Int. J. Appl. Math. Comput. Sci., 2002, Vol.12, No.1, 127–137

ANALYSIS OF OPTIMALITY CONDITIONS FOR SOME TOPOLOGY OPTIMIZATION PROBLEMS IN ELASTICITY

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The subject of topology optimization has undergone an enormous practical development since the appearance of the paper by Bendsøe and Kikuchi (1988), where some ideas from homogenization theory were put into practice. Since then, several engineering applications as well as different approaches have been developed successfully. However, it is difficult to find in the literature some analytical examples that might be used as a test in order to assess the validity of the solutions obtained with different algorithms. As a matter of fact, one is often faced with numerical instabilities requiring a fine tuning of the algorithm for each specific case. In this work, we develop a family of analytical solutions for very simple topology optimization problems, in the framework of elasticity theory, including bending and extension of rods, torsion problems as well as plane stress and plane strain elasticity problems. All of these problems are formulated in a simplified theoretical framework. A key issue in this type of problems is to be able to evaluate the sensitivity of the homogenized elastic coefficients with respect to the microstructure parameter(s). Since we are looking for analytical solutions, we use laminates for which an explicit dependence of the homogenized coefficients on the microstructure is known.

Keywords: topology optimization, optimality conditions, elasticity

1. Abstract Setting

Those readers who are familiar with the type of problems under consideration may wish to skip this section and go directly to the discussion examples, starting in the next section.

The type of problems we wish to address can be described by the following abstract setting: Let $\mathbf{V}$ be a Hilbert space (state space), Λ be a Banach space and Φ be an open subset of Λ (control space). We are given the functionals:

- $$a: \quad \Phi \times \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{R}, \quad (\varphi, u, v) \longmapsto a(\varphi; u, v)$$
- $$l: \quad \Phi \times \mathbf{V} \longrightarrow \mathbb{R}, \quad (\varphi, v) \longmapsto l(\varphi; v)$$
- $$J: \quad \Phi \times \mathbf{V} \longrightarrow \mathbb{R}, \quad (\varphi, v) \longmapsto J(\varphi; v).$$

For each φ, $a(\varphi; \cdot, \cdot)$ is supposed to be bilinear, continuous, symmetric, coercive in u and v; l is supposed to be linear and continuous in v. Both are supposed to be of class $\mathcal{C}^1$ with respect to φ, in the *spaces of continuous bilinear functionals* and *continuous linear functionals*, respectively. As for J, it is supposed to be of class $\mathcal{C}^1$ with respect to the pair (φ, v).

The problem we are studying is the following: We consider $u^\varphi \in \mathbf{V}$ uniquely defined by the state equation

$$a(\varphi; u^\varphi, v) = l(\varphi; v), \quad \forall v \in \mathbf{V},$$

and $j(\varphi) = J(\varphi, u^\varphi)$. We wish to compute

$$\min_{\varphi \in \Phi} j(\varphi).$$

More precisely, in order to use descent-type methods or to compute the necessary stationarity conditions, we want to differentiate $j(\varphi)$ with respect to φ. We have the following classical result (cf. Chenais, 1987):

Theorem 1. *Under the above conditions, the functions $\varphi \longmapsto u^\varphi$ and $\varphi \longmapsto j(\varphi)$ from Φ into $\mathbf{V}$ and $\mathbb{R}$, respectively, are of class $\mathcal{C}^1$. Moreover, for any $\delta\varphi \in \Lambda$ we have*

$$\frac{\mathrm{d}j}{\mathrm{d}\varphi}(\varphi)\delta\varphi = \frac{\partial J}{\partial \varphi}(\varphi; u^\varphi)\delta\varphi - \frac{\partial a}{\partial \varphi}(\varphi; u^\varphi, p^\varphi)\delta\varphi + \frac{\partial l}{\partial \varphi}(\varphi; p^\varphi)\delta\varphi, \tag{1}$$

where p^φ is the adjoint state variable, which is given as the unique solution of the equation:

$$p^\varphi \in \mathbf{V}, \quad a(\varphi; w, p^\varphi) = \frac{\partial J}{\partial v}(\varphi; u^\varphi)w, \quad \forall w \in \mathbf{V}. \tag{2}$$

This result can be generalized to the case when one has several control variables $\varphi_1, \varphi_2, \ldots, \varphi_m$.

2. One-Dimensional Examples—Extension and Bending of a Rod

Consider a rod occupying the interval $]0, L[\subset \mathbb{R}$, of a constant cross section area A, fixed at one of the ends, e.g. at $x = 0$, and subjected to a load F applied to the free end $x = L$.

Let us assume that the material the rod is made of possesses a microstructure formed by the combination of two layered materials, which are supposed to be homogeneous and isotropic of Young's moduli E^+ and E^-, in proportions γ and $(1-\gamma)$, respectively, with

$$0 \le \gamma \le 1. \tag{3}$$

We denote by ρ^+ and ρ^- the specific masses of these materials, respectively. For simplicity, we will assume that the materials are well ordered, i.e.

$$E^+ > E^-, \quad \rho^+ > \rho^-, \tag{4}$$

and that the macroscopic properties are obtained via homogenization theory.

In topology optimization problems one usually considers the problem of minimizing the work of the applied forces, i.e. $Fu(L)$, with respect to the volumic fraction γ, where u stands for the axial displacement of the rod (Bendsøe, 1995; Bendsøe and Kikuchi, 1988).

The solution of this minimization problem is simple and given by $\gamma = 1$. As a result, the whole rod will be made of the strongest material of elasticity modulus E^+. However, it may happen that the cost of adding the stiffest material will be extremely high and, as a consequence, in some parts of the rod, or in all of it, it has to be made of a material with Young's modulus E^- in a proportion to be determined. Moreover, there is no reason to expect that the value of γ in a certain cross section is the same as the value of γ in another cross section. That is, one may have $\gamma = \gamma(x)$.

Let us consider the problem of minimizing the functional

$$j(\gamma) = F\,u(L) + k\int_0^L \rho^H\,A\,\mathrm{d}x, \tag{5}$$

where the constant k, whose units are work/mass, represents the work done in order to add a unit of mass to the rod. The quantity ρ^H represents the macroscopic (homogenized) specific mass of the rod and is a function of the volumic fraction γ.

In order for the structure to be in equilibrium, the displacement field u must satisfy the equilibrium equations, which we write down in variational form, i.e. $u \in V = \{v \in H^1(0,L) : v(0) = 0\}$:

$$\underbrace{\int_0^L E^H\,A\,u'\,v'\,\mathrm{d}x}_{a(\gamma;u^\gamma,v)} - \underbrace{F\,v(L)}_{l(\gamma;v)} = 0, \quad \forall v \in V, \tag{6}$$

where E^H represents the macroscopic (homogenized) Young modulus and primes stand for the derivatives with respect to the axial variable x.

Summarizing, the topology optimization problem under consideration consists in the minimization of functional (5), with respect to function γ, subject to conditions (3) and (6) and, for the present case, for materials verifying (4). This problem fits into the presented abstract setting.

