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FACTORIZATION OF THE POPOV FUNCTION OF A MULTIVARIABLE LINEAR DISTRIBUTED PARAMETER SYSTEM IN THE NON-COERCIVE CASE: A PENALIZATION APPROACH[†]

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We study the construction of an outer factor to a positive definite Popov function $\Pi(i\omega)$ of a distributed parameter system. We assume that $\Pi(i\omega)$ is a non-negative definite matrix with non-zero determinant. Coercivity is not assumed. We present a penalization approach which gives an outer factor just in the case when there exists any outer factor.

Keywords: linear distributed systems, dissipative systems, factorization, outer factor, Popov function

1. Introduction

Let $\Pi(i\omega)$ be a matrix-valued function defined on the imaginary axis, and let $\Pi(i\omega) = \Pi^*(i\omega) \geq 0$. The 'factorization problem' consists in the following: we wish to find a 'factor' $M(z)$, with suitable properties, such that

$$\Pi(i\omega) = M^*(i\omega)M(i\omega), \tag{1}$$

where * denotes both the conjugate of a complex number and the conjugate of a matrix or an operator.

The first result of this type is the well-known Fejer-Riesz theorem (Riesz and Sz-Nagy, 1955), which states that if $\Pi(i\omega)$ is a (scalar) polynomial, then we can choose a polynomial $M(z)$ which, additionally, does not have zeros in the right half-plane. Extensions to polynomial, hence also to rational matrices are known and have been widely used in systems theory (Anderson 1967; Balakrishnan, 1995; Francis, 1987; Kalman, 1963; Yakubovich, 1973; Youla, 1961).

The fact that we want to stress is the following: let us assume for a moment that $\Pi(i\omega)$ is a rational (square) matrix function, and that $R = \lim_{|\omega|\to+\infty} \Pi(i\omega)$ is an invertible matrix, and hence a positive definite matrix. In this case the factorization

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problem is equivalent to the solution of a special Algebraic Riccati Equation, and hence to the solution of a suitable Quadratic Regulator Problem. This observation has been used in both ways: it has been proposed to solve the Riccati equation in order to solve the factorization problem or, alternatively, to solve the factorization problem in an independent way and then to use the factor in order to solve the quadratic regulator problem (Callier and Winkin, 1990; 1992).

In this paper we are interested in the factorization problem and in the study of the properties of the factor in the case when the function $\Pi(i\omega)$ is *matrix*-valued and *non-rational*, but it is the *Popov function* (equivalently, the *spectral density*) of the pair of a quadratic form and a distributed control system. The control may enter the system as a distributed or a boundary control. In fact, we shall see that even the boundary control case can be reduced to the analysis of an equivalent distributed control system.

Finally, let us discuss the properties to be imposed on the factor. It is clear that if $\Pi(i\omega) \geq 0$, then $M(i\omega) = \sqrt{\Pi(i\omega)}$ is a factor of $\Pi(i\omega)$; but this kind of factor is of little use. We require that the factor admit a holomorphic extension to the right half-plane and, furthermore, that the factor be an *outer factor.* The properties of outer factors are discussed in the next section. A factorization of the form (1) is called a *spectral factorization* if the factor $M(z)$ is an *outer* function.

The factorization problem is empty if $\Pi(i\omega) \equiv 0$, so that we consider the case, where $\Pi(i\omega) \not\equiv 0$. In fact, we assume even more, i.e. that

$$\det \Pi(i\omega) \not\equiv 0.$$

It is known (Nikolski, 1982) that the determinant of an outer square matrix-valued function may be identically zero; but if it is not identically zero, then it is a scalar-valued outer function. Hence it does not have unstable zeros. For this reason, in Section 5 we study the construction of a factor of $\Pi(i\omega)$ without zeros in the right half-plane. In Section 6 we give conditions under which this factor without right half-plane zeros is in fact an outer factor. General properties of outer functions, introduced in Section 2, are used for this purpose. In Section 3 we present some examples of Popov functions of boundary control systems. In fact, in Section 4 we show that, in order to construct a factor, a boundary control system can always be reduced to an equivalent distributed control system. The results and methods of this paper are an adaptation of the results of (Pandolfi, 1998, sec. 5.1).

