^1(\Gamma_3), \\ +\infty & \text{otherwise.} \end{cases} \tag{13}$$

In the sequel, we suppose that:

$$V \text{ is a closed linear subspace in } H_1, \text{ is dense in } H \text{ and contains } \mathcal{D}(\Omega)^d; \tag{14}$$

$$j \text{ is a proper, convex and lower semicontinuous functional on } V. \tag{15}$$

Since $|\Gamma_1| > 0$, Korn's inequality implies that there exists a constant $C_K > 0$, depending only on Ω and Γ_1, such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq C_K \|\boldsymbol{v}\|_{H_1}, \quad \forall \boldsymbol{v} \in V. \tag{16}$$

A proof of Korn's inequality can be found in (Nečas and Hlavaček, 1981, p. 79). We consider the inner product on V given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{17}$$

and let $\|\cdot\|_V$ be the norm associated with the inner product (17), i.e.

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}, \quad \forall \boldsymbol{v} \in V. \tag{18}$$

From (16) it follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore, by (14), $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by combining Sobolev's trace theorem and (16), there exists a constant C_0, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le C_0 \|\boldsymbol{v}\|_V, \quad \forall \boldsymbol{v} \in V. \tag{19}$$

We suppose that the viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies the following conditions:

(i_1) $\mathcal{A}(x,\cdot)$ is monotone on S_d, i.e.

$$(\mathcal{A}(x,\boldsymbol{\tau}_1) - \mathcal{A}(x,\boldsymbol{\tau}_2)) \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) \ge 0, \qquad \forall\, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in S_d, \ \text{a.e. } x \in \Omega;$$

(i_2) there exist $r \in L^\infty(\Omega)$ and $s \in L^2(\Omega)$ such that

$$|\mathcal{A}(x,\boldsymbol{\tau})| \le r(x)|\boldsymbol{\tau}| + s(x), \qquad \forall\, \boldsymbol{\tau} \in S_d, \ \text{a.e. } x \in \Omega;$$

(i_3) $\mathcal{A}(x,\cdot)$ is continuous on S_d, a.e. $x \in \Omega$;

(i_4) $\mathcal{A}(\cdot,\boldsymbol{\tau})$ is Lebesgue measurable on Ω for all $\boldsymbol{\tau} \in S_d$.

The elasticity tensor $\mathcal{G} : \Omega \times S_d \to S_d$ is assumed to satisfy the usual properties of ellipticity and symmetry, i.e.

(j_1) there exists a constant $m_{\mathcal{G}} > 0$ such that

$$\mathcal{G}(x,\boldsymbol{\tau}) \cdot \boldsymbol{\tau} \ge m_{\mathcal{G}}\, |\boldsymbol{\tau}|^2, \quad \forall\, \boldsymbol{\tau} \in S_d, \ \text{a.e. } x \in \Omega;$$

(j_2) $\mathcal{G}(x,\boldsymbol{\tau})\cdot\boldsymbol{\sigma} = \boldsymbol{\tau}\cdot\mathcal{G}(x,\boldsymbol{\sigma})$, $\forall\, \boldsymbol{\tau}, \boldsymbol{\sigma} \in S_d$, a.e. $x \in \Omega$;

(j_3) $g_{ijkl} \in L^\infty(\Omega)$ for all i,j,k,l.

We suppose that the mass density satisfies

$$\rho \in L^\infty(\Omega) \text{ and there exists } \rho^* > 0 \text{ such that } \rho(x) \ge \rho^* \text{ a.e. } x \in \Omega. \tag{20}$$

In the sequel, we define a new inner product on H given by

$$((\boldsymbol{u},\boldsymbol{v}))_H = (\rho \boldsymbol{u},\boldsymbol{v})_H, \quad \forall \boldsymbol{u},\boldsymbol{v} \in H, \tag{21}$$

and let $|||\cdot|||_H$ be the associated norm, i.e.

$$|||\boldsymbol{v}|||_H = (\rho\,\boldsymbol{v},\boldsymbol{v})_H^{1/2}, \quad \forall\, \boldsymbol{v} \in H. \tag{22}$$

Using assumption (20), from (22) it follows that $|||\cdot|||_H$ and $\|\cdot\|_H$ are equivalent norms on H. Moreover, by (14), the inclusion mapping of $(V, \|\cdot\|_V)$ into $(H, |||\cdot|||_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write $V \subset H \subset V'$. We use the notation $\langle\cdot,\cdot\rangle_{V'\times V}$ to represent the duality pairing between V' and V. We have

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle_{V'\times V} = ((\boldsymbol{u},\boldsymbol{v}))_H, \quad \forall\, \boldsymbol{u} \in H, \quad \forall\, \boldsymbol{v} \in V. \tag{23}$$

We assume that the volume forces and tractions satisfy

$$\boldsymbol{f}_0 \in W^{1,1}(0,T;H) \text{ and } \boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{24}$$

Let us define the functional $J : V \to \mathbb{R} \cup \{+\infty\}$ by

$$J(\boldsymbol{v}) = j(\boldsymbol{v}) - \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, \mathrm{d}a, \quad \forall\, \boldsymbol{v} \in V. \tag{25}$$

We note that by (24) the integral in (25) is well defined.