In order to solve this problem, one may apply Theorem 1 or, equivalently, consider the Lagrangian

$$\begin{aligned}\mathcal{L} &= F\,u(L) + k\int_0^L \rho^H\,A\,\mathrm{d}x \\ &\quad + \lambda\left[\int_0^L E^H\,A\,u'\,v'\,\mathrm{d}x - F\,v(L)\right] \\ &\quad + \int_0^L \tau^+\,(\gamma-1)\,\mathrm{d}x - \int_0^L \tau^-\,\gamma\,\mathrm{d}x,\end{aligned} \tag{7}$$

where λ, τ^+ and τ^- are the Lagrange multipliers associated with constraints (6), $\gamma \le 1$ and $0 \le \gamma$, respectively.

From the necessary conditions of stationarity, we obtain, besides (6) and for $x \in]0, L[$:

$$v = -\frac{1}{\lambda}\,u,$$

$$k\,\frac{\mathrm{d}\rho^H}{\mathrm{d}\gamma}\,A - \frac{\mathrm{d}E^H}{\mathrm{d}\gamma}A|u'|^2 + \tau^+ - \tau^- = 0, \tag{8}$$

$$\tau^+ \ge 0, \tag{9}$$

$$\tau^- \ge 0, \tag{10}$$

$$\tau^+\,(\gamma - 1) = 0, \tag{11}$$

$$\tau^-\,\gamma = 0. \tag{12}$$

We remark that (8) is exactly what we obtain applying (1) and (2). So far, we have not specified the type of microstructure the bar is made of. Let us consider the simplest case of a laminated microstructure made of two materials with Young's moduli E^+ and E^- oriented along a system of axes Oy_1 parallel to Ox_1 and Oy_2 parallel to Ox_2, in proportions γ and $(1-\gamma)$, respectively. From homogenization theory (Bendsøe, 1995), we get

$$\rho^H = \gamma\,\rho^+ + (1-\gamma)\,\rho^-, \qquad \frac{1}{E^H} = \frac{\gamma}{E^+} + \frac{1-\gamma}{E^-}.$$

Therefore (8) becomes

$$k\,(\rho^+ - \rho^-)\,A - \frac{E^+\,E^-\,(E^+ - E^-)}{[\gamma\,E^- + (1-\gamma)\,E^+]^2}\,|u'|^2\,A + \tau^+ - \tau^- = 0.$$

On the other hand, from the differential expression of the equilibrium equation (6), one has $(E^H\ A\ u')^2 = F^2$, which, after substitution in the previous equation, leads to

$$\underbrace{k\ (\rho^+ - \rho^-)\ A - F^2\ \frac{(E^+ - E^-)}{A\ E^+\ E^-}}_{\chi} = -\tau^+ + \tau^-. \quad (13)$$

Together with (9)–(12), this is the equation that leads to the solution to the problem. As a matter of fact, if the cost k is low, one has $\chi < 0$ and, as a consequence, $\tau^- = 0$, $\tau^+ > 0$, which implies, from (11), that $\gamma = 1$. In other words, if the cost of adding material is low, the rod will be made of the hardest material with Young's modulus E^+ (usually the more expensive one). On the other hand, if the cost of adding material is high, one gets $\chi > 0$ and, consequently, $\tau^- > 0$, $\tau^+ = 0$, which implies, from (12), that $\gamma = 0$. That is, the rod is made of the weakest material of Young's modulus E^-.

Summarizing, we have

$$\begin{aligned} \chi &> 0 \Longrightarrow \tau^+ = 0, \tau^- > 0 \Longrightarrow \gamma = 0 \\ &\Longrightarrow E^H = E^-, \end{aligned} \quad (14)$$

$$\begin{aligned} \chi &< 0 \Longrightarrow \tau^+ > 0, \tau^- = 0 \Longrightarrow \gamma = 1 \\ &\Longrightarrow E^H = E^+, \end{aligned} \quad (15)$$

$$\begin{aligned} \chi &= 0 \Longrightarrow \tau^+ = 0, \tau^- = 0 \Longrightarrow 0 \le \gamma \le 1 \\ &\Longrightarrow E^H = \frac{1}{\dfrac{\gamma}{E^+} + \dfrac{1-\gamma}{E^-}}, \end{aligned} \quad (16)$$

or, in an equivalent manner,

$$\begin{aligned} F^2 &< k\ (\rho^+ - \rho^-)\ A^2\ \frac{E^+\ E^-}{E^+ - E^-} \\ &\Longrightarrow \gamma = 0 \Longrightarrow E^H = E^-, \end{aligned} \quad (17)$$

$$\begin{aligned} F^2 &> k\ (\rho^+ - \rho^-)\ A^2\ \frac{E^+\ E^-}{E^+ - E^-} \\ &\Longrightarrow \gamma = 1 \Longrightarrow E^H = E^+, \end{aligned} \quad (18)$$

$$\begin{aligned} F^2 &= k\ (\rho^+ - \rho^-)\ A^2\ \frac{E^+\ E^-}{E^+ - E^-} \\ &\Longrightarrow 0 \le \gamma \le 1 \Longrightarrow E^H = \frac{1}{\dfrac{\gamma}{E^+} + \dfrac{1-\gamma}{E^-}}. \end{aligned} \quad (19)$$

From the above equations, we conclude that when the applied force is such that

$$F^2 = k\ (\rho^+ - \rho^-)\ A^2\ \frac{E^+\ E^-}{E^+ - E^-}, \quad (20)$$

the uniqueness is lost and the rod may be formed by any combination of the two materials. This may be a source of difficulties if one tries to solve these types of problems numerically. Namely, some oscillation in the values of γ and, consequentely, in the values of the displacement field may be found from one iteration to another and in adjacent finite elements. This fact means that for this value of the applied force, the minimizing functional is independent of γ. As a matter of fact, for this case we have

$$j(\gamma) = \frac{F^2}{A}\ \frac{1}{E^+\ E^-}\ \frac{(E^+\ \rho^+ - E^-\ \rho^-)}{\rho^+ - \rho^-}\ L.$$

In conclusion, we see that as the force increases, keeping the cost of adding material fixed (or as the cost diminishes keeping the force F constant), the solution to the problem is given by a rod whose properties are (E^-, ρ^-) until the relationship (20) is reached, where a multiplicity of solutions exists, characterized by $0 \le \gamma \le 1$. Increasing the force even further one obtains a rod made of a material with properties (E^+, ρ^+).