2. Properties of Outer Matrix-Valued Functions

In this section we recall the properties of an outer matrix-valued function $M(z)$. We recall that H^2_m is the Hilbert space of the Laplace transformations of functions in $L^2(0,+\infty;\mathbb{R}^m)$. Alternatively, $f(z) \in H^2_m$ if and only if

$$\|f\|^2_{H^2} = \sup_{x>0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \|f(x+iy)\|^2 \, \mathrm{d}y < \infty.$$

Definition 1. Let $M(z)$ be an $m \times m$ matrix-valued function which is holomorphic and bounded on the half-plane $\Re e\, z > 0$ (i.e. let it belong to H^{∞}). We say that it is an *outer* function when

$$\mathrm{cl} M(z) H^2_m = H^2_m.$$

The previous definition can be weakened in at least two directions. The first one removes the assumption, implicit in the previous definition, that $M(z)$ is a square matrix whose determinant is not identically zero. The second direction removes the boundedness assumption on $M(z)$. We do not insist on this.

The previous definition applies in both the matrix and in the scalar cases, and it can be useful for actual verification that a given $M(z)$ is outer, see Example 1 in the following section. However, we can give two further tests for a function to be outer. First, we give a test for *scalar*-valued functions which are bounded (Garnett, 1981, p.67):

Theorem 1. *Let $M(z)$ be scalar-valued, holomorphic and bounded in the right half-plane $\Re e\, z > 0$. The function $M(z)$ is outer if and only if there exists $z_0 = x_0 + iy_0$ with $x_0 > 0$ such that*

$$\log \big| M(z_0) \big| = \frac{1}{\pi} \int_{-\infty}^{+\infty} \log \big| M(it) \big| \frac{x_0}{x_0^2 + (y_0 - t)^2} \,\mathrm{d}t. \tag{2}$$

If this is the case, then (2) holds for every $x_0 > 0$, $y_0 \in \mathbb{R}$.

As a consequence of Theorem 1, the integral is finite for every $x_0 > 0$, $y_0 \in \mathbb{R}$. Hence, if $M(z)$ satisfies the assumption of the theorem, then $M(z_0) \neq 0$ for every $z_0 = x_0 + iy_0$, $x_0 > 0$. It may have zeros on the imaginary axis.

In the rational case (and only in this case!) the converse holds: if $M(z)$ is rational and without zeros in $\Re e\, z > 0$, then it is an outer function.

The following test applies to matrix-valued functions (Nikolski, 1982, p. 22):

Theorem 2. *Let $M(z)$ be a bounded square matrix-valued function which is holomorphic in $\Re e\, z > 0$. Let $\det M(z) \not\equiv 0$. In this case the* matrix-*valued function $M(z)$ is outer if and only if the* scalar-*valued function $\det M(z)$ is outer.*

In this paper we give some conditions under which the Popov function of a linear system admits a spectral factorization. Moreover, we present an iterative construction of the factor.

3. Problem Description and Examples of Popov Functions

We consider the following system in a Hilbert space:

$$\dot{x} = Ax + Bu. \tag{3}$$

Here $x \in X$, $u \in U$, and A generates a C_0-semigroup on X *which is exponentially stable.* For the moment let $B \in \mathcal{L}(U,X)$.

We shall assume that U is a *finite-dimensional* space, $U = \mathbb{R}^m$. We introduce the quadratic form

$$F(x,u) = \langle x, Qx\rangle + 2\Re e\,\langle x, Su\rangle + \langle u, Ru\rangle, \tag{4}$$

where $Q = Q^*$, $R = R^*$ and S are bounded linear operators between the appropriate spaces. We are not assuming any positivity on the operators Q and R.

The function $\Pi(i\omega)$ defined below will be called the 'Popov function':

$$u^*\Pi(i\omega)u = F\big((i\omega I - A)^{-1}Bu, u\big). \tag{5}$$

This function is well-defined on the imaginary axis since the semigroup e^{At} is exponentially stable.

We note a special case: if $Q = 0$ and $R = 0$, then $\Pi(i\omega)$ is the real part of the *transfer function*

$$T(i\omega) = S^*(i\omega I - A)^{-1}B.$$

The Popov function is a quadratic form which may or may not be sign definite. It must be non-negative definite for the solutions of important problems (Anderson and Vongpanitlered, 1973; Pandolfi, 1997), and if it is non-negative definite, then it is important to understand its factorization properties.