We suppose that the initial data of Problem P^c satisfy

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in D(\partial J), \tag{26}$$

where ∂J denotes the subdifferential of J and $D(\partial J)$ represents its domain.

We also assume that there exists $\boldsymbol{h} \in H$ such that

$$(\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + c\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_0))_{\mathcal{H}} + J(\boldsymbol{v}) - J(\boldsymbol{v}_0) \ge ((\boldsymbol{h}, \boldsymbol{v} - \boldsymbol{v}_0))_H, \quad \forall\, \boldsymbol{v} \in V. \tag{27}$$

For instance, in the case when we have

$$(\boldsymbol{\sigma}_0^c, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_0))_{\mathcal{H}} + J(\boldsymbol{v}) - J(\boldsymbol{v}_0) \ge ((\boldsymbol{f}_0(0), \boldsymbol{v} - \boldsymbol{v}_0))_H, \quad \forall\, \boldsymbol{v} \in V,$$

with $\boldsymbol{\sigma}_0^c := \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + c\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0)$, the condition (27) is satisfied.

We turn now to derive a variational formulation for the mechanical problem P^c. To this end, let us fix $c \ge 0$. We suppose in the following that $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ are regular functions satisfying (6)–(11) and such that $\varphi(\dot{\boldsymbol{u}}) \in L^1(\Gamma_3)$. Let $\boldsymbol{w} \in V$ with $\varphi(\boldsymbol{w}) \in L^1(\Gamma_3)$ and $t \in [0,T]$. Applying (3) to $\boldsymbol{\sigma}$ for $\boldsymbol{v} = \boldsymbol{w} - \dot{\boldsymbol{u}}(t)$ and using (6), we get

$$(\rho\, \ddot{\boldsymbol{u}}(t) - \boldsymbol{f}_0(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_H + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} = \int_{\Gamma} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{w} - \dot{\boldsymbol{u}}(t))\, \mathrm{d}a.$$

Using (8), (9), (21) and (23), we obtain

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} = (\boldsymbol{f}_0(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_H + (\boldsymbol{f}_2, \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_{L^2(\Gamma_2)^d} + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{w} - \dot{\boldsymbol{u}}(t))\, \mathrm{d}a. \tag{28}$$

Combining (28), (10) and (13), we conclude that

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + j(\boldsymbol{w}) - j(\dot{\boldsymbol{u}}(t)) \ge (\boldsymbol{f}_0(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_H + (\boldsymbol{f}_2, \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_{L^2(\Gamma_2)^d}. \tag{29}$$

Taking into account (13), we observe that (29) remains true for all $\boldsymbol{w} \in V$. Consequently, combining (29) and (25), we deduce that

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + J(\boldsymbol{w}) - J(\dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}_0(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_H, \quad \forall\, \boldsymbol{w} \in V, \ \text{a.e. } t \in (0,T).$$

Therefore, keeping in mind (7) and (11), we are led to the following variational formulation of the mechanical problem P^c, for each $c \geq 0$:

Problem $\boldsymbol{P_V^c}$: *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}_1$ *such that*

$$\boldsymbol{\sigma}(t) = c\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \ \textit{a.e.}\ t \in (0,T), \tag{30}$$

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + J(\boldsymbol{w}) - J(\dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}_0(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_H, \quad \forall\, \boldsymbol{w} \in V, \ \textit{a.e. } t \in (0,T), \tag{31}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \ \textit{ in }\ \Omega. \tag{32}$$

We state now our existence and uniqueness result.

Theorem 2. *Assume that (14), (15),* (i_1)*–*(i_4)*,* (j_1)*–*(j_3)*, (20), (24), (26) and (27) hold. Then for each* $c \geq 0$ *there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *to Problem* P_V^c *such that*

$$\boldsymbol{u} \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H), \tag{33}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \mathrm{Div}\,\boldsymbol{\sigma} \in L^\infty(0,T;H). \tag{34}$$

We conclude that, under the assumptions of Theorem 2, Problem P^c has a unique weak solution $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ having the regularity (33), (34).

Proof. Let us fix $c \geq 0$. We consider the Hilbert spaces $H = L^2(\Omega)^d$ and V given by (12). We introduce the operator $A : V \to V'$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V'\times V} = (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{35}$$

Using (18), (j_2) and (j_3), we see that $A \in L(V, V')$, and (j_1) implies that A satisfies the condition (4) with $\alpha = 0$ and $\omega = m_{\mathcal{G}}$.

