

## REDUCED ORDER CONTROLLERS FOR BURGERS' EQUATION WITH A NONLINEAR OBSERVER

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A method for reducing controllers for systems described by partial differential equations (PDEs) is applied to Burgers' equation with periodic boundary conditions. This approach differs from the typical approach of reducing the model and then designing the controller, and has developed over the past several years into its current form. In earlier work it was shown that functional gains for the feedback control law served well as a dataset for reduced order basis generation via the proper orthogonal decomposition (POD). However, the test problem was the two-dimensional heat equation, a problem in which the physics dominates the system in such a way that controller efficacy is difficult to generalize. Here, we additionally incorporate a nonlinear observer by including the nonlinear terms of the state equation in the differential equation for the compensator.

**Keywords:** Burgers' equation, reduced order controllers, proper orthogonal decomposition, minmax control design, stabilized finite elements

### 1. Introduction

Over the past thirty or more years, the design of low order controllers has been a frequent topic of research in control theory. Not only is the concept important for large scale systems of ordinary differential equations (ODE), but it is imperative for real-time control of systems of partial differential equations (PDE). A typical approach is to reduce the model and design a control using standard designs such as linear-quadratic-gaussian (LQG), minmax, or  $H_\infty$ . We call this approach "reduce-then-design" (see Fig. 1.1(a)). In addition to allowing the use of existing computational and control design techniques, an added benefit of this approach may be the production of a low order model that may be desired for other purposes, such as simulation. For ODE systems, balanced realization and truncation is one commonly used model reduction technique (Moore, 1981; Mustafa and Glover, 1991).

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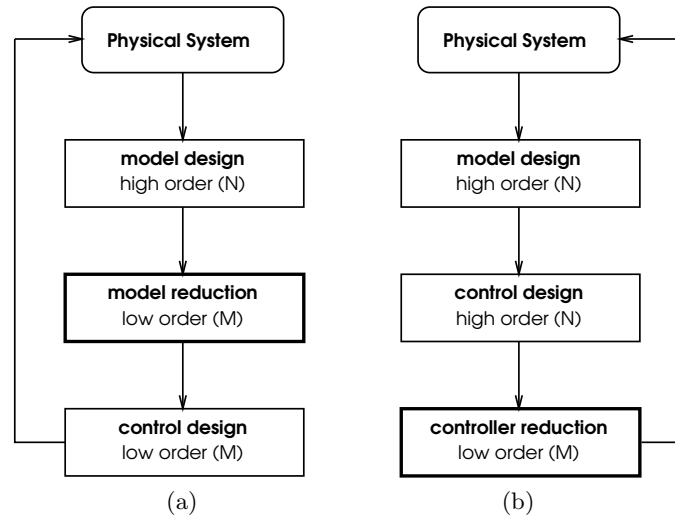


Fig. 1. (a) Reduce-then-design method, (b) design-then-reduce method.

For PDE systems, the first step in the computation of a control design is typically numerical approximation by a large-scale ODE system; then the tools for ODE systems can be applied. In recent literature, model reduction techniques based on the proper orthogonal decomposition (POD) have been investigated (Atwell and King, 2001; Banks *et al.*, 2000; Fahl, 1999; Kunisch and Volkwein, 1999; Ly and Tran, 2001).

An alternative approach when the real goal is the design of a low order feedback control may be to reduce the controller instead of the model, as shown in Fig. 1(b). In this “design-then-reduce” approach, model design is followed by controller design, and then by reduction of the controller (Atwell and King, 2002; 2001; Burns and King, 1998; King, 1998; King and Sachs, 2000). A benefit to this philosophy is that robust low order systems (Atwell and King, 2002; Burns and King, 1998; King and Sachs, 2000) and insight into sensor placement and design (Burns and Rubio, 1997; Faulds and King, 2000) can be obtained. This approach is based on the observation that vital “information” or “physics” contained in the higher order model are available for control design if reduction is postponed.

The favorable results in (Atwell and King, 1999; 2002) were obtained for the two dimensional heat equation with boundary control. The physics of the heat equation dominates the control system in a such a way that it becomes difficult to conclude if the results are truly significant and widely applicable, or merely good for that example. Therefore, we apply the methodology developed in these two papers to the design of low order compensator based controllers for Burgers’ equation.

In (Atwell and King, 2002; 2001; Burns and King, 1998; King, 1998; King and Sachs, 2000), a framework is outlined to construct reduced order compensator-based feedback controllers for systems described by PDEs. The first step of the framework involves a high order (converged) approximation to the PDE control problem. In the

next step, a high order controller is computed for this approximate system. In (Atwell and King, 2002), reduced order controllers were obtained by projecting the controller onto a low order basis. Although this approach works well, it requires computing the solutions of two large scale Riccati equations. In (Atwell and King, 1999), a viable “hybrid” approach involves projection combined with computation of solutions of Riccati equations using a low order basis. The specifics are a bit tricky to explain and will be discussed in detail in Section 2.3. The main benefit is that computational requirements are reduced and results compare favorably with the previous method.

To form the low order basis, the POD technique—which has been widely discussed in the literature as a tool for model reduction—is used. The process is also known as Principle Component Analysis or Karhunen-Loève decomposition. It has been successfully used for simulation (Berkooz, 1991; Berkooz *et al.*, 1993; Chambers *et al.*, 1988; Holmes, 1991; Kirby *et al.*, 1990; Sirovich, 1987), and more recently in optimization (Fahl, 1999; Ly and Tran, 2001; Theodoropoulou *et al.*, 1998) and feedback control (Atwell and King, 2002; 2001; Banks *et al.*, 2000; Kunisch and Volkwein, 1999), for example. POD provides an orthonormal basis for a set of data which we refer to as an “input collection.” The input collection can be composed of theoretical, experimental or computed data. An often used input collection is one composed of “time snapshots”—data collected from numerical simulation at various time points (Sirovich, 1987). An important consideration when using POD for control design is obtaining a basis that also reflects controller dynamics. In (Atwell and King, 2002), a novel input collection was chosen, one that contains controller characteristics rather than model characteristics. Specifically, an integral representation of the feedback law for the PDE control system is used as a foundation for low order approximations. The kernels, called functional gains, reflect into controller dynamics. Functional gains are an infinite dimensional analog to gain matrices in finite dimensional control problems. Atwell and King showed that for certain parabolic systems, controllers designed using this input collection perform favorably when compared with those designed using the time snapshot approach. Here, we apply the ideas used therein to Burgers' equation with a periodic boundary condition. The feedback control problem for the model is presented in Section 2. We discuss numerical approximations for the PDE problem and present the framework for reduced order controllers in Section 3. Section 4 contains a brief overview of POD. We present numerical results in Section 4 and make concluding remarks in Section 5.