The previous arguments have been presented under the assumption that $B \in \mathcal{L}(U,X)$ is a distributed control action. It may well be possible that B takes values in a space which is larger than X and, nevertheless, the formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s$$

can be suitably interpreted. This is the case of the classes identified in (Lasiecka and Triggiani, 2000), i.e. the case where $\operatorname{im} B \subseteq (\operatorname{dom}(-A^*)^\gamma)'$, $\gamma < 1$, if A generates a holomorphic semigroup, and the case where

$$\operatorname{im} B \subseteq \big(\operatorname{dom}(-A^*)\big)' \text{ and } \int_0^T \|B^* e^{A^*t}x\|^2\,\mathrm{d}t \le M\|x\|^2 \quad \forall x \in \operatorname{dom} A^*, \ \ T > 0.$$

Large classes of boundary control systems fit in the previous models (Lasiecka and Triggiani, 2000).

Example 1. Let

$$\Pi(i\omega) = \frac{e^2 + 1 - 2e\cos\omega + 2}{1+\omega^2}.$$

It is clear that this function is non-negative, and

$$\Pi(i\omega) = T^*(i\omega)\cdot T(i\omega), \quad T(z) = \frac{e^z + e^{-z}}{1+z}.$$

The function $T(z)$ is the transfer function of the system

$$\begin{cases} x_t = -x_s + x, \quad t > 0, \ s \in (0,1), \\ x(t,0) = u(t), \\ y(t) = \dfrac{1}{e} \displaystyle\int_0^1 x(s)\, \mathrm{d}s. \end{cases} \tag{6}$$

Hence the function $T(z)$ is the transfer function of a boundary control problem.

$\Pi(i\omega)$ is the Popov function that we obtain from the quadratic form $F(x,u) = \langle x, Qx \rangle$, $Q = C^*C$ with $Cx(\cdot) = y$, as described in (6). We note that $\Pi(i\omega) \geq 0$ for every ω and $\lim_{|\omega| \to +\infty} \Pi(i\omega) = 0$.

We presented a special factorization of $\Pi(i\omega)$. We observe that it is a spectral factorization, since $T(z)$ is an outer function. In fact, $T(z)f(z) = g(z)$ can be solved in H^2 for every $g(z)$ which is the Laplace transform of a function $\check{g}(t)$ being differentiable on $\mathbb{R}$ and square integrable, with compact support in $(0, +\infty)$. The functions $\check{g}(t)$ with these properties are dense in $L^2(0, +\infty)$ so that their Laplace transforms $g(z)$ are dense in H^2. ♦

The next examples are from (Pandolfi, 1999a). The first one shows that the Popov function may be *coercive* even if the operator R in the quadratic form is *negative* definite, while the last example is an example of a matrix-valued Popov function, which is easily derived from a system described by partial differential equations.

Example 2. We consider a system described by

$$x_t = -x_\theta, \quad 0 < \theta < 1, \quad t > 0, \quad x(t,0) = u(t).$$

This system is exponentially stable since the free evolution is zero for $t > 1$.

The functional $F(x,u)$ which we associate to this system is

$$F(x,u) = \|x(\cdot)\|^2_{L^2(0,1)} - \alpha |u|^2,$$

so that

$$J(x_0; u) = \int_0^{+\infty} \Big\{ \|x(t,\cdot)\|^2_{L^2(0,1)} - \alpha \big|u(t)\big|^2 \Big\} \, \mathrm{d}t.$$

If $x(0,\theta) \equiv 0$, then

$$\hat{x}(z,\theta) = e^{-z\theta} \hat{u}(z).$$

Hence

$$\langle u, \Pi(i\omega) u \rangle = [1 - \alpha] \cdot |u|^2.$$

This is non-negative for each $\alpha \leq 1$ in spite of the fact that $R = -\alpha$ can be negative. ♦

We observe that in the previous example the limit of $\Pi(i\omega)$ for $|\omega| \to +\infty$ exists, but it is not equal to the operator R of the quadratic form $F(x,u)$. In contrast with this, in the 'parabolic' case, i.e. when A generates a holomorphic semigroup and $\operatorname{im} B \subseteq (\operatorname{dom}(-A^*)^\gamma)'$, we have

$$R = \lim_{\omega\to+\infty} \Pi(i\omega).$$

Hence in the parabolic case $R \geq 0$ is implied by $\Pi(i\omega) \geq 0$ for large ω.