Define now the set-valued operator M_c: $V \to V'$ by

$$M_c = B_c + \partial J, \tag{36}$$

where $B_c : V \to V'$ is given by

$$\langle B_c\boldsymbol{u}, \boldsymbol{v}\rangle_{V'\times V} = c(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{37}$$

From (37) and (i_1), we have

$$\langle B_c\boldsymbol{u} - B_c\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle_{V'\times V} = c(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \geq 0, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

so the operator B_c is monotone. Using (37) and (18), we have

$$\|B_c\boldsymbol{u} - B_c\boldsymbol{v}\|_{V'} \leq c\|\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

and, keeping in mind (i_2), (i_3), (i_4) and Krasnoselski's theorem (see Kavian, 1993, p. 60), we find that $B_c : V \to V'$ is a continuous operator. Using again (37) and (i_2), we find that B_c is bounded.

From (15) and (25) we deduce that J is proper, convex and lower semicontinuous, which implies that ∂J is maximal monotone. Consequently, since B_c is monotone, bounded and hemicontinuous from V to V', we conclude (Barbu, 1976, p. 39) that $M_c = B_c + \partial J$ is maximal monotone.

Moreover, the initial data $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ satisfy (5) due to (26) and (27). Thus, all the requirements of Theorem 1, with A defined by (35), $M = M_c$ given in (36) and $\boldsymbol{g} = \boldsymbol{f}_0$, are satisfied. By defining $\boldsymbol{\sigma}$ by (30), it follows that there exists a unique solution $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ to Problem P_V^c satisfying (33).

It remains to show (34) for $\boldsymbol{\sigma}$. From (30), (i_2), (j_2), (j_3) and (33) it follows that $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$. Let $t \in [0,T]$ and $\boldsymbol{\psi} \in \mathcal{D}(\Omega)^d$. Since $J(\dot{\boldsymbol{u}}(t) \pm \boldsymbol{\psi}) = J(\dot{\boldsymbol{u}}(t)) < +\infty$, choosing $\boldsymbol{w} = \dot{\boldsymbol{u}}(t) \pm \boldsymbol{\psi} \in V$ (see (14)) in (31), using (3), (21) and (23), we obtain

$$\rho\ddot{\boldsymbol{u}} = \mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 \ \text{ in }\ H.$$

Now, taking into account (33) and (24), we arrive at $\mathrm{Div}\,\boldsymbol{\sigma} \in L^\infty(0,T;H)$, and thus (34) is satisfied. The uniqueness of the solution follows from Theorem 1. The proof of Theorem 2 is now complete. ■

4. Convergence as Viscosity Vanishes

In this section we investigate the behaviour of the solution to the viscoelastic problem P_V^c when the viscosity operator converges to zero and when the frictional potential function tends to the potential of the corresponding elastic problem. We suppose in the sequel that (14), (i_2)–(i_4), (j_1)–(j_3), (20), (24) hold and the following additional property is satisfied:

(i_5) $\mathcal{A}(x,\cdot)$ is strongly monotone on S_d, i.e. there exists $m_{\mathcal{A}} > 0$ such that

$$(\mathcal{A}(x,\boldsymbol{\tau}_1) - \mathcal{A}(x,\boldsymbol{\tau}_2)) \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) \geq m_{\mathcal{A}}|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2|^2, \quad \forall\, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in S_d, \ \text{a.e. } x \in \Omega.$$

We focus our attention on the convergence to frictional elasticity and the continuity with respect to the friction potential function. Thus, we consider a sequence of problems $P_V^{c_n}$ obtained from Problem P_V^c in which we set $c = c_n$, where (c_n) is a sequence of viscosity coefficients such that $c_n \to 0$ as $n \to \infty$; $\varphi = \varphi_n$, where $\varphi_n : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ are given functions; $\boldsymbol{u}_{n0}$ and $\boldsymbol{v}_{n0}$ stand for the initial displacements and velocities, respectively. We have the following variational problem, for each n:

Problem $\boldsymbol{P_V^{c_n}}$: *Find a displacement field $\boldsymbol{u}_n : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_n : [0,T] \to \mathcal{H}_1$ such that*

$$\boldsymbol{\sigma}_n(t) = c_n \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) \text{ a.e. } t \in (0,T), \quad (38)$$

$$\langle \ddot{\boldsymbol{u}}_n(t), \boldsymbol{w} - \dot{\boldsymbol{u}}_n(t)\rangle_{V' \times V} + (\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)))_{\mathcal{H}} + J_n(\boldsymbol{w}) - J_n(\dot{\boldsymbol{u}}_n(t)) \geq (\boldsymbol{f}_0(t), \boldsymbol{w} - \dot{\boldsymbol{u}}_n(t))_H, \quad \forall\, \boldsymbol{w} \in V, \text{ a.e. } t \in (0,T), \quad (39)$$

$$\boldsymbol{u}_n(0) = \boldsymbol{u}_{n0}, \quad \dot{\boldsymbol{u}}_n(0) = \boldsymbol{v}_{n0} \text{ in } \Omega. \quad (40)$$

Here J_n is defined by (25) for $j = j_n$, where j_n is given by (13) for $\varphi = \varphi_n$.

Next we consider the elastic problem P_V^0 obtained from P_V^c for $c = 0$ and the data $\varphi : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$, $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ given.