## 2. Feedback Control of Burgers' Equation

Burger's equation can be thought of as a one dimensional analog of the Navier-Stokes equations which model the behavior of viscous fluids. As such, it is a useful model equation on which to investigate techniques that might be applied to complicated fluid flow problems. Moreover, Burgers' equation is nonlinear, which provides an interesting challenge for the control design. The controller here is designed using the MinMax approach. This methodology is typically applied to linear systems, but it can be applied to Burgers equation by first linearizing the model about an equilibrium, then applying MinMax theory. Additionally, we do not use full-state feedback, which is

impractical for PDE control problems (and many large-scale ODE systems) since it requires full knowledge of the state. Instead, we incorporate a nonlinear compensator which provides a state estimate based upon measurements of the state. This was also done in (Burns and King, 1998; King, 1998; King and Sachs, 2000), and provided favorable results especially in the context of reduced order controllers.

### 2.1. The Model Problem

Define the spatial domain  $\Omega = [0, \ell]$ . The viscous form of Burgers' equation can be written as

$$w_t(t, x) = \epsilon w_{xx}(t, x) - w(t, x)w_x(t, x) + u(t, x) + f(t, x) \quad \text{for } x \in \Omega, \quad t > 0, \quad (1)$$

where  $u(t, x)$  is a control term and  $f(t, x)$  models disturbances to the system. This type of control term was considered in (Burns and King, 1995) for a structural control problem, in (King, 1995) for the heat equation, and can be thought of as a distributed control in which the control can be applied at every point of the spatial domain. We select this system because the control input operator is the identity, which allows us to consider issues such as optimal sensor location. We take  $f(t, x)$  to be of the form  $d(x)\eta(t)$ . We apply periodic boundary conditions as in many fluid applications of interest; these are given as

$$w(t, 0) = w(t, \ell), \quad w_x(t, 0) = w_x(t, \ell). \quad (2)$$

An initial condition is provided by

$$w(0, x) = w_0(x). \quad (3)$$

Implementation of distributed parameter control theory requires the abstract form of the PDE which can be obtained as follows (Burns and Kang, 1991; Marrekchi, 1993). Let  $w(t) = w(t, \cdot)$  be the state with state space  $W = L_2(\Omega)$ . Define the linear operator  $A_\epsilon$  as  $A_\epsilon w = \epsilon w''$ , for all  $w \in \mathcal{D}(A_\epsilon) = H_{per}^1(\Omega) \cap H^2(\Omega)$ , where  $H_{per}^1(\Omega) = \{w + c : w \in H_0^1(\Omega) \text{ and } c(x) \text{ is constant}\}$ . The disturbance,  $f(t, x)$ , can be written as  $D\eta(t)$  where  $D \in \mathcal{L}(\mathbb{R}, W)$ . The abstract form of the controlled model problem with disturbance can then be written as the initial value problem

$$\frac{\partial}{\partial t} w(t) = A_\epsilon w(t) + G(w(t)) + Bu(t) + D\eta(t), \quad w(0) = w_0, \quad \text{for } t > 0 \quad (4)$$

on the space  $W$ , where  $G(w) = -ww'$  is defined on  $H_{per}^1(\Omega)$ . It is known that  $A_\epsilon$  is the infinitesimal generator of an analytic semigroup on  $W$ . Additionally, mild solutions of the system exist (Burns and Kang, 1991). Note that  $A_\epsilon$  is not stable.

As has previously been mentioned, we do not assume that we have knowledge of the full state. Instead, we assume a state measurement of the form

$$y(t) = Cw(t) + E\eta(t), \quad (5)$$

where  $C \in \mathcal{L}(W, \mathbb{R}^m)$  and  $E \in \mathcal{L}(\mathbb{R}, \mathbb{R}^m)$ . We can apply the theory and results from (Burns and Kang, 1991; Marrekchi, 1993) to show that a stabilizing compensator-based controller can be applied to the system. This can be done as discussed in the following section.

## 2.2. MinMax Control Design

MinMax control theory provides a controller which allows for limited state measurement and feeding back an estimate of the state that is based upon that measurement. Such a controller can be computed via the solution of two algebraic Riccati equations as we shall outline in this section. It is typically applied to linear systems, resulting in a linear controller. However, it can be applied to nonlinear systems by first linearizing the state equation, designing the standard feedback control law, and then incorporating a nonlinear term in the equation for the state estimate. Note that this term is modeled after the nonlinearity in the state equation, and results in an overall nonlinear feedback controller. This approach was applied in (Burns and King, 1998; King, 1998; King and Sachs, 2000) to a structural control problem and was used in the context of reduced order controllers.

Given a system in the abstract form

$$\dot{w}(t) = Aw(t) + G(w(t)) + Bu(t) + D\eta(t), \quad w(0) = w_0, \quad (6)$$

assume that a linearization of the system about an equilibrium solution,  $w_e(t)$ , is given by

$$\dot{z}(t) = \tilde{A}z(t) + Bu(t) + D\eta(t), \quad z(0) = z_0, \quad (7)$$

where  $\tilde{A}$  is composed of  $A$  and the linearization of  $G$  about  $w_e(t)$  (which we denote by  $G_\ell$ ).

Suppose that a measurement of the state is given by

$$y(t) = Cw(t) + E\eta(t). \quad (8)$$

MinMax design as applied to the linearization in (7) (and assuming the measurement for the linearized state,  $z(t)$ , takes the form of (8)) provides a linear equation for a state estimate,  $z_c(t)$ , and a feedback law. These are given as

$$\dot{z}_c(t) = A_c z_c(t) + Fz(t), \quad z_c(0) = z_{c_0} \quad (9)$$

and

$$u(t) = -Kz_c(t), \quad (10)$$

respectively. In (10), the operator  $K$  is called the feedback operator. To account for the nonlinearity in the state equation (6), we form the equation for the state estimate

$$\dot{w}_c(t) = A_c w_c(t) + \tilde{G}(w_c(t)) + Fy(t), \quad w_c(0) = w_{c_0}, \quad (11)$$

where  $\tilde{G} = G - G_\ell$ .