Let us now look at some other examples where the same type of phenomena may occur. Let us consider, in the first place, the case when the rod is also subjected to a distributed force f over its domain $]0, L[$. The functional to minimize is now

$$j(\gamma) = F\ u(L) + \int_0^L f\ u\,\mathrm{d}x + k\ \int_0^L \rho^H\ A\,\mathrm{d}x. \quad (21)$$

The equilibrium equation, written in its variational form, is given by $u \in V = \{v \in H^1(0, L) : v(0) = 0\}$:

$$\int_0^L E^H\ A\ u'\ v'\,\mathrm{d}x - \int_0^L f\ v\,\mathrm{d}x - F\ v(L) = 0, \quad \forall v \in V. \quad (22)$$

As before, from (1) and (2) we obtain

$$\begin{aligned} &k\ (\rho^+ - \rho^-)\ A \\ &- \left[F + \int_x^L f(s)\,\mathrm{d}s\right]^2 \frac{E^+ - E^-}{A\ E^+\ E^-} + \tau^+ - \tau^- = 0, \end{aligned} \quad (23)$$

together with (9)–(12). As a result,

$$\begin{aligned} &\left[F + \int_x^L f(s)\,\mathrm{d}s\right]^2 < k\ (\rho^+ - \rho^-)\ A^2\ \frac{E^+\ E^-}{E^+ - E^-} \\ &\qquad \Longrightarrow \gamma = 0, \end{aligned} \quad (24)$$

$$\begin{aligned} &\left[F + \int_x^L f(s)\,\mathrm{d}s\right]^2 > k\ (\rho^+ - \rho^-)\ A^2\ \frac{E^+\ E^-}{E^+ - E^-} \\ &\qquad \Longrightarrow \gamma = 1, \end{aligned} \quad (25)$$

$$\begin{aligned} &\left[F + \int_x^L f(s)\,\mathrm{d}s\right]^2 = k\ (\rho^+ - \rho^-)\ A^2\ \frac{E^+\ E^-}{E^+ - E^-} \\ &\qquad \Longrightarrow 0 \le \gamma \le 1, \end{aligned} \quad (26)$$

which reduce to (17)–(19) whenever $f \equiv 0$ and where similar questions of nonuniqueness may arise. If, e.g., $F \equiv 0$ and if $f \neq 0$ but is constant, then

$$L - x < \frac{A}{f} \sqrt{\frac{k\,(\rho^+ - \rho^-)\,E^+\,E^-}{E^+ - E^-}} \Longrightarrow \gamma = 0,$$

$$L - x > \frac{A}{f} \sqrt{\frac{k\,(\rho^+ - \rho^-)\,E^+\,E^-}{E^+ - E^-}} \Longrightarrow \gamma = 1,$$

$$L - x = \frac{A}{f} \sqrt{\frac{k\,(\rho^+ - \rho^-)\,E^+\,E^-}{E^+ - E^-}} \Longrightarrow 0 \le \gamma \le 1.$$

We then conclude that the sections where the stiffest material will be located are the regions where the stress is higher, i.e. near the end $x = 0$.

If, e.g., $F \equiv 0$ and $f(x) = \sin(\pi x/L)$, we have

$$\Big(1 + \cos\Big(\frac{\pi x}{l}\Big)\Big)^2 < k\,(\rho^+ - \rho^-)\,\frac{A^2\,\pi^2}{L^2}\,\frac{E^+\,E^-}{E^+ - E^-} \Longrightarrow \gamma = 0,$$

$$\Big(1 + \cos\Big(\frac{\pi x}{l}\Big)\Big)^2 > k\,(\rho^+ - \rho^-)\,\frac{A^2\,\pi^2}{L^2}\,\frac{E^+\,E^-}{E^+ - E^-} \Longrightarrow \gamma = 1,$$

$$\Big(1 + \cos\Big(\frac{\pi x}{l}\Big)\Big)^2 = k\,(\rho^+ - \rho^-)\,\frac{A^2\,\pi^2}{L^2}\,\frac{E^+\,E^-}{E^+ - E^-} \Longrightarrow 0 \le \gamma \le 1.$$

Again, the cross section where the stress is higher is the one where the stiffest material will be located.

Many other situations may now be considered. If we wish to consider a bar subjected to its own weight, i.e. if $f = -\rho^H g A$, where g stands for the gravity constant, similar reasoning can be performed, but in this case, in order to have an analytical solution it is necessary to show that $u(x) < 0$ for all $x \in]0, L[$, which can be done using the maximum principle.

It is possible to consider layers with a different orientation and, in fact, later on we shall consider any orientation and seek the optimility with respect to the layer orientation as well. Let us suppose that the layers are such that their orientation is defined by a system of coordinates Oy_1y_2 in such a way that $Oy_1 \parallel Ox_2$ and $Oy_2 \parallel Ox_1$. In this case one has $E^H = \gamma E^+ + (1-\gamma)E^-$ and

$$(E^H)^2 > \frac{E^+ - E^-}{k\,(\rho^+ - \rho^-)}\,\frac{1}{A^2}\left[F + \int_x^L f(s)\,\mathrm{d}s\right]^2 \Longrightarrow \gamma = 0,$$

$$(E^H)^2 < \frac{E^+ - E^-}{k\,(\rho^+ - \rho^-)}\,\frac{1}{A^2}\left[F + \int_x^L f(s)\,\mathrm{d}s\right]^2 \Longrightarrow \gamma = 1,$$

$$(E^H)^2 = \frac{E^+ - E^-}{k\,(\rho^+ - \rho^-)}\,\frac{1}{A^2}\left[F + \int_x^L f(s)\,\mathrm{d}s\right]^2 \Longrightarrow 0 \le \gamma \le 1.$$

For example, if $F \equiv 0$ and if $f \neq 0$ is constant, we obtain

$$\left[F + \int_x^L f(s)\,\mathrm{d}s\right]^2 = f^2\,(L-x)^2,$$

and the solution will be such that $0 \le x \le x_1$, $\gamma = 1$ for $x_1 < x < x_2$, $0 < \gamma < 1$, and for $x_2 \le x \le L$ we will have $\gamma = 0$. We remark that, in this case, the solution is unique, and x_1 and x_2 depend only upon the data.

Clearly, we can now consider several other situations involving either different orientations of the microstructure, different loadings or even other problems. As an example, we shall write down the solution obtained if one considers the bending of a simply supported rod subjected to a transverse load of intensity q:

$$x^2\,(L-x)^2 < \frac{4\,k\,A\,I}{q^2}\,\frac{E^+ - E^-}{E^+ - E^-}(\rho^+ - \rho^-) \Longrightarrow \gamma = 0,$$

$$x^2\,(L-x)^2 > \frac{4\,k\,A\,I}{q^2}\,\frac{E^+ - E^-}{E^+ - E^-}(\rho^+ - \rho^-) \Longrightarrow \gamma = 1,$$

$$x^2\,(L-x)^2 = \frac{4\,k\,A\,I}{q^2}\,\frac{E^+ - E^-}{E^+ - E^-}(\rho^+ - \rho^-) \Longrightarrow 0 \le \gamma \le 1,$$

where I stands for the second moment of area with respect to a principal axis perpendicular to the bending plane of the rod. One observes that the region where the bending moment is higher is the one where the stiffest material will be located.

3. Two-Dimensional Examples—The Torsion Problem

Let us now consider the problem of maximizing the torsion constant as a function of the proportion $\gamma = \gamma(x_1, x_2)$ of the components of the microstructure at each point $(x_1, x_2) \in \omega \subset \mathbb{R}^2$ of a rod of cross section ω made by lamination of two materials as has previously been described. Since the optimal configuration may lead to a heterogeneous and anisotropic material, one has to set up the torsion problem in all its generality.