Problem $\boldsymbol{P_V^0}$: *Find a displacement field $\boldsymbol{u} : [0,T] \to V$ and a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}_1$ such that*

$$\boldsymbol{\sigma}(t) = \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \text{ a.e. } t \in (0,T), \quad (41)$$

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V' \times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + J(\boldsymbol{w}) - J(\dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}_0(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t))_H, \quad \forall\, \boldsymbol{w} \in V, \text{ a.e. } t \in (0,T), \quad (42)$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \text{ in } \Omega. \quad (43)$$

Here, the functionals J and j are defined by (25) and (13), respectively.

We assume that j satisfies (15);

$$j_n \text{ is a proper, convex and lower semicontinuous function on } V \text{ for all } n; \quad (44)$$

there exist $m \in [1,2)$ and $\alpha > 0$ such that, for all n,

$$|\varphi_n(x, \boldsymbol{y}) - \varphi(x, \boldsymbol{y})| \leq \alpha c_n |\boldsymbol{y}|^m, \quad \forall\, \boldsymbol{y} \in \mathbb{R}^d, \text{ a.e. } x \in \Gamma_3; \quad (45)$$

there exists $\boldsymbol{h} \in H$ such that

$$(\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_0))_{\mathcal{H}} + J(\boldsymbol{v}) - J(\boldsymbol{v}_0) \geq ((\boldsymbol{h}, \boldsymbol{v} - \boldsymbol{v}_0))_H, \quad \forall\, \boldsymbol{v} \in V; \quad (46)$$

for each n there exists $\boldsymbol{h}_n \in H$ such that

$$(\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n0}) + c_n \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{n0}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{n0}))_{\mathcal{H}} + J_n(\boldsymbol{v}) - J_n(\boldsymbol{v}_{n0}) \geq ((\boldsymbol{h}_n, \boldsymbol{v} - \boldsymbol{v}_{n0}))_H, \quad \forall\, \boldsymbol{v} \in V; \quad (47)$$

and $\boldsymbol{v}_0 \in D(\partial J)$, $\boldsymbol{v}_{n0} \in D(\partial J_n)$. Finally,

$$\boldsymbol{u}_{n0} \to \boldsymbol{u}_0 \text{ in } V, \quad \boldsymbol{v}_{n0} \to \boldsymbol{v}_0 \text{ in } H \text{ as } n \to \infty. \quad (48)$$

Let us remark that if we have, for all n, $\boldsymbol{v}_{n0} = \boldsymbol{v}_0$ and

$$(\boldsymbol{\sigma}_{n0}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_0))_{\mathcal{H}} + J_n(\boldsymbol{v}) - J_n(\boldsymbol{v}_0) \geq ((\boldsymbol{f}_0(0), \boldsymbol{v} - \boldsymbol{v}_0))_H, \quad \forall \boldsymbol{v} \in V,$$

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_0))_{\mathcal{H}} + J(\boldsymbol{v}) - J(\boldsymbol{v}_0) \geq ((\boldsymbol{f}_0(0), \boldsymbol{v} - \boldsymbol{v}_0))_H, \quad \forall \boldsymbol{v} \in V,$$

with $\boldsymbol{\sigma}_{n0} := \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n0}) + c_n \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{n0})$, $\boldsymbol{\sigma}_0 := \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)$, then the assumptions (46)–(48) are satisfied.

From Theorem 2 it follows that, for each n, Problem $P_V^{c_n}$ has a unique solution $\{\boldsymbol{u}_n, \boldsymbol{\sigma}_n\}$ with regularity $\boldsymbol{u}_n \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H)$, $\boldsymbol{\sigma}_n \in L^2(0,T;\mathcal{H})$, $\operatorname{Div} \boldsymbol{\sigma}_n \in L^\infty(0,T;H)$, and Problem P_V^0 has a unique solution $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ with regularity $\boldsymbol{u} \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H)$, $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$, $\operatorname{Div} \boldsymbol{\sigma} \in L^\infty(0,T;H)$.

We are now in a position to formulate our convergence result.

Theorem 3. *Let (c_n) be a sequence in $(0,+\infty)$ such that $c_n \to 0$ as $n \to \infty$. Suppose that (14), (15), (i_2)–(i_5), (j_1)–(j_3), (20), (24), (44)–(47) and denote by $\{\boldsymbol{u}_n, \boldsymbol{\sigma}_n\}$, $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ the unique solutions to Problems $P_V^{c_n}$ and P_V^0, respectively. Then there exists a constant $C > 0$, depending on $\boldsymbol{u}$ and on the data, but independent of n, such that for all n we have*

$$\|\boldsymbol{u}_n - \boldsymbol{u}\|_{C([0,T];V)} + \|\boldsymbol{u}_n - \boldsymbol{u}\|_{C^1([0,T];H)} + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}\|_{L^2(0,T;\mathcal{H})} \leq C(\|\boldsymbol{u}_{n0} - \boldsymbol{u}_0\|_V + \|\boldsymbol{v}_{n0} - \boldsymbol{v}_0\|_H + \sqrt{c_n}).$$