Control design means determining  $K$ ,  $F$  and  $A_c$ ; MinMax theory provides these operators as follows. Define a non-negative definite, self-adjoint state weighting operator  $Q : W \rightarrow W$  and a positive definite, self-adjoint control weighting operator  $R : U \rightarrow U$ . Given  $\theta \geq 0$ , the algebraic Riccati equation

$$A^* \Pi + \Pi A - \Pi [BR^{-1}B^* - \theta^2 DD^*] \Pi + Q = 0 \quad (12)$$

is solved for the non-negative definite, self-adjoint solution,  $\Pi$ . Then, given a positive definite self-adjoint weighting operator  $H$ ,  $P$  is found as the non-negative definite, self-adjoint solution to

$$AP + PA^* - P[C^*H^{-1}C - \theta^2Q]P + DD^* = 0. \quad (13)$$

When  $\theta = 0$ , the classic LQG controller is obtained. In this case, one can see from (12) and (13) that the operator  $\Pi$  has no “knowledge” of the physics of the disturbance in the system. Hence, we expect larger values of  $\theta$  to yield a controller which is more robust to disturbances. There is a limit as to how large  $\theta$  can be and maintain solvability of the Riccati equations. We note that the operators  $R$  and  $H$  and the scalar  $\theta$  are design parameters.

If the solutions  $P$  and  $\Pi$  exist and  $(I - \theta^2P\Pi)^{-1}P$  is non-negative, we define

$$\begin{aligned} K &= R^{-1}B^*\Pi, \\ F &= (I - \theta^2P\Pi)^{-1}PC^*H^{-1}, \\ A_c &= A - BK - FC + \theta^2DD^*\Pi. \end{aligned} \quad (14)$$

Using (6), (8), (11) and (10), the closed loop system can be written in matrix form as

$$\begin{aligned} \begin{bmatrix} \dot{w}(t) \\ \dot{w}_c(t) \end{bmatrix} &= \begin{bmatrix} A & -BK \\ FC & A_c \end{bmatrix} \begin{bmatrix} w(t) \\ w_c(t) \end{bmatrix} + \begin{bmatrix} G(w(t)) \\ \tilde{G}(w_c(t)) \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & FE \end{bmatrix} \eta(t), \\ \begin{bmatrix} w(0) \\ w_c(0) \end{bmatrix} &= \begin{bmatrix} w_0 \\ w_{c_0} \end{bmatrix}. \end{aligned} \quad (15)$$

The optimal control in (10) exists for the problem (4) defined in Section 2 (Marrekchi, 1993); specifically, the arguments therein can be modified to address the periodic boundary conditions. In addition, the control law can be written as

$$u(t, \xi) = -[Kw_c(t)](\xi) = \int_{\Omega} k(\xi, x)w_c(t, x) dx, \quad (16)$$

where the kernel  $k(\xi, x) \in L_2(\Omega \times \Omega)$  is called the functional gain. Since the heat equation is the linearization of Burger’s equation used in this paper for the design of  $K$ , the results in (King, 1995) can be applied to ensure the existence of the functional gains. This representation is of interest in the design of reduced order controllers for several reasons. First, the integral representation has the effect of smoothing the integrand. Thus, coarse approximations of  $k(\xi, x)$  and  $w_c(t, x)$  may be sufficient to compute the control accurately. Second, for any  $\xi$ , the effect on  $u(t, \xi)$  of the state estimate  $w_c(t, x)$  is influenced by the magnitude of the functional gain at  $(\xi, x)$ . Regions over which the functional gain is large correspond to a larger contribution of the state estimate value to the control; where the functional gain is zero, the state estimate makes no contribution to the control. The functional gains have been used

in (Burns and Rubio, 1997; Faulds and King, 2000) to place sensors. In (Atwell and King, 1999; 2002), they were used in the proper orthogonal decomposition to derive a low order basis for reduced order controller computation with favorable results. Here, we will apply the same ideas to Burgers' equation.

### 2.3. Reduced Order Compensators

Implementation of a controller for a PDE system requires a numerical discretization. For example, a finite element method provides semi-discrete finite dimensional approximations of (6) and (8) of order  $N$  (where the order refers to the number of basis elements), given by

$$\dot{w}^N(t) = A^N w^N(t) + G^N(w^N(t)) + B^N u^N(t) + D^N \eta^N(t), \quad w^N(0) = w_0^N, \quad (17)$$

$$y^N(t) = C^N w^N(t) + E^N \eta^N(t). \quad (18)$$

In a full order compensator design, the order  $N$  approximations are used to compute  $K^N$ ,  $F^N$  and  $A_c^N$ . Then finite dimensional approximations of the compensator equation (11) and control law (10) are respectively given by

$$\dot{w}_c^N(t) = A_c^N w_c^N(t) + \tilde{G}^N(w_c^N(t)) + F^N y^N(t), \quad w_c^N(0) = w_{c_0}^N, \quad (19)$$

$$u^N(t) = -K^N w_c^N(t). \quad (20)$$

The approximation to the closed-loop LQG system (which will henceforth be referred to as full order) is given by

$$\begin{aligned} \begin{bmatrix} \dot{w}^N(t) \\ \dot{w}_c^N(t) \end{bmatrix} &= \begin{bmatrix} A^N & -B^N K^N \\ F^N C^N & A_c^N \end{bmatrix} \begin{bmatrix} w^N(t) \\ w_c^N(t) \end{bmatrix} + \begin{bmatrix} G^N(w^N(t)) \\ \tilde{G}^N(w_c^N(t)) \end{bmatrix} \\ &\quad + \begin{bmatrix} D^N & 0 \\ 0 & F^N E^N \end{bmatrix} \eta(t), \\ \begin{bmatrix} w^N(0) \\ w_c^N(0) \end{bmatrix} &= \begin{bmatrix} w_0^N \\ w_{c_0}^N \end{bmatrix}. \end{aligned} \quad (21)$$

Convergence of the finite dimensional approximations to the distributed parameter system in (15) is addressed in (Marrekchi, 1993).