In order to do so, we use the methods of asymptotic analysis described in (Trabucho and Viaño, 1996), which we are now going to summarize. With the usual notation in elasticity theory, where Latin indices take the values $1, 2$ or 3 while Greek indices take the values 1 or 2, and with the usual summation convention on repeated indices, we have

$$e_{3\alpha}(u) = 2\, a^H_{3\alpha 3\beta}\, \sigma_{3\beta}, \quad \sigma_{3\beta} = 2\, A^H_{3\beta 3\alpha}\, e_{3\alpha}(u),$$

where $a^H_{3\beta 3\alpha}$ and $A^H_{3\beta 3\alpha}$ represent the elasticity coefficients obtained through homogenization theory. For example, in the homogeneous isotropic case, with $E^+ = E^- = E$, this corresponds to

$$A^H_{3131} = A^H_{3232} = \frac{E}{2(1+\nu)}, \quad A^H_{3132} = A^H_{3231} = 0,$$

where E and ν stand for Young's modulus and Poisson's ratio, respectively. Let us consider the following notation:

$$D = \begin{vmatrix} A^H_{3131} & A^H_{3132} \\ A^H_{3231} & A^H_{3232} \end{vmatrix} = A^H_{3131}\, A^H_{3232} - A^H_{3132}\, A^H_{3231},$$
$$\xi_1 = -x_2, \quad \xi_2 = x_1. \tag{27}$$

Then it is possible to show that the warping function $w \in H^1(\omega)$ is a solution to the following problem:

$$-\partial_\beta [A^H_{3\beta 3\alpha}\, (\partial_\alpha w + \xi_\alpha)] = 0 \quad \text{in } \omega,$$
$$A^H_{3\beta 3\alpha}\, (\partial_\alpha w + \xi_\alpha)\, n_\beta = 0 \quad \text{on } \partial\omega,$$
$$\int_\omega w \,\mathrm{d}\omega = 0, \tag{28}$$

and that Prandtl's potential function, $\Psi \in H^1_0(\omega)$, satisfies

$$-\partial_\beta \left(\frac{A^H_{3\beta 3\alpha}}{D}\, \partial_\alpha \Psi \right) = 2 \quad \text{in } \omega,$$
$$\Psi = 0 \quad \text{on } \partial\omega. \tag{29}$$

The relationship between these two functions is given by

$$\partial_2 \Psi = A^H_{313\alpha}\, (\partial_\alpha w + \xi_\alpha), \quad \partial_1 \Psi = -A^H_{323\alpha}\, (\partial_\alpha w + \xi_\alpha). \tag{30}$$

The torsion constant is given by

$$J = -\int_\omega x_\alpha\, \partial_\alpha \Psi \,\mathrm{d}\omega = 2 \int_\omega \Psi \,\mathrm{d}\omega$$
$$= \int_\omega \frac{1}{D}\, A^H_{3\beta 3\alpha}\, \partial_\alpha \Psi\, \partial_\beta \Psi \,\mathrm{d}\omega. \tag{31}$$

Using (30), we obtain

$$J = \int_\omega \left(A^H_{3232}\, x_1^2 + A^H_{3131}\, x_2^2 - 2\, A^H_{3231}\, x_1\, x_2 \right) \mathrm{d}\omega$$
$$- \int_\omega A^H_{3\beta 3\alpha}\, \partial_\alpha w\, \partial_\beta w \,\mathrm{d}\omega, \tag{32}$$

or, equivalently,

$$J = \int_\omega A^H_{3\beta 3\alpha}\, (\partial_\alpha w + \xi_\alpha)\, (\partial_\beta w + \xi_\beta) \,\mathrm{d}\omega. \tag{33}$$

The problem of maximizing the torsion constant while taking into account the cost of adding material is equivalent to the one of minimizing the functional

$$j(\gamma) = -2 \int_\omega \Psi \,\mathrm{d}\omega + k \int_\omega \rho^H \, d\omega, \tag{34}$$

where Prandtl's potential function, $\Psi \in H^1_0(\omega)$, satisfies the equilibrium equation, which we write in its variational form as follows:

$$\int_\omega \frac{A^H_{3\beta 3\alpha}}{D}\, \partial_\alpha \Psi\, \partial_\beta v \,\mathrm{d}\omega = \int_\omega 2\, v \,\mathrm{d}\omega, \quad \forall v \in H^1_0(\omega). \tag{35}$$

Proceeding as before, eqns. (1) and (2) give, for this case,

$$k\, (\rho^+ - \rho^-) + \frac{\partial}{\partial \gamma} \left(\frac{A^H_{3\beta 3\alpha}}{D} \right) \partial_\alpha \Psi\, \partial_\beta \Psi$$
$$+ \tau^+ - \tau^- = 0, \tag{36}$$

which has to be taken into account together with (9)–(12).

Having written it in terms of the warping function, instead of (36) we have

$$k\, (\rho^+ - \rho^-) - \frac{\partial}{\partial \gamma} \left(\frac{A^H_{3\beta 3\alpha}}{D} \right) (\partial_\alpha w + \xi_\alpha)(\partial_\beta w + \xi_\beta)$$
$$+ \tau^+ - \tau^- = 0, \tag{37}$$

In the particular case when the laminates are oriented parallel to the principal axes of the cross section, we obtain

$$k\, (\rho^+ - \rho^-) - \chi + \tau^+ - \tau^- = 0, \tag{38}$$

where

$$\chi = 2\,(1+\nu)\,(E^+ - E^-)\Big\{\frac{1}{[\gamma E^+ + (1-\gamma)E^-]^2}|\partial_1\Psi|^2 + \frac{1}{E^+E^-}|\partial_2\Psi|^2\Big\}. \tag{39}$$

As a result, we get

$$\frac{k\,(\rho^+ - \rho^-)}{2\,(1+\nu)\,(E^+ - E^-)} > \frac{|\partial_1\Psi|^2}{(E^-)^2} + \frac{|\partial_2\Psi|^2}{E^+E^-} \Longrightarrow \gamma = 0, \tag{40}$$

$$\frac{k\,(\rho^+ - \rho^-)}{2\,(1+\nu)\,(E^+ - E^-)} < \frac{|\partial_1\Psi|^2}{(E^+)^2} + \frac{|\partial_2\Psi|^2}{E^+E^-} \Longrightarrow \gamma = 1, \tag{41}$$

$$\frac{k\,(\rho^+ - \rho^-)}{2\,(1+\nu)\,(E^+ - E^-)} = \frac{|\partial_1\Psi|^2}{[\gamma E^+ + (1-\gamma)E^-]^2} + \frac{|\partial_2\Psi|^2}{E^+E^-} \Longrightarrow 0 \le \gamma \le 1. \tag{42}$$

From the maximum principle, it is possible to show that, for sufficiently smooth domains, Prandt's potential function is always positive, vanishes on the boundary and possesses its maximum in the interior of the domain. Consequentely, we see that in a neighbourhood of a point where the function has its maximum one must have $\gamma = 0$, and in a neighbourhood of the border, where the gradient is higher, one should have $\gamma = 1$. This is illustrated on the left panel of Figs. 1 and 2. In these pictures each finite element is coloured with a gray scale corresponding to the correct value of γ. The darker tonalities correspond to higher values of γ. At the points where (42) holds γ may take any value in the interval $[0, 1]$. Once again, the discretization of this problem may lead to some difficulties in elements for which such a multiple value of γ may occur.