Consequently, if (48) holds, then as $n \to \infty$,

$$\boldsymbol{u}_n \to \boldsymbol{u} \text{ in } C([0,T];V) \cap C^1([0,T];H),$$

$$\boldsymbol{\sigma}_n \to \boldsymbol{\sigma} \text{ in } L^2(0,T;\mathcal{H}).$$

Proof. Let $t \in [0,T]$. Taking $\boldsymbol{w} = \dot{\boldsymbol{u}}(t)$ in (39) and using (38), we have

$$\begin{aligned}
&\langle \ddot{\boldsymbol{u}}_n(t), \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_n(t)\rangle_{V'\times V} \\
&\quad + c_n(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)))_{\mathcal{H}} \\
&\quad + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)))_{\mathcal{H}} \\
&\quad + j_n(\dot{\boldsymbol{u}}(t)) - j_n(\dot{\boldsymbol{u}}_n(t)) \\
&\geq (\boldsymbol{f}_0(t), \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_n(t))_H \\
&\quad + (\boldsymbol{f}_2, \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_n(t))_{L^2(\Gamma_2)^d},
\end{aligned}$$

and taking $\boldsymbol{w} = \dot{\boldsymbol{u}}_n(t)$ in (42), using (41), we obtain

$$\begin{aligned}
&\langle \ddot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}_n(t) - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V} \\
&\quad + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} \\
&\quad + j(\dot{\boldsymbol{u}}_n(t)) - j(\dot{\boldsymbol{u}}(t)) \\
&\geq (\boldsymbol{f}_0(t), \dot{\boldsymbol{u}}_n(t) - \dot{\boldsymbol{u}}(t))_H \\
&\quad + (\boldsymbol{f}_2, \dot{\boldsymbol{u}}_n(t) - \dot{\boldsymbol{u}}(t))_{L^2(\Gamma_2)^d}.
\end{aligned}$$

Adding the last two inequalities, we deduce that for each $t \in [0,T]$,

$$\begin{aligned}
&\langle \ddot{\boldsymbol{u}}_n(t) - \ddot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}_n(t)\rangle_{V'\times V} \\
&\quad + c_n(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)))_{\mathcal{H}} \\
&\quad + c_n(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)))_{\mathcal{H}} \\
&\quad + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t)))_{\mathcal{H}} \\
&\quad + j_n(\dot{\boldsymbol{u}}(t)) - j_n(\dot{\boldsymbol{u}}_n(t)) + j(\dot{\boldsymbol{u}}_n(t)) - j(\dot{\boldsymbol{u}}(t)) \geq 0.
\end{aligned}$$

Integrating this inequality on $[0,t]$ and using (23), (j_2), (i_5), (18), (40), (43), we conclude that

$$\begin{aligned}
&\frac{1}{2}|||\dot{\boldsymbol{u}}_n(t) - \dot{\boldsymbol{u}}(t)|||_H^2 + c_n m_{\mathcal{A}}\int_0^t \|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)\|_V^2\,\mathrm{d}s \\
&\quad + \frac{1}{2}(\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\
&\leq \frac{1}{2}|||\boldsymbol{v}_{n0} - \boldsymbol{v}_0|||_H^2 \\
&\quad + c_n\int_0^t \|\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s))\|_{\mathcal{H}}\|\dot{\boldsymbol{u}}(s) - \dot{\boldsymbol{u}}_n(s)\|_V\,\mathrm{d}s \\
&\quad + \frac{1}{2}(\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{n0}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_0)), \boldsymbol{\varepsilon}(\boldsymbol{u}_{n0}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_0))_{\mathcal{H}} \\
&\quad + \int_0^t |j_n(\dot{\boldsymbol{u}}(s)) - j_n(\dot{\boldsymbol{u}}_n(s)) + j(\dot{\boldsymbol{u}}_n(s)) - j(\dot{\boldsymbol{u}}(s))|\,\mathrm{d}s.
\end{aligned}$$

Using (j_1), (j_3) and (18), and since by (i_2), (i_3) and (i_4) we have $\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) \in L^2(0,T;\mathcal{H})$, we deduce that

$$\begin{aligned}
&\frac{1}{2}|||\dot{\boldsymbol{u}}_n(t) - \dot{\boldsymbol{u}}(t)|||_H^2 \\
&\quad + c_n m_{\mathcal{A}}\int_0^t \|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)\|_V^2\,\mathrm{d}s \\
&\quad + \frac{m_{\mathcal{G}}}{2}\|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V^2 \\
&\leq \frac{1}{2}|||\boldsymbol{v}_{n0} - \boldsymbol{v}_0|||_H^2 + \frac{c_n}{2m_{\mathcal{A}}}\int_0^t \|\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s))\|_{\mathcal{H}}^2\,\mathrm{d}s \\
&\quad + \frac{c_n m_{\mathcal{A}}}{2}\int_0^t \|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)\|_V^2\,\mathrm{d}s \\
&\quad + \frac{C}{2}\|\boldsymbol{u}_{n0} - \boldsymbol{u}_0\|_V^2 + \int_0^t (|j_n(\dot{\boldsymbol{u}}(s)) - j(\dot{\boldsymbol{u}}(s))| \\
&\quad + |j_n(\dot{\boldsymbol{u}}_n(s)) - j(\dot{\boldsymbol{u}}_n(s))|)\,\mathrm{d}s, \qquad (49)
\end{aligned}$$