Real-time control using the full order compensator may be impossible for many physical problems in that they may require large discretized systems for adequate approximation. Therefore, a reduced order compensator is required. A “reduce-then-design” approach to reduced order compensators uses model reduction to produce a system given by (17) and (18) with a small value of  $N$ . A control is then designed for the reduced system (Fig. 1(a)). Again, a potential drawback of this approach is that important physics or information contained in the model can be lost before obtaining the controller.

On the other hand, as was previously shown in (Atwell and King, 1999; 2002; Burns and King, 1998; King and Sachs, 2000), the design-then-reduce approach can yield robust low order controllers which outperform the standard full order controller. A key observation is that the control, when implemented, will be coupled with a real physical system rather than a discretized differential equation. Thus, it is a low order controller rather than a low order model which is the goal. A “design-then-reduce” approach computes a high order, well-resolved approximation to (6) that retains more of the important physical properties of the system. A controller is designed based on this high order model, and then reduced (again, see Fig. 1(b)).

Specifically, (17) and (18) are computed using a high order finite element basis (large  $N$ ) to give the finite dimensional system. These high order approximations are used to compute matrices needed in the compensator equation (19) and the control law (20) via the algebraic Riccati equations in (12) and (13). In (Atwell and King, 1999; 2001) it was shown that for the reduced order system to be stable, the low order  $K^M$  (where  $M \ll N$ ) should be formed by projecting the high order  $K^N$  that arises from  $\Pi^N$  onto a low order basis (order  $M$ ), rather than computed using the low order bases (as is done in “reduce-then-design” methods). The high order  $F^N$  and  $A_c^N$  (from  $\Pi^N$  and  $P^N$ ) could also be projected, or computed by approximating (12) and (13) with the low order bases. It is the latter approach that we adopt here. By computing (12), (13) with the low order POD basis, we obtain the low order controller

$$\dot{w}_c^M(t) = A_c^M w_c^M(t) + G^M(w_c^M(t)) + F^M y^N(t), \quad w_c^M(0) = w_{c0}^M, \quad (22)$$

$$u^M(t) = -K^M w_c^M(t). \quad (23)$$

To investigate the performance of the design, a simulation of the system consisting of the full order ( $N$ ) state and reduced order ( $M$ ) state estimate given by

$$\begin{aligned} \begin{bmatrix} \dot{w}^N(t) \\ \dot{w}_c^M(t) \end{bmatrix} &= \begin{bmatrix} A^N & -B^N K^M \\ F^M C^N & A_c^M \end{bmatrix} \begin{bmatrix} w^N(t) \\ w_c^M(t) \end{bmatrix} + \begin{bmatrix} G^N(w^N(t)) \\ \tilde{G}^M(w_c^M(t)) \end{bmatrix} \\ &+ \begin{bmatrix} D^N & 0 \\ 0 & F^M E^M \end{bmatrix} \eta(t) \\ \begin{bmatrix} w^N(0) \\ w_c^M(0) \end{bmatrix} &= \begin{bmatrix} w_0^N \\ w_{c0}^M \end{bmatrix} \end{aligned} \quad (24)$$

is typically computed. As we shall show in our section on numerics, relying on current simulation tools and codes to test a design may be a shortcoming of current technology. To form the low order basis for the reduced order compensator, we apply the proper orthogonal decomposition (see, e.g., (Atwell and King, 2002)).

### 3. Proper Orthogonal Decomposition (POD)

POD, described in detail in (Berkooz, 1991; Berkooz *et al.*, 1993; Holmes, 1991; Sirovich, 1987), takes a given input collection and creates an orthogonal basis (which



we refer to as the *POD basis*) for the data. In this section, we give a brief overview of the procedure and optimality properties of POD. Sirovich (1987), Berkooz (1991), and others describe POD in a statistical setting using an averaging operation for application to turbulent flows. POD can also be derived as an application of singular value decomposition (SVD) (Kunisch and Volkwein, 1999).

### 3.1. General Procedure

Suppose that an *input collection* is composed of a set of functions on a spatial domain  $\Omega$ . In addition, assume that each member of the input collection can be written in terms of an  $n$ -th order Galerkin global finite element basis  $\{\phi_i(s)\}_{i=1}^n$ . Denote each element of the input collection by

$$y_j \in L_2(\Omega), \quad j = 1, \dots, m.$$

The entire collection can then be written in vector form as

$$\mathcal{Y} = [y_1, \dots, y_m] \in L_2^m(\Omega). \quad (25)$$

A finite dimensional approximation of the collection can be represented as an  $n \times m$  matrix  $Y$  where the  $j$ -th column of  $Y$ ,  $Y_{\cdot,j}$ , contains the coefficients for the  $j$ -th element of the input collection in terms of the finite element basis, i.e.,

$$y_j(s) = \sum_{i=1}^n Y_{i,j} \phi_i(s). \quad (26)$$

The first step of the POD process is to find  $\psi \in L_2(\Omega)$  with

$$\max_{\psi, \|\psi\|=1} \sum_{j=1}^m |(y_j, \psi)|^2. \quad (27)$$

Assume that  $\psi$  is also in the linear space spanned by  $\{\phi_i(s)\}_{i=1}^n$ , i.e.,  $\psi(s) = \sum_{i=1}^n \vec{\psi}_i \phi_i(s)$ . Denote by  $M$  the positive definite mass matrix corresponding to the finite element basis. The Fritz John necessary conditions for optimality of  $\vec{\psi}$  require

$$Y Y^T M \vec{\psi} = \lambda \vec{\psi} \quad (28)$$

for some  $\lambda \neq 0$ . Multiply both the sides of (28) by  $M^{1/2}$  and write  $\hat{Y} = M^{1/2} Y$  and  $u = M^{1/2} \vec{\psi}$ . This yields

$$\hat{Y} \hat{Y}^T u = \lambda u. \quad (29)$$

Since  $\hat{Y} \hat{Y}^T$  is non-negative and symmetric, all eigenvalues  $\lambda$  must be non-negative and real, i.e.,  $\lambda = \sigma^2$ .