3.1. Rotation in Torsion

We can easily see that the above analytical solutions can be obtained for the case of layered materials of higher rank. Due to the lack of space, we shall not write down the corresponding equations but will rather consider a related issue, namely, the orientation of each layer. In order to be more specific, let us consider only a rank-one layered material and try to find out not only the correct amount of strong material γ, but also the orientation θ made by the microstructure reference axes Oy_1 and Oy_2 with respect to the macroscopic axes Ox_1 and Ox_2.

Proceeding as before, and using an immediate generalization of Theorem 1 for the case of two parameters, it is possible to show that, in addition to the stationarity condition (36), one now also gets

$$\frac{\partial}{\partial\theta}\left(\frac{A^H_{3\beta3\alpha}}{D}\right)\partial_\alpha\Psi\,\partial_\beta\Psi = 0. \tag{43}$$

We remark that solving this equation can be seen as trying to solve the following problem: Given a vector field $\nabla\Psi$, what must be the orientation of the 2×2 coefficient matrix $A^H_{3\beta3\alpha}/D$ so that

$$\left(\frac{A^H_{3\beta3\alpha}}{D}\right)\partial_\alpha\Psi\,\partial_\beta\Psi$$

is a maximum?

Solving this problem, together with the previous one, and denoting by α the angle formed by vector $\nabla\Psi$ with the Ox_1 axis, we obtain that vector $\nabla\Psi$ should be aligned with the principal axes of the base material characteristic 2×2 matrix $A^H_{3\beta3\alpha}/D$. The final results are illustrated on the right panels of Figs. 1 and 2. Each finite element was not only coloured with the corresponding gray tonality (a solution to (36)), but we also rotated it accordingly to the angle α (a solution to (43)).

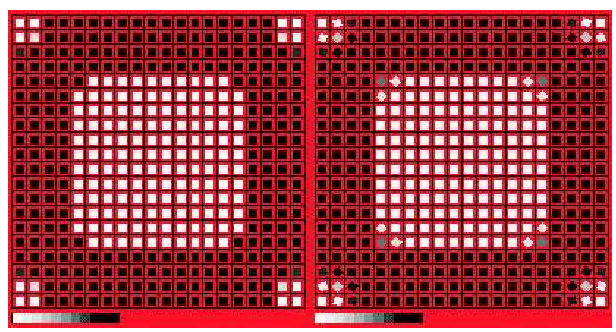

Fig. 1. Optimization of the torsion constant for a square cross section.

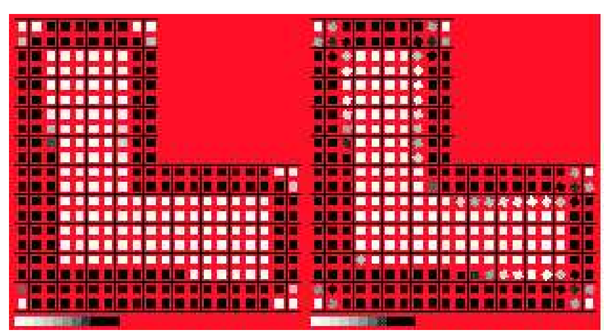

Fig. 2. Optimization of the torsion constant for an L-shaped cross section.

4. Three-Dimensional Examples—Some Elasticity Problems

We shall now consider the same type of problems for the elasticity case. Let us consider a solid occupying a volume $\Omega \subset \mathbb{R}^n$, an open bounded simply-connected subset of $\mathbb{R}^n$, $n = 2, 3$. Moreover, assume that the body is fixed in a part of its surface denoted by Γ_0 and that $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let $f = (f_i)$ and $g = (g_i)$, $i = 1, 2, 3$ denote the forces per unit volume and the forces per unit surface area of Γ_1, applied to the body, respectively. Denoting by $u = (u_i)$, $i = 1, 2, 3$ the displacement field, we consider the problem of maximizing the work done by the external loads taking into account the cost of adding material, i.e. the problem of minimizing the functional

$$j(\gamma) = \int_\Omega f_i\, u_i \,\mathrm{d}x + \int_{\Gamma_1} g_i\, u_i \,\mathrm{d}s + k \int_\Omega \rho^H \,\mathrm{d}x, \quad (44)$$

subjected to the equilibrium condition, written in variational form, $u \in V = \{v \in [H^1(\Omega)]^3 : v_i = 0 \text{ on } \Gamma_0,\ i = 1, 2, 3\}$:

$$\int_\Omega E^H_{ijkl}\, e_{kl}(u)\, e_{ij}(v) \,\mathrm{d}x = \int_\Omega f_i\, u_i \,\mathrm{d}x + \int_{\Gamma_1} g_i\, u_i \,\mathrm{d}s, \quad \forall v \in V, \quad (45)$$

where E^H_{ijkl} $(i, j, k, l = 1, 2, 3)$ denotes the homogenized elasticity coefficients and where one also considers (9)–(12).

The Lagrangian is now of the following form:

$$\begin{aligned}\mathcal{L} = &\int_\Omega f_i\, u_i \,\mathrm{d}x + \int_{\Gamma_1} F_i\, u_i \mathrm{d}s + k \int_\Omega \rho^H \,\mathrm{d}x \\ &+ \lambda \Big[\int_\Omega E^H_{ijkm}\, e_{km}(u)\, e_{ij}(v) \mathrm{d}x - \int_\Omega f_i\, v_i \,\mathrm{d}x \\ &- \int_{\Gamma_1} F_i\, v_i \,\mathrm{d}s \Big] + \int_\Omega \tau^+\, (\gamma - 1) \,\mathrm{d}x - \int_\Omega \tau^-\, \gamma \,\mathrm{d}x. \quad (46)\end{aligned}$$

Once again, Theorem 1, or the necessary conditions for optimality, lead to

$$k\, \frac{\partial \rho^H}{\partial \gamma} - \frac{\partial E^H_{ijkl}}{\partial \gamma}\, e_{kl}(u)\, e_{ij}(u) = -\tau^+ + \tau^-, \quad (47)$$

together with (9)–(12) and (45).