where C is a positive constant independent of n and may change from line to line. From the fact that $\varphi_n(\dot{\boldsymbol{u}}_n(s)) \in L^1(\Gamma_3)$, $\varphi(\dot{\boldsymbol{u}}(s)) \in L^1(\Gamma_3)$, by using (45), it follows that $\varphi_n(\dot{\boldsymbol{u}}(s)) \in L^1(\Gamma_3)$, $\varphi(\dot{\boldsymbol{u}}_n(s)) \in L^1(\Gamma_3)$. Consequently, using (45), Young's inequality and (19), we can write, for all $s \in [0,T]$,

$$\begin{aligned}
&|j_n(\dot{\boldsymbol{u}}(s)) - j(\dot{\boldsymbol{u}}(s))| + |j_n(\dot{\boldsymbol{u}}_n(s)) - j(\dot{\boldsymbol{u}}_n(s))| \\
&\leq \alpha c_n\int_{\Gamma_3} |\dot{\boldsymbol{u}}(s)|^m\,\mathrm{d}a + \alpha c_n\int_{\Gamma_3} |\dot{\boldsymbol{u}}_n(s)|^m\,\mathrm{d}a \\
&\leq \alpha(1 + 2^{m-1})c_n\int_{\Gamma_3} |\dot{\boldsymbol{u}}(s)|^m\,\mathrm{d}a \\
&\quad + \alpha 2^{m-1}c_n\int_{\Gamma_3} |\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)|^m\,\mathrm{d}a \\
&\leq \alpha(1 + 2^{m-1})\,c_n\int_{\Gamma_3} |\dot{\boldsymbol{u}}(s)|^m\,\mathrm{d}a \\
&\quad + c_n\int_{\Gamma_3}\Big[\frac{2-m}{2}\Big(\frac{\alpha 2^{m-1}}{\tau}\Big)^{\frac{2}{2-m}} \\
&\quad + \frac{m}{2}\,\tau^{\frac{2}{m}}|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)|^2\Big]\,\mathrm{d}a \\
&\leq C(c_n\|\dot{\boldsymbol{u}}(s)\|_{L^2(\Gamma_3)^d}^m + c_n) \\
&\quad + c_n\frac{m}{2}\,\tau^{\frac{2}{m}}\|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)\|_{L^2(\Gamma_3)^d}^2 \\
&\leq C(c_n C_0^m\|\dot{\boldsymbol{u}}(s)\|_V^m \\
&\quad + c_n) + c_n\frac{m}{2}\,\tau^{\frac{2}{m}}C_0^2\|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)\|_V^2,
\end{aligned}$$

where $\tau > 0$ is a constant that will be chosen below.

Integrating this inequality on $[0,t]$ and using Hölder's inequality, we obtain

$$\int_0^t (|j_n(\dot{\boldsymbol{u}}(s)) - j(\dot{\boldsymbol{u}}(s))| + |j_n(\dot{\boldsymbol{u}}_n(s)) - j(\dot{\boldsymbol{u}}_n(s))|)\,\mathrm{d}s$$

$$\le Cc_n + c_n \frac{m}{2}\tau^{\frac{2}{m}} C_0^2 \int_0^t \|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)\|_V^2\,\mathrm{d}s$$

$$\le Cc_n + c_n \frac{m_{\mathcal{A}}}{3}\int_0^t \|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)\|_V^2\,\mathrm{d}s, \qquad (50)$$

for τ chosen such that $(m/2)\tau^{2/m}C_0^2 \le m_{\mathcal{A}}/3$. Combining (49) with (50) and using the equivalence of the norms $|||\cdot|||_H$ and $\|\cdot\|_H$ on H, we infer that

$$\|\dot{\boldsymbol{u}}_n(t) - \dot{\boldsymbol{u}}(t)\|_H^2 + c_n \int_0^t \|\dot{\boldsymbol{u}}_n(s) - \dot{\boldsymbol{u}}(s)\|_V^2\,\mathrm{d}s$$

$$+ \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V^2$$

$$\le C(\|\boldsymbol{v}_{n0} - \boldsymbol{v}_0\|_H^2 + \|\boldsymbol{u}_{n0} - \boldsymbol{u}_0\|_V^2 + c_n).$$