The coefficients of the first POD basis function,  $\psi^1$ , with respect to the finite element basis are given by  $\vec{\psi}^1 = M^{-1/2} u$ , where  $u$  is the normalized eigenvector of  $\hat{Y} \hat{Y}^T$  corresponding to the largest eigenvalue. That is,

$$\vec{\psi}^1 = [\psi_1^1, \psi_2^1, \dots, \psi_n^1]^T = M^{-1/2} u \quad \text{and} \quad \psi^1(s) = \sum_{i=1}^n \psi_i^1 \phi_i(s).$$

The second POD basis element,  $\psi^2$ , is found by minimizing (27) subject to the additional constraint  $(\psi^1, \psi^2) = 0$ . The coefficients of  $\psi^2$  are given by  $\vec{\psi}^2 = M^{-1/2}u$ , where  $u$  is the normalized eigenvector in (29) corresponding to the second largest eigenvalue. This process is repeated until all non-zero eigenvalues of  $\hat{Y}\hat{Y}^T$  are exhausted. For simplicity of notation, assume that  $Y$  has rank  $m$ . The POD basis in matrix form is

$$\Psi = \begin{bmatrix} \vec{\psi}^1 & \dots & \vec{\psi}^m \end{bmatrix}.$$

If  $\text{rank}(Y) = k < m$ , the POD basis has only  $k$  elements.

The relative magnitude of each eigenvalue with respect to the others gives a measure of the importance of the corresponding eigenfunction in representing elements of the input collection. POD inherently orders the basis elements by their relative importance; this is further clarified in the context of SVD.

The normalized eigenvectors in (29) are the left singular vectors in the SVD of  $\hat{Y}$ , i.e., the eigenvectors are the columns of  $U$  where

$$\hat{Y} = U\Sigma V^T. \quad (30)$$

Consider the reduced SVD, where the number of columns in  $U$  and  $V$  is equal to  $\text{rank}(\hat{Y})$  (which is equal to  $\text{rank}(Y)$ ).  $U$  and  $V$  have orthonormal columns, and  $\Sigma$  is square and diagonal. The diagonal elements of  $\Sigma$  are the non-zero singular values,  $\sigma_i$  of  $\hat{Y}$  ordered from largest to smallest. This intrinsic ordering of singular values results in the ordering of POD basis elements by the magnitude of their corresponding eigenvalues.

The input collection  $Y$  can be written as

$$Y = M^{-1/2}\hat{Y} = M^{-1/2}U\Sigma V^T = \Psi B, \quad (31)$$

where  $\Psi = M^{-1/2}U$  and  $B = \Sigma V^T$ . The column  $B_{\cdot,j}$  contains coefficients for the  $j$ -th element of the input collection in the expansion with respect to the POD basis and  $\Psi_{\cdot,j} = \vec{\psi}^j$ . The POD basis vectors (in matrix representation) can be obtained by finding the left singular vectors of  $\hat{Y}$  (or solving the eigenvalue problem (29)) and multiplying on the left by  $M^{-1/2}$ .

**Remark.** The elements of  $Y^TMY$  are  $\int_{\Omega} y_i(x)y_j(x) dx$ , i.e.,  $Y^TMY$  is the covariance matrix developed in the statistical context (Ly and Tran, 2002).

The following results can be found in terms of the statistical framework in (Berkooz, 1991) and others. They are found in terms of SVD in (Kunisch and Volkwein, 1999). Here, they are stated in the context of an input collection represented by a finite element basis (as in (26)).

**Property 1.** (Span of POD basis) *Suppose that the finite element approximation of the input collection is represented by a matrix  $Y \in \mathbb{R}^{n \times m}$ , and  $\Psi$  represents the corresponding POD basis. If  $Y$  has rank  $k$ , then*

$$\text{span}\{\Psi_{\cdot,j}\}_{j=1}^k = \text{span}\{Y_{\cdot,j}\}_{j=1}^m.$$

This implies that the span of the POD basis is the smallest linear subspace containing all of the input collection. Thus, the POD basis elements exhibit linear properties that are shared by all elements of the input collection, such as boundary conditions, periodicity, and incompressibility (as in fluid flows) (Berkooz, 1991). In addition, working with the POD basis limits the problem to a small linear space that has properties observed in the input collection.

**Property 2.** (Uncorrelated Coefficients) *If the finite element approximation of the input collection is represented by  $Y$ , and  $B = \Sigma V^T$  as in (31), then*

$$(B_{i,\cdot}, B_{j,\cdot}) = \delta_{ij} \sigma_i^2.$$

Property 2 follows directly from the orthonormal properties of the matrix  $V$  in the SVD. It says that the coefficients of the  $i$ -th POD basis element in the expansion of the input collection are not linearly related to the coefficients of the  $j$ -th POD basis element for  $i \neq j$ .

Suppose that we wish to approximate the elements of an input collection  $\mathcal{Y}$  by a basis of order  $p$ , where  $p < n$  and  $p < m$ . The *reduced POD basis* formed by truncation is the optimal basis of order  $p$  for approximating the input collection.

To show this optimality, one begins by using the reduced POD basis to obtain the approximate input collection,  $Y^p$ . Then the mean square error between the original input collection and the approximate input collection is calculated. Finally, it can be concluded that this error is minimal over all order  $p$  approximations. To see the details, refer to (Atwell and King, 2002).

**Property 3.** (Least Error in Truncation) *Let  $Y^p$  be the approximation to an input collection  $Y$  given by the reduced POD basis of order  $p$ , and let  $Z^p$  be any other order  $p$  approximation to the input collection. Then*

$$\|Y - Y^p\|_F^2 \leq \|Y - Z^p\|_F^2,$$

where  $\|\cdot\|_F$  denotes the Frobenius norm. That is, approximation of the input collection by the reduced POD basis introduces a smaller mean square error than any other approximation of the same order.

Property 3 has important implications for model reduction. A criterion can be set to limit the error in approximation in terms of a fraction  $P$ . We seek  $p \ll m$  so that

$$\frac{\sum_{i=1}^p \sigma_i^2}{\sum_{i=1}^m \sigma_i^2} > P. \tag{32}$$

Then  $p$  gives the number of POD basis functions necessary for an approximation with mean square error less than  $(1 - P) \sum_{i=1}^m \sigma_i^2$ . We think of  $p$  as the number of basis functions required to retain  $P * 100$  percent of the “essential information” in the input collection. The POD basis is optimal for model reduction since no other basis can represent more of the essential information in a fixed number of terms.