It is now necessary to calculate the derivatives with respect to the volumic fraction γ. As an example, for the two-dimensional case and for simple laminates, formed by well-ordered materials with elastic properties E^+ and E^-, in proportions γ and $1 - \gamma$, respectively, we have (cf. (Bendsøe, 1995) and references therein):

$$E^H_{1111} = \frac{E^+_{1111}\, E^-_{1111}}{\gamma\, E^-_{1111} + (1-\gamma)\, E^+_{1111}},$$

$$E^H_{1212} = \frac{E^+_{1212}\, E^-_{1212}}{\gamma\, E^-_{1212} + (1-\gamma)\, E^+_{1212}},$$

$$E^H_{1122} = \left[\gamma\, \frac{E^+_{1122}}{E^+_{1111}} + (1-\gamma)\, \frac{E^-_{1122}}{E^-_{1111}}\right] \times \frac{E^+_{1111}\, E^-_{1111}}{\gamma\, E^-_{1111} + (1-\gamma)\, E^+_{1111}},$$

$$\begin{aligned}E^H_{2222} = &\ \gamma\, E^+_{2222} + (1-\gamma)\, E^-_{2222} \\ &- \left[\gamma\, \frac{(E^+_{1122})^2}{E^+_{1111}} + (1-\gamma)\, \frac{(E^-_{1122})^2}{E^-_{1111}}\right] \\ &+ \left[\gamma\, \frac{E^+_{1122}}{E^+_{1111}} + (1-\gamma)\, \frac{E^-_{1122}}{E^-_{1111}}\right]^2 \\ &\times \frac{E^+_{1111}\, E^-_{1111}}{\gamma\, E^-_{1111} + (1-\gamma)\, E^+_{1111}}. \end{aligned} \quad (48)$$

For the case of plane stress these expressions can be simplified, and so one gets

$$\begin{aligned} E^H_{1111} &= \frac{1}{1-\nu^2} I_1, \quad E^H_{1212} = \frac{1}{2\,(1+\nu)} I_1, \\ E^H_{1122} &= \frac{\nu}{1-\nu^2} I_1, \quad E^H_{2222} = I_2 + \frac{\nu^2}{1-\nu^2} I_1, \end{aligned} \quad (49)$$

where

$$I_1 = \frac{E^+\, E^-}{\gamma\, E^- + (1-\gamma)\, E^+}, \quad I_2 = \gamma\, E^+ + (1-\gamma)\, E^-, \quad (50)$$

and ν stands for Poisson's ratio, which, for simplicity, is supposed to be the same for both the materials. As a result, (47) becomes

$$\begin{aligned} k(\rho^+ - \rho^-) - &\frac{(E^+ - E^-)}{E^+\, E^-} \Big[\frac{1}{1-\nu^2} \\ &\times I_1^2 \big[\big(e_{11}(u) + \nu e_{22}(u)\big)\big]^2 + E^+\, E^- \big(e_{22}(u)\big)^2 \\ &+ \frac{2}{(1+\nu)} I_1^2 \big(e_{12}(u)\big)^2 \Big] = -\tau^+ + \tau^-. \end{aligned} \quad (51)$$

In a similar way, in the case of plane strain, we would have

$$k\,(\rho^+ - \rho^-) - \frac{(E^+ - E^-)}{E^+\,E^-}\,\Big[\frac{1}{(1-\nu^2)\,(1-2\nu)} \times I_1^2\big[(e_{11}(u) + \nu\, e_{22}(u))\big]^2 + \frac{E^+\,E^-}{(1-\nu^2)}\,\big(e_{22}(u)\big)^2 + \frac{2}{(1+\nu)} I_1^2\big(e_{12}(u)\big)^2\Big] = -\tau^+ + \tau^-. \tag{52}$$

We are now in a position to build some analytical examples. Consider, e.g., a rectangular plate with principal axes Ox_1 and Ox_2 subjected to uniform loads on its boundary and in a plane stress state, namely, a normal stress of intensity σ along the direction Ox_1, a normal stress of intensity Σ along the direction Ox_2 and a shear stress of intensity τ all over the boundary. Thus $\sigma_{11} = \sigma$, $\sigma_{22} = \Sigma$ and $\sigma_{12} = \sigma_{21} = \tau$. For this case and for the associated strain field, we have

$$e_{11}(u) = \sigma\left(\frac{1-\nu^2}{I_1} + \frac{\nu^2}{I_2}\right) - \Sigma\,\frac{\nu}{I_2},$$

$$e_{22}(u) = \Sigma\,\frac{1}{I_2} - \sigma\,\frac{\nu}{I_2}, \quad e_{12}(u) = -\tau\,\frac{1+\nu}{I_1}.$$

Equation (51) now takes the form

$$k\,(\rho^+ - \rho^-) - \frac{(E^+ - E^-)}{E^+\,E^-}\Big[(1-\nu^2)\sigma^2 + E^+E^- \times \frac{(\Sigma - \nu\,\sigma)^2}{I_2^2} + 2\,(1+\nu)\,\tau^2\Big] = -\tau^+ + \tau^-. \tag{53}$$

According to the relative values of σ, Σ and τ, we have the following load history:

$$k(\rho^+ - \rho^-) - \frac{(E^+ - E^-)}{E^+E^-}\Big[(1-\nu^2)\sigma^2 + \frac{E^+}{E^-}(\Sigma - \nu\sigma)^2 + 2(1+\nu)\tau^2\Big] \geq 0 \Longrightarrow \gamma = 0,$$

$$k(\rho^+ - \rho^-) - \frac{(E^+ - E^-)}{E^+E^-}\Big[(1-\nu^2)\sigma^2 + \frac{E^+}{E^-}(\Sigma - \nu\sigma)^2 + 2(1+\nu)\tau^2\Big]$$

$$< (\rho^+ - \rho^-) - \frac{(E^+ - E^-)}{E^+E^-}\Big[(1-\nu^2)\sigma^2 + E^+E^-\frac{(\Sigma - \nu\sigma)^2}{I_2^2} + 2(1+\nu)\tau^2\Big] = 0$$

$$< k(\rho^+ - \rho^-) - \frac{(E^+ - E^-)}{E^+E^-}\Big[(1-\nu^2)\sigma^2 + \frac{E^-}{E^+}(\Sigma - \nu\sigma)^2 + 2(1+\nu)\tau^2\Big]$$

$$\Longrightarrow \gamma = \frac{1}{(E^+ - E^-)} \times \left[\sqrt{\frac{E^+E^-(\Sigma - \nu\sigma)^2}{k(\rho^+ - \rho^-)\frac{E^+E^-}{(E^+ - E^-)} - (1-\nu^2)\sigma^2 - 2(1+\nu)\,\tau^2}} - E^-\right],$$

$$k(\rho^+ - \rho^-) - \frac{(E^+ - E^-)}{E^+E^-}\Big[(1-\nu^2)\sigma^2 + \frac{E^-}{E^+}(\Sigma - \nu\sigma)^2 + 2(1+\nu)\tau^2\Big] \leq 0 \Longrightarrow \gamma = 1.$$

From these expressions, we conclude that if $\Sigma = \nu\sigma$, there will be no region for which $0 < \gamma < 1$. Moreover, the presence of $\tau \neq 0$ contributes to the incresing value of γ (keeping the applied normal stresses, σ and Σ, at the same values).

Using now (52) and the corresponding equations for the plane strain case, a similar expression could be obtained. As a matter of fact, for the plane strain case we can conclude that if $\Sigma = \nu/(1-\nu)$, then there is no region for which $0 < \gamma < 1$.