Finally, we consider a third example, taken from Pandolfi (1999a).

Example 3. We consider the acoustics equations (on the interval $0 < s < 1$)

$$\begin{cases} x_t = -x_s, & x(t,0) = u(t) + ay(t,0), \\ y_t = y_s, & y(t,1) = v(t). \end{cases} \tag{7}$$

Let

$$F\big(x(\cdot), y(\cdot), u, v\big) = |u|^2 + |v|^2 - k\int_0^1 \big|x(s)\big|^2\,\mathrm{d}s, \quad k > 0.$$

We compute the Popov function: we excite the system with the signal $u(t) = e^{zt}u_0$, $v(t) = e^{zt}v_0$, and search for solutions of the form $x(t,s) = e^{zt}x_0(s)$, $y(t,s) = e^{zt}y_0(s)$. It is easily seen that $x_0(s)$ and $y_0(s)$ are given by $y_0(s) = e^{z(s-1)}$, $x_0(s) = e^{-zs}[ae^{-z}v_0 + u_0]$ so that the Popov function is

$$\begin{aligned}\Pi(i\omega) &= u_0^2 + v_0^2 - k\{ae^{-i\omega}\overline{v}_0 + \overline{u}_0\}\cdot\{ae^{i\omega}v_0 + u_0\} \\ &= |u_0|^2 + |v_0|^2 - k\big\{a^2|v_0|^2 + |u_0|^2 + 2a\cos\omega u_0 v_0\big\}.\end{aligned}$$

It is easily seen that $\Pi(i\omega)$ is positive definite if and only if k satisfies the following inequality:

$$0 < k < k_0 = \inf_\omega \frac{(1+a^2) - \sqrt{(1+a^2)^2 - 4a^2\sin^2\omega}}{2a^2\sin^2\omega}.$$

It is clear that the right-hand side is non-negative. In order for a number $k > 0$ to satisfy the previous inequality, we must ascertain that the infimum is strictly positive. This is easily seen by noting that the function

$$\frac{(1+a^2) - \sqrt{(1+a^2)^2 - 4a^2\sin^2\omega}}{2a^2\sin^2\omega}$$

is periodic in ω and continuous, for $\omega \neq n\pi$, and that its limit for $\omega \to n\pi$ is $1/[1+a^2] > 0$ for each integer n. Consequently, there exists a *positive* number k_0 such that if $k \in (0, k_0)$, we are in the *coercive* case; if $k = k_0$, the Popov function is non-negative. If $k > k_0$, the Popov function is not sign-definite. ♦

The examples above are examples of boundary control systems. It is by now well understood that the factorization problem for systems with boundary controls can be reduced to the solution of a factorization problem for a system with *distributed control action*. This is recalled in the next section, see (Pandolfi, 1994; 1995; 1998; 1999a; 1999b) for more details.

4. From Boundary to Distributed Control Actions

A general model for the analysis of boundary control systems, now widely used, was proposed by Fattorini (1968). It is based on the following variation-of-constants formula:

$$x(t) = e^{At}x_0 - A\int_0^t e^{A(t-s)}Du(s)\,\mathrm{d}s. \tag{8}$$

In spite of the operator A in front of the integral, this formula makes sense in X for large classes of systems (Lasiecka and Triggiani, 2000).

For simplicity, we assume that e^{At} is exponentially stable. (See (Pandolfi, 1998) for the stabilizable case.) Let $u(\cdot)$ be continuously differentiable and $x(0) - Du(0) \in \operatorname{dom} A$. Then $\xi(t) = x(t) - Du(t)$ is a classical solution to

$$\dot{\xi} = A\xi - D\dot{u}, \quad \xi_0 = \xi(0) = x(0) - Du(0), \tag{9}$$

and conversely.

We 'augment' (9) and we study the system

$$\begin{cases} \dot{\xi} = A\xi - Dv, \\ \dot{u} = v. \end{cases} \tag{10}$$

Here we consider formally $v(\cdot)$ as a new 'input', see (Pandolfi, 1994; 1995). Moreover, we note that, e^{At} being exponentially stable, it is possible to stabilize the previous system with the simple feedback $v = -u$.