## 4. Numerical Computations and Results

For numerical computations, the viscosity coefficient was taken to be  $\epsilon = .001$ ; this is analogous to taking a relatively large Reynolds number in the Navier-Stokes equations. The spatial interval is taken to be  $\Omega = [0, 1]$ . The control input operator is  $B = I$ , and the control space  $U = W$ . It is often necessary in applications to control certain regions in the spatial domain more accurately than others. This can be done by defining the state weighting operator (used in Riccati equation calculations) as  $Q = q(x)I_W$ , where  $I_W$  is the identity operator on  $W$  and  $q(x)$  is large on the regions over which control must be most accurate. In our examples, we chose  $q(x) = 10$  on the region given by  $0.7 \leq x \leq 0.9$  and  $q(x) = 1$  over the rest of the domain. This causes the control to influence the state more heavily over  $[.7, .9]$ . The control weighting operator is  $R = I_U$  where  $I_U$  represents the identity operator on  $U$ .

### 4.1. Finite Element Systems

A high order finite dimensional approximation of the PDE system is computed as a foundation for reduced order computations. Specifically, we apply a linear Galerkin finite element approximation by subdividing the domain  $[0, 1]$  into  $N$  uniform subintervals; applying the periodic boundary conditions gives a basis of size  $N$ .

One can apply the results in (Marrekchi, 1993; Ito and Morris, 1998) to show that the finite dimensional system converges to the PDE as  $N \rightarrow \infty$ , so the matrices  $K^N$ ,  $F^N$  and  $A_c^N$  defining the approximated controller converge to the corresponding operators in (14) which define the controller for the infinite dimensional system. For the approximate system, there are a discrete number of actuator locations; the finite element scheme chosen for this problem has  $N$  actuator locations, one for each grid point. Therefore, there are  $N$  corresponding functional gains  $k(\xi, x)$ . These gains represent the approximate feedback operator  $K^N$ . Figure 2 shows the functional gains,  $k(\xi, x)$  for actuator locations,  $\xi$  as computed with the finite element approximation of the MinMax controller with  $\theta = .9$ . The gains for the LQG system, i.e.,  $\theta = 0$ , have the same form, but are slightly smaller in magnitude. The discretization  $N = 80$  was chosen to capture behavior in simulations, and is more than needed to obtain gain convergence. Thus, these approximate gains are assumed to represent the infinite dimensional feedback law. Note that the periodic boundary conditions are reflected in the periodicity of the gains with respect to  $\xi$  and with respect to  $x$ .

### 4.2. POD Basis Formulation

As in (Atwell and King, 2002), we choose our input collection to be the set of functional gains from the control law. Therein, we found promising results and we wish to see how these might be extended to other applications. In this work, we chose a POD basis which captured 90% of the information in the input collection of functional gains; this resulted in 10 basis functions in Fig. 3 (see (32)).

To obtain the low order approximation,  $K^M$  ( $M=10$ ), we project  $K^N$  onto this basis. The low order compensator equation (i.e., the matrices  $A_c^M$  and  $F^M$ ) is obtained by using this low order basis to compute solutions  $\Pi^{10}$ ,  $P^{10}$  to (12) and (13).

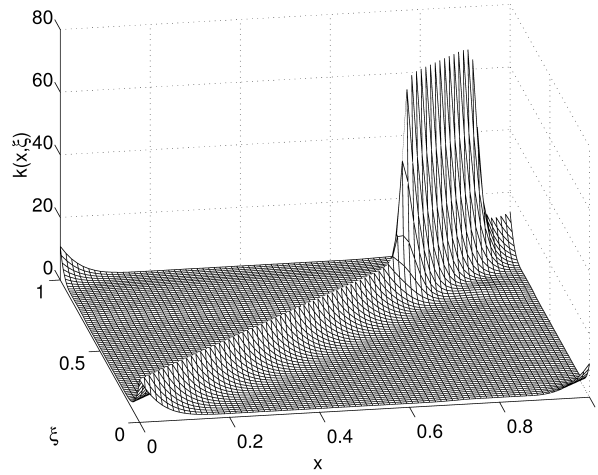


Fig. 2. Functional gains,  $k(\xi, x)$ .

### 4.3. Simulations

In this section, simulations of the various systems discussed in this paper are presented. The initial condition for the state is taken to be

$$w(0, x) = \begin{cases} .5 \sin(2\pi x) & \text{if } 0 < x \leq .5, \\ 0 & \text{if } .5 \leq x < 1. \end{cases}$$

The simulations are based on a stabilized finite element code as described in (Atwell and King, 2000; Franca and Frey, 1992a; 1992b). This allows for a very small choice of  $\epsilon$  for which the system could not otherwise be simulated. The disturbance operator  $D$  is taken to be

$$D\eta(t) = \int_0^1 \eta(t) \, dx$$

and  $\eta(t) = .75 \cos(10t)$ .

Figure 4 shows the dynamics of the uncontrolled, i.e., open loop system. One can notice the initial disturbance located at  $0 \leq x \leq .5$  propagating to the right, and then, via the periodic boundary conditions, entering at the left at about five seconds and propagating again toward the right. The disturbance provided by  $\eta(t)$  is evident.

To examine the difference between LQG and MinMax controls in attenuating the disturbance, we first consider full order systems, i.e., systems in which both the state and compensator equations are approximated on the finite element grid arising from  $N = 80$ . To simulate the full order LQG system, an initial condition is needed for the state estimate. We use  $w_c(0, x) = 1.05w_0(x)$ , that is, the initial condition for the