Let us now consider another example, consisting in the bending of a clamped rectangular beam of length L, width h and thickness b. That is, the beam occupies the region $]0, L[\times]-h/2, +h/2[\times]-b/2, +b/2[$ along axes Ox_1, Ox_2 and Ox_3, respectively. Assume that we have a constant distributed transverse load P applied at $x_1 = 0$, and that the beam is clamped at $x_1 = L$. Then, from the equilibrium equations, the stress distribution is given by

$$\sigma_{11} = -\frac{P}{I}\,x_1\,x_2, \quad \sigma_{22} = 0,$$

$$\sigma_{12} = \frac{P}{2\,I}\Big[x_2 - \Big(\frac{h}{2}\Big)^2\Big], \quad I = \frac{b\,h^3}{12}.$$

Proceeding as before, (1) and (2) lead to

$$k(\rho^+ - \rho^-) - \frac{(E^+ - E^-)}{E^+E^-}\frac{P^2}{I^2} \times \Big\{\underbrace{\Big[(1-\nu^2) + \frac{E^+E^-\nu^2}{I_2^2}\Big]x_1^2x_2^2 + \frac{(1+\nu)}{2}\Big[x_2^2 - \Big(\frac{h}{2}\Big)^2\Big]^2}_{F(\nu,\gamma,x_1,x_2)}\Big\} = -\tau^+ + \tau^-. \tag{54}$$

By studying now the function $F(\nu, \gamma, x_1, x_2)$, several conclusions can be drawn. For instance, if we start with a zero load and successively increase it, it can be observed that at the beginning $\gamma = 0$ all over the beam. As the load starts increasing around the corners $(L, \pm h/2)$, we have $0 < \gamma < 1$. While increasing the load slightly further, these regions propagate along the beam, and around the same corners we have a region with $\gamma = 1$ while in a neighbourhood of the beam's axis, $x_2 = 0$, we always have $\gamma = 0$. It is possible to show that the darker regions move longitudinally, approaching the ellipse whose equation is

$$\left(\frac{x_1}{\sqrt{\frac{(1+\nu)(h/2)^2}{(1-\nu^2)(E^+ \; E^- \; \nu^2)/I_2^2}}}\right)^2 + \left(\frac{x_2}{\sqrt{h^2/2}}\right)^2 = 1. \quad (55)$$

Increasing the load still further, the solution becomes extremely complex but its behaviour is completely determined by the function $F(\nu, \gamma, x_1, x_2)$. These features are illustrated in Figs. 3–6.

4.1. Rotation in Elasticity

Once again, it is possible to include not only the optimization with respect to higher rank laminates, but also the orientation of the cell. As an illustration and as in the torsion problem, for the plane case, if we consider rank-one laminates and the orientation of the base cell with respect to the macroscopic axes, denoted by θ, then not only must we solve (47) together with (9)–(12) and (45), but also

$$-\frac{\partial E^H_{\alpha\beta\gamma\delta}}{\partial\theta}\, e_{\gamma\delta}(u)\, e_{\alpha\beta}(u) = 0. \quad (56)$$

As in the torsion case, this can be interpreted in the following way:

(i) we assume that the strain field $e_{\alpha\beta}$ is given in a system of axes Ox_1x_2, which allows one to calculate the principal strains e_1 and e_2, as well as the principal direction characterized by the angle α made by the eigenvector e_1 with the axis Ox_1,

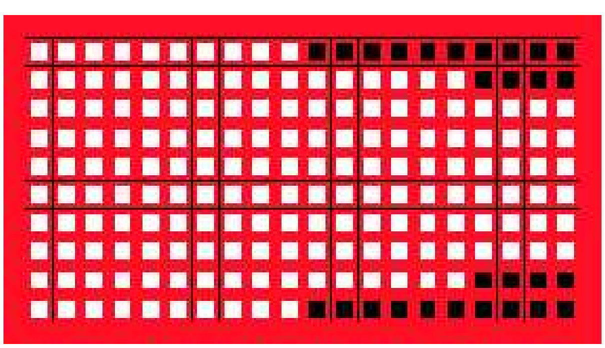

Fig. 3. Optimization of the bending problem for a small force F_0.

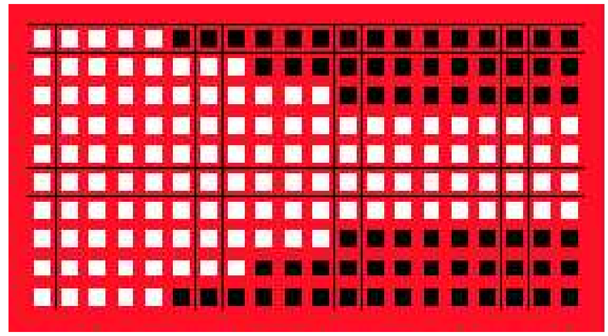

Fig. 4. Optimization of the bending problem for a force $F_1 > F_0$.

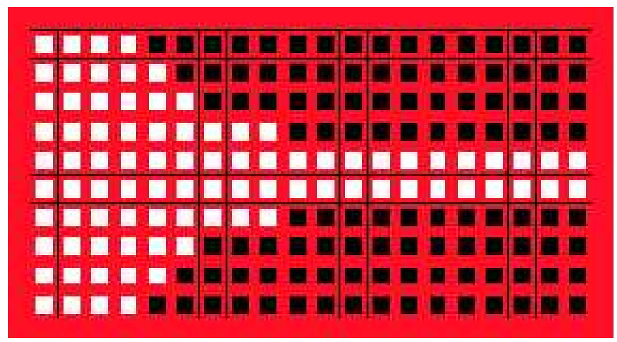

Fig. 5. Optimization of the bending problem for a force $F_2 > F_1$.

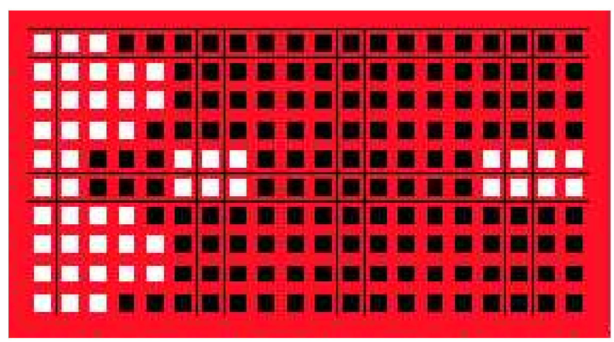

Fig. 6. Optimization of the bending problem for a force $F_3 > F_2$.

(ii) we assume that the material properties $\overline{E}^H_{ijkl}$ are given with respect to a coordinate system $O\overline{x}_1\overline{x}_2$, and we wish to find out the orientation of this system, i.e. the angle θ between $O\overline{x_1}$ and Ox_1, such that

$$\frac{1}{2}\overline{E}^H_{\alpha\beta\gamma\delta}\, \overline{e}_{\gamma\delta}\, \overline{e}_{\alpha\gamma}$$

is a maximum with respect to θ, and where $\overline{e}_{\alpha\beta}$ stands for the strain field components written with respect to the $O\overline{x}_1\overline{x}_2$ system.