From the continuity of the embedding $V \subset H$, it follows that

$$\|\boldsymbol{u}_n - \boldsymbol{u}\|_{C([0,T];V)}^2 + \|\boldsymbol{u}_n - \boldsymbol{u}\|_{C^1([0,T];H)}^2$$

$$+ c_n \|\dot{\boldsymbol{u}}_n - \dot{\boldsymbol{u}}\|_{L^2(0,T;V)}^2$$

$$\le C(\|\boldsymbol{v}_{n0} - \boldsymbol{v}_0\|_H^2 + \|\boldsymbol{u}_{n0} - \boldsymbol{u}_0\|_V^2 + c_n). \qquad (51)$$

It remains to prove that for all n we have

$$\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}\|_{L^2(0,T;\mathcal{H})}$$

$$\le C(\|\boldsymbol{u}_{n0} - \boldsymbol{u}_0\|_V + \|\boldsymbol{v}_{n0} - \boldsymbol{v}_0\|_H + \sqrt{c_n}). \qquad (52)$$

From (38) and (41) we deduce that

$$\|\boldsymbol{\sigma}_n(t) - \boldsymbol{\sigma}(t)\|_{\mathcal{H}}^2 \le 2c_n^2 \|\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n(t))\|_{\mathcal{H}}^2$$

$$+ 2\|\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|_{\mathcal{H}}^2,$$

a.e. $t \in (0,T)$. Condition (i_2) implies that there exist two positive constants δ_1 and δ_2 such that $\|\mathcal{A}\boldsymbol{\tau}\|_{\mathcal{H}}^2 \le \delta_1\|\boldsymbol{\tau}\|_{\mathcal{H}}^2 + \delta_2$, $\forall \boldsymbol{\tau} \in \mathcal{H}$. It follows that

$$\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}\|_{L^2(0,T;\mathcal{H})}^2$$

$$\le 2c_n^2\delta_1 \|\dot{\boldsymbol{u}}_n\|_{L^2(0,T;V)}^2 + 2T\delta_2 c_n^2$$

$$+ 2\int_0^T \|\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) - \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|_{\mathcal{H}}^2\,\mathrm{d}t.$$

Using again (j_3) and (18), we deduce that

$$\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}\|_{L^2(0,T;\mathcal{H})}^2$$

$$\le 4c_n^2\delta_1\|\dot{\boldsymbol{u}}_n - \dot{\boldsymbol{u}}\|_{L^2(0,T;V)}^2$$

$$+ 4c_n^2\delta_1\|\dot{\boldsymbol{u}}\|_{L^2(0,T;V)}^2 + 2T\delta_2 c_n^2$$

$$+ 2C\int_0^T \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V^2\,\mathrm{d}t.$$

Keeping in mind (51), we arrive at

$$\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}\|_{L^2(0,T;\mathcal{H})}^2 \le C(c_n+1)(\|\boldsymbol{u}_{n0} - \boldsymbol{u}_0\|_V^2$$

$$+ \|\boldsymbol{v}_{n0} - \boldsymbol{v}_0\|_H^2 + c_n) + Cc_n^2,$$

so, since $c_n \to 0$ as $n \to \infty$, we obtain (52). This completes the proof. ■

We conclude by Theorem 3 that the weak solution to the frictional elastic problem P^0 can be approached by the weak solution to the frictional viscoelastic problem P^{c_n} when the coefficient of viscosity is small enough and the corresponding friction potential functions satisfy (45). In addition to the mathematical interest in the convergence properties proved in Theorem 3, this is of importance from the mechanical point of view, because the frictional elasticity appears as a limit case of frictional viscoelasticity.

Remark 1. A similar argument to the one used in the proof of Theorem 3 shows that Theorem 3 remains true when we replace the condition (45) with the following property: there exist an integer $p \ge 1$ and numbers $m_1,\dots,m_p \in [1,2)$, $\alpha_1,\dots,\alpha_p > 0$ such that for every n one has

$$|\varphi_n(x,\boldsymbol{y}) - \varphi(x,\boldsymbol{y})| \le c_n \sum_{i=1}^p \alpha_i |\boldsymbol{y}|^{m_i},$$

$$\forall \boldsymbol{y} \in \mathbb{R}^d \text{ a.e. } x \in \Gamma_3. \qquad (53)$$

5. Examples of Subdifferential Contact Condition

In this section we present some examples of contact friction laws which lead to an inequality of the form (10) and for which (14) and (15) hold. Using Theorem 2, we conclude that the boundary value problem P_V^c corresponding to each of the examples below has a unique weak solution. Moreover, the convergence result given in Theorem 3 is applicable in all the concrete examples below.

Example 1. We consider the bilateral contact with Tresca's friction law. This contact condition can be found in (Duvaut and Lions, 1976; Panagiotopoulos, 1985) and, more recently, in (Amassad *et al.*, (1999); Chau *et al.*, 2001b). We use the following boundary condition:

$$\begin{cases} u_\nu = 0, \quad |\boldsymbol{\sigma}_\tau| \le g, \\ |\boldsymbol{\sigma}_\tau| < g \implies \dot{\boldsymbol{u}}_\tau = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau| = g \implies \dot{\boldsymbol{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau, \text{ for some } \lambda \ge 0, \end{cases}$$

$$\text{on } \Gamma_3 \times (0, T).$$

Here g represents the friction bound, i.e. the magnitude of the limiting friction traction at which slip begins. We assume that $g \in L^\infty(\Gamma_3)$, $g \ge 0$ a.e. on Γ_3.