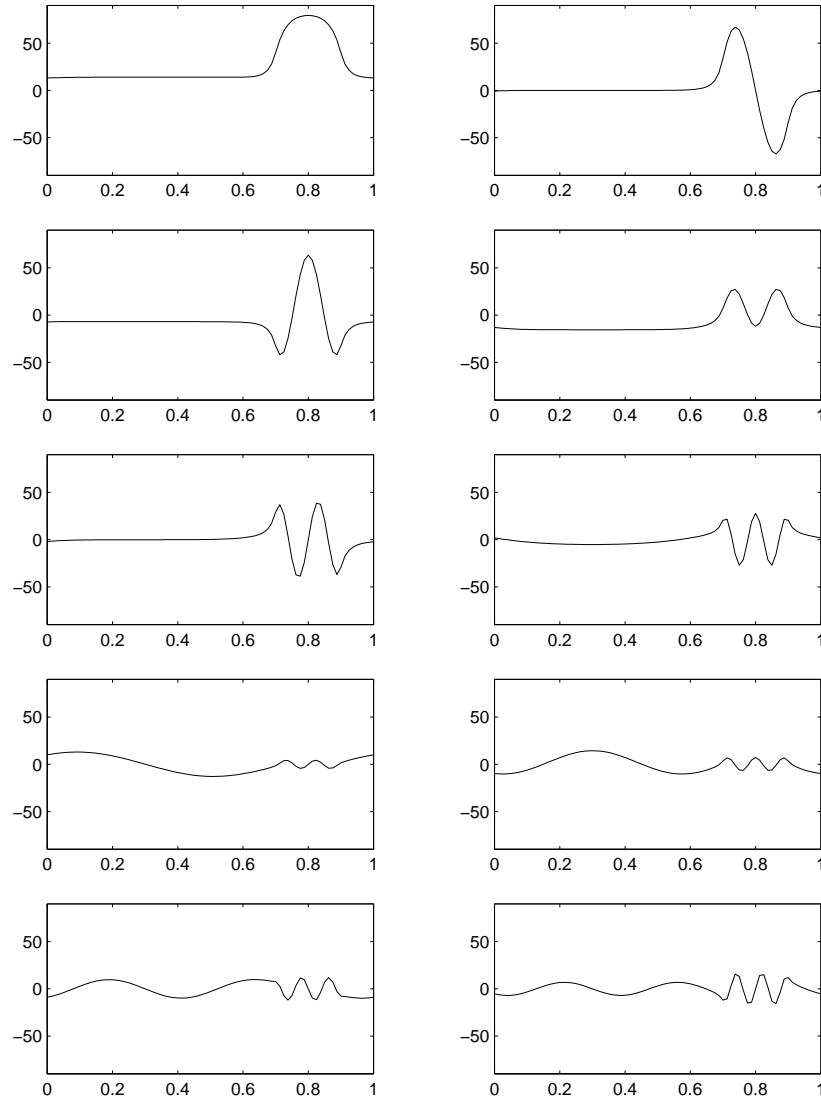


Fig. 3. POD basis functions from functional gain input collection.

state plus a five percent error in that initial condition. For  $w \in X$ , the measurement operator is taken as  $Cw = (\tilde{w}(\bar{x}_1), \tilde{w}(\bar{x}_2), \dots, \tilde{w}(\bar{x}_1))$ , where

$$\tilde{w}(\bar{x}_i) = \frac{1}{2\delta} \int_{\bar{x}_i - \delta}^{\bar{x}_i + \delta} w(x) dx.$$

The operators  $E$  and  $H$  are taken to be zero and the identity, respectively. In Fig. 5, we see that as the sine curve enters the region in which the state weighting operator is

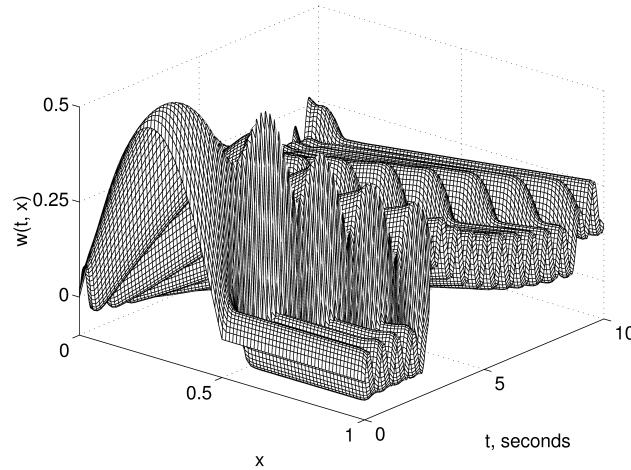


Fig. 4. Open loop (uncontrolled) simulation.

greater ( $x \in [.7, .9]$ ), an increased level of attenuation over the uncontrolled dynamics occurs and is maintained.

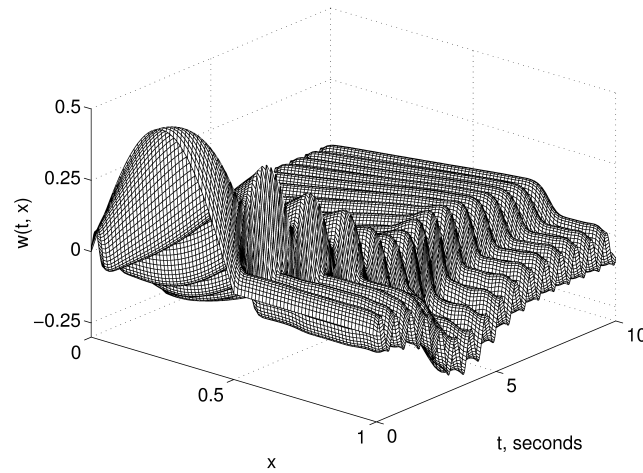


Fig. 5. Full order LQG simulation.

The MinMax controller was computed with  $\theta = .9$ , which was the largest value of  $\theta$  which could be chosen and still solve the algebraic Riccati equations (given the specific weighting matrices and various operators). We note that in our numerical work for this problem, as well as earlier problems, this value is highly sensitive to the machine which is used for the computations. The full order MinMax system simulation is shown in Fig. 6. There is a greater attenuation of the disturbance as compared with the LQG system, as is expected.

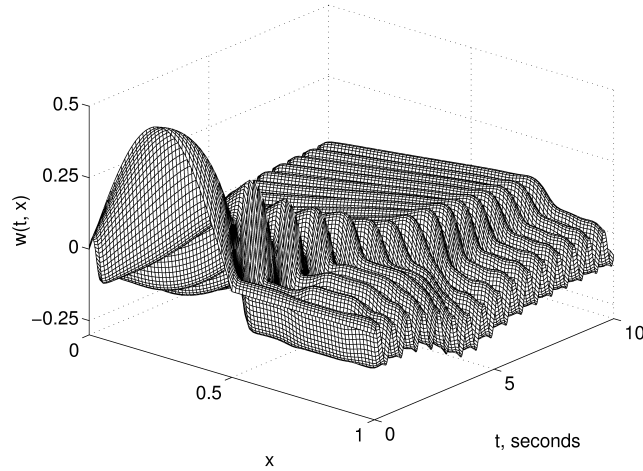


Fig. 6. Full order MinMax simulation.