Setting $\Psi = \theta - \alpha$, the solution to this problem is given by (cf. (Bendsøe, 1995) and references therein):

$$e_1 = e_2 \Longrightarrow \quad \Psi \quad \text{may take any value,}$$

$$e_1 \neq e_2 \Longrightarrow \sin 2\Psi = 0$$

or

$$\cos 2\Psi = \frac{(e_1 + e_2)}{(e_1 - e_2)} \frac{(\overline{E}_{2222} - \overline{E}_{1111})}{(\overline{E}_{1111} + \overline{E}_{2222} - 2\overline{E}_{1122} - 4\overline{E}_{1212})}.$$

For the plane stress case, taking into account (49)–(51), these conditions reduce to

$$e_1 = e_2 \Longrightarrow \quad \Psi \quad \text{may take any value,}$$

$$e_1 \neq e_2 \Longrightarrow \sin 2\Psi = 0 \quad \text{or} \quad \cos 2\Psi = \frac{(e_1 + e_2)}{(e_1 - e_2)}.$$

For the same case, as is illustrated in Figs. 3–6, we obtain now the results depicted in Figs. 7–10.

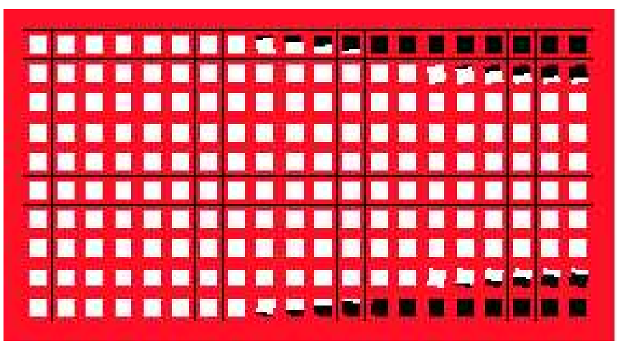

Fig. 7. Optimization of the bending problem for a small distributed force F_0, with rotation.

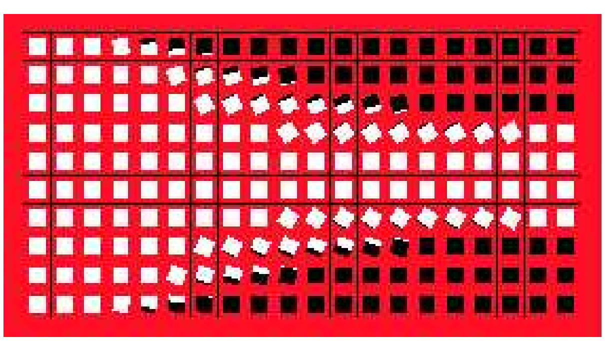

Fig. 8. Optimization of the bending problem for a distributed force $F_1 > F_0$, with rotation.

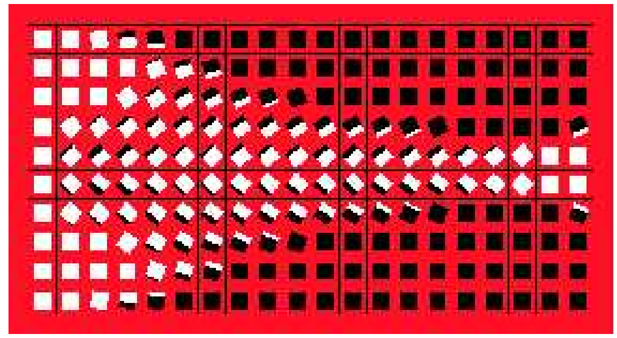

Fig. 9. Optimization of the bending problem for a distributed force $F_2 > F_1$, with rotation.

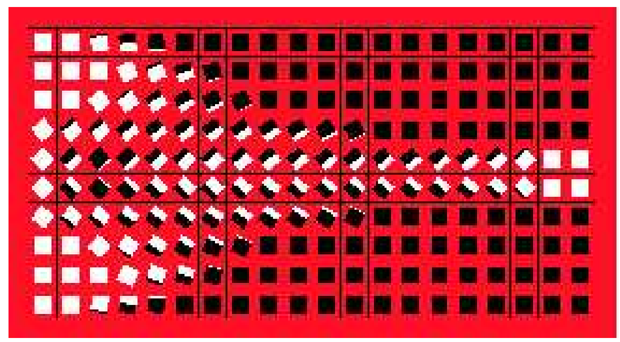

Fig. 10. Optimization of the bending problem for a distributed force $F_3 > F_2$.

Many other examples could be obtained. As an illustration, we consider only the case of a load concentrated in the middle of the left end $x_1 = 0$. The results are illustrated in Fig. 11.

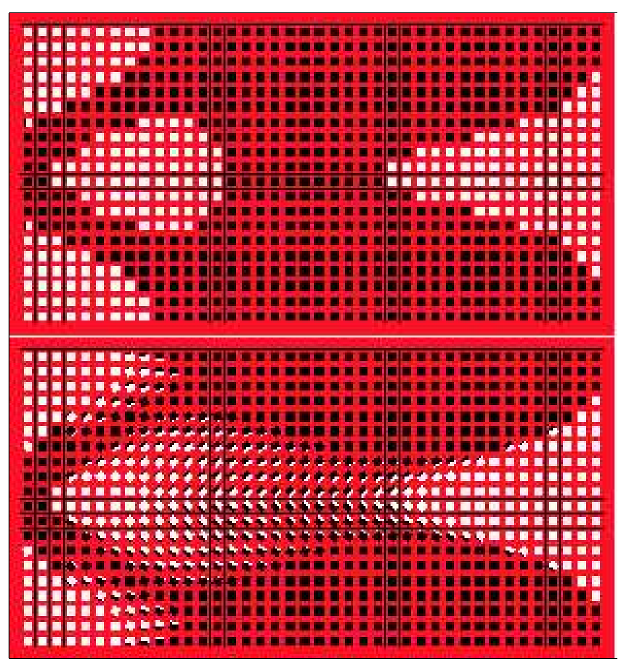

Fig. 11. Optimization of the bending problem for a concentrated force.

If we choose a higher ratio E^+/E^-, then the result shown in Fig. 12 is obtained.

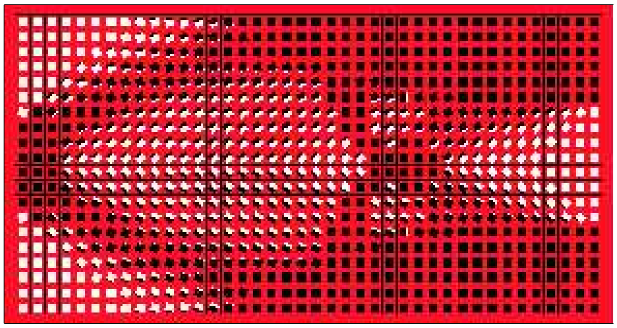

Fig. 12. Optimization of the bending problem for a concentrated force and a large relationship between the elastic moduli.

Acknowledgements

The numerical results were performed by Gaspar Machado from the Mathematics Department of the University of Minho, in Guimarães, Portugal. The author would also like to thank for the comments and suggestions from the referees.

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