We set $U = \{\boldsymbol{v} \in H_1 \,|\, v_\nu = 0 \text{ on } \Gamma_3\}$ and deduce from (12) that

$$V = \{\boldsymbol{v} \in H_1 \,|\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3\}.$$

We see that (10) holds with the choice

$$\varphi(x, \boldsymbol{y}) = g(x) |\boldsymbol{y}_{\tau(x)}|, \quad \forall\, x \in \Gamma_3, \quad \boldsymbol{y} \in \mathbb{R}^d,$$

where $\boldsymbol{y}_{\tau(x)} := \boldsymbol{y} - y_{\nu(x)} \boldsymbol{\nu}(x)$, $y_{\nu(x)} := \boldsymbol{y} \cdot \boldsymbol{\nu}(x)$, with $\boldsymbol{\nu}(x)$ as the unit normal at $x \in \Gamma_3$. In the convergence result, denote by g the friction bound for the elastic problem P_V^0 and by g_n the one for the viscoelastic problem $P_V^{c_n}$. If the functions $g_n, g \in L^\infty(\Gamma_3)$ satisfy, for all n,

$$\|g_n - g\|_{L^\infty(\Gamma_3)} \le \alpha c_n,$$

for some $\alpha > 0$, the assumption (45) is satisfied with $m = 1$. ♦

Example 2. We model the bilateral contact with a viscous friction condition defined by a tangential damped response. We use the following boundary condition:

$$u_\nu = 0, \quad \boldsymbol{\sigma}_\tau = -\mu |\dot{\boldsymbol{u}}_\tau|^{q-1} \dot{\boldsymbol{u}}_\tau \ \text{ on } \Gamma_3 \times (0, T),$$

where $0 < q < 1$ and $\mu \in L^\infty(\Gamma_3)$, $\mu \ge 0$ a.e. on Γ_3.

We let $U = \{\boldsymbol{v} \in H_1 \,|\, v_\nu = 0 \text{ on } \Gamma_3\}$, and then

$$V = \{\boldsymbol{v} \in H_1 \,|\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3\}.$$

Thus (10) is satisfied with

$$\varphi(x, \boldsymbol{y}) = \frac{1}{q+1} \mu(x) \,|\boldsymbol{y}_{\tau(x)}|^{q+1}, \quad \forall\, x \in \Gamma_3, \quad \boldsymbol{y} \in \mathbb{R}^d.$$

In the convergence result denote by μ the coefficient of friction for the elastic problem P_V^0 and by μ_n the one for the viscoelastic problem $P_V^{c_n}$. If the positive functions $\mu_n, \mu \in L^\infty(\Gamma_3)$ satisfy, for all n,

$$\|\mu_n - \mu\|_{L^\infty(\Gamma_3)} \le \alpha (q+1) c_n,$$

for some $\alpha > 0$, the assumption (45) is satisfied with $m = q + 1$. ♦

Example 3. We consider a model of damped response contact with Tresca's friction law (see, e.g., Jarušek and Eck, 1999; Rochdi and Shillor, 2001c). We use the following boundary condition:

$$\begin{cases} -\sigma_\nu = k |\dot{u}_\nu|^{q-1} \dot{u}_\nu, \quad |\boldsymbol{\sigma}_\tau| \le g, \\ |\boldsymbol{\sigma}_\tau| < g \implies \dot{\boldsymbol{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = g \implies \dot{\boldsymbol{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau, \quad \text{for some } \lambda \ge 0, \end{cases}$$

on $\Gamma_3 \times (0, T)$. Here $0 < q < 1$ and $g, k \in L^\infty(\Gamma_3)$, $g, k \ge 0$.

We have $U = H^1(\Omega)^d$ and

$$V = \{\boldsymbol{v} \in H^1(\Omega)^d \,|\, \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1\}.$$

Thus (10) is satisfied with

$$\varphi(x, \boldsymbol{y}) = \frac{1}{q+1}\, k(x)\, |y_{\nu(x)}|^{q+1} + g(x) |\boldsymbol{y}_\tau(x)|,$$

$$\forall x \in \Gamma_3, \quad \boldsymbol{y} \in \mathbb{R}^d.$$

In the convergence result denote by g, k the parameters for the elastic problem P_V^0 and by g_n, k_n those for the viscoelastic problem $P_V^{c_n}$. If the positive functions $g_n, k_n, g, k \in L^\infty(\Gamma_3)$ satisfy, for all n,

$$\|k_n - k\|_{L^\infty(\Gamma_3)} \le \alpha_1 (q+1) c_n,$$

$$\|g_n - g\|_{L^\infty(\Gamma_3)} \le \alpha_2 c_n,$$

for some $\alpha_1, \alpha_2 > 0$, then (53) is satisfied with $p = 2$, $m_1 = q + 1$, $m_2 = 1$. ♦

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