For reduced order compensators, we use the bases as previously shown: the basis from LQG gains for reduction of the LQG system, and those from the MinMax system for the MinMax reduction. The simulation of the reduced order system with  $\theta = 0$  (i.e., LQG design) that uses the POD basis from the functional gain input collection is shown in Fig. 7. The initial condition for the state estimate is obtained by projecting the initial condition for the full order state estimate onto the reduced basis. Although the performance of the system differs significantly from that of the full order system shown in Fig. 5, the behavior of the system in the region  $x \in [.7, .9]$  does show good attenuation. One could conclude that if that is truly the region in which one cares about disturbance attenuation, one might be satisfied with the “erroneous” behavior in the rest of the spatial domain.

For the MinMax design, we were not able to simulate the systems that we could design. Specifically, we could design a reduced order controller using the previously described methodology in Section 2.3 with  $\theta = .9$ . However, in order to simulate the system,  $\theta$  in the computations of the reduced compensator equation needed to be reduced to .01. Larger values of  $\theta$  led to nonconvergence of the simulations. Recall that in our methodology, the low order  $F^M$  and  $A_C^M$  are obtained by computing  $\Pi^M$  and  $P^M$  using reduced order matrices.

In (Atwell and King, 1999; 2002) we found that in some cases the reduced order systems took longer to simulate than the full order systems did. We point out that this was for smaller systems which were more robust and performed better than the full order systems. This seems to indicate to us that these “nonconventional” systems might require special tools for simulation.

In Fig. 8, the simulation of the reduced order system with  $\theta = .9$  for the calculation of  $K^N$  and POD basis and  $\theta = .01$  for the calculation of  $F^M$  and  $A_C^M$  is shown. Specifically, in computing  $\Pi^N$  in (12) we used  $\theta = .9$ , and in the computation of  $P^M$



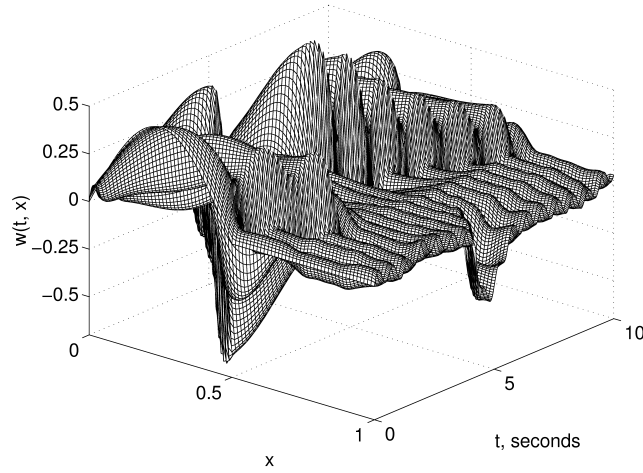


Fig. 7. LQG simulation with reduced order state estimate.

in (13) we used  $\theta = .01$ . We note that although the MinMax parameter in (13) had to be significantly reduced to simulate the resulting system, the performance of this system surpasses that of the reduced LQG system in Fig. 7 and resembles more the behavior of the full order systems. This observation leads us to ponder the efficiency of the MinMax controller when  $\theta = .9$  is used to compute both  $\Pi$  and  $P$  in the reduced order controllers.

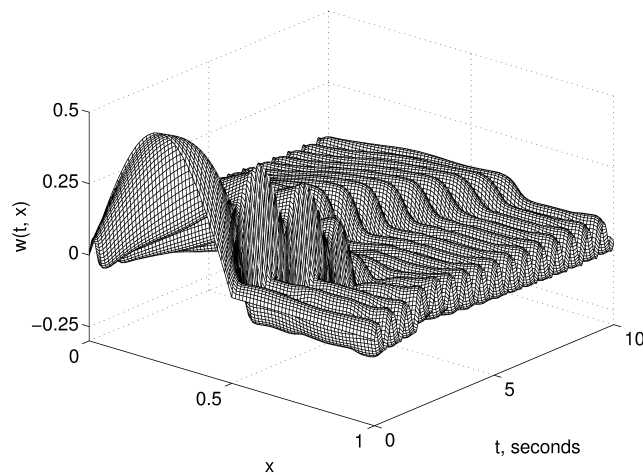


Fig. 8. MinMax simulation with reduced order state estimate.

#### 4.4. Stability Radius

One measure of robustness in a system is the unstructured stability radius (Demmel, 1987), also known as the robust stability radius. This gives the largest unstructured perturbation to the system that can occur without destabilizing the system. It was observed in (Atwell and King, 2002; Burns and King, 1998; King and Sachs, 2000) that reduced order systems can have stability radii larger than those for full order systems. Moreover, in (Atwell and King, 2002), the systems designed using the method applied in this paper had larger stability radii than systems with reduced order controls from “reduce-then-design” methods. Stability radii are listed in Table 1. We note that the reduced order MinMax system with  $\theta = .9$  has the largest stability radii of all systems, while the MinMax system with the two values of  $\theta$  needed for simulation ( $\theta = .9$  and  $\theta = .01$ ) has the smallest. This reinforces the desire to be able to test the performance of the MinMax system designed only with  $\theta = .9$ .

Table 1. Stability margins for various systems.

System	Stability Margin
LQG	.0169
MinMax	.0169
Reduced LQG	.0203
Reduced MinMax, $\theta = .9$	.0222
Reduced MinMax, $\theta = .9, .01$	.0082

## 5. Conclusion

A method for low order compensator-based control design for large scale systems presented in (Atwell and King, 2002) has been applied to Burgers’ equation. The method uses a low order basis obtained through POD as applied to functional gains to design a low order controller. The integral representation of the feedback control law motivates the use of a set of computed functional gains as an input collection to the POD process.

Numerical experiments show that the reduced order controllers perform well (compared with full order controllers) in attenuating disturbances in a target area over which control is to be focused. Erroneous behavior that was observed in the reduced LQG system (as compared to the full order system) is not observed in the MinMax system. Although the reduced order MinMax system could be designed for  $\theta = .9$ , simulations of this system were not possible, even with the stabilized finite element code. This combined with earlier experience with simulations of reduced order systems that were computationally time-consuming leads us to postulate that

new tools for simulation of closed loop control systems merit investigation. Otherwise, robust designs with improved performance over traditional designs might be discounted.

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