We associate the following quadratic form

$$F(\xi + Du, u) \tag{11}$$

with the augmented system (10) (the quadratic form $F(x, u)$ is in (4)). This quadratic form does not depend explicitly on the new input $v(\cdot)$: it is a quadratic form of the state, which is now $\Xi = [\xi, u]$.

It is easy to expect that the factorization problems for the original and augmented systems are related. In order to see this, we apply the stabilizing feedback $v = -u+\nu$ (a transformation which does not affect the problem), and we write down the Popov function for the stabilized augmented system. A simple computation shows that the Popov function of the augmented system is

$$P(i\omega) = \frac{\Pi(i\omega)}{1+\omega^2}, \tag{12}$$

where $\Pi(i\omega)$ is the Popov function of the original system.

Hence, *the transformations outlined above from the original to the augmented system do not affect the positivity of the Popov function. Moreover, if $\omega^s\Pi(i\omega)$ is bounded from below, then so is $\omega^{s+2}P(i\omega)$*. This observation is crucial in the analysis of the quadratic regulator problem in the non-coercive case and in the study of the factorization problem. Owing to this observation, from now on we shall study the factorization problem under the further, nonrestrictive assumption that the input acts as a distributed control.

5. Factorization: A Factor without Right Half-Plane Zeros

We recapitulate: our system is described by (3) with $B \in \mathcal{L}(U,X)$ and $F(x,u) = \langle x, Qx\rangle$, so that

$$\big\langle u, \Pi(i\omega)u\big\rangle = \big\langle (i\omega I - A)^{-1}Bu, Q(i\omega I - A)^{-1}Bu\big\rangle.$$

We recall that the factorization problem for a boundary control system can always be reduced to the factorization problem for a *distributed* control system, so that the assumption $B \in \mathcal{L}(U,X)$ is not restrictive in this context.

By assumption, we have

$$\Pi(i\omega) \geq 0 \quad \forall \omega \in \mathbb{R}\ quad\text{and} \quad \Pi(i\omega) \not\equiv 0.$$

The approach that we adopt to the factorization is via *penalization.* This approach was used in (Pandolfi, 1998) in the scalar case.

We consider the quadratic forms $F_n(x,u) = \langle x, Qx\rangle + \|u\|^2/n$ and the associated Popov functions $\Pi_n(i\omega) = \Pi(i\omega) + 1/n$. It is known (Louis and Wexler, 1991) that the following Algebraic Riccati equation admits a maximal solution $P = P^*$ such that $A - nBB^*P$ generates an exponentially stable semigroup

$$\langle Ax, Py\rangle + \langle Px, Ay\rangle + \langle x, Qy\rangle - n\langle B^*Px, B^*Py\rangle = 0, \tag{13}$$

and the function

$$M_n(z) = \sqrt{n}B^*P_n(zI - A)^{-1}B + 1/\sqrt{n}$$

is a spectral factor of $\Pi_n(i\omega)$,

$$\Pi_n(i\omega) = M_n^*(i\omega)M_n(i\omega).$$

In fact, it has another property: its inverse $M_n^{-1}(z)$ is bounded in the right half-plane:

$$M_n^{-1}(z) = \sqrt{n} - n^{3/2}B^*P_n(zI - A - \sqrt{n}BB^*P_n)^{-1}B.$$

The matrix function $M_n^{-1}(z)$ is bounded in the right half-plane because P_n is a *stabilizing* solution of the Riccati equation. See in particular Lemmas 2.6–2.8 in (Louis and Wexler, 1991) for the previous properties.

We note explicitly that the factor $M_n(z)$ has a realization in terms of the same operators A and B which appear in the description of the system. Our strategy is now to study the limit of the sequence $\{M_n(z)\}$. Due to the factor $\sqrt{n}$, it is not at all clear that $\{M_n(z)\}$ is a bounded sequence. In fact, we prove that this is indeed the case. This can be proved as follows: we note that $M_n(z)$ is an H^∞ function, and hence

$$\sup_{\Re e\, z>0} \|M_n(z)\|^2 = \sup_{\omega\in\mathbb{R}} \|M_n(i\omega)\|^2 = \sup_{\omega\in\mathbb{R}} \|\Pi(i\omega) + 1/n\| \leq \text{const.}$$

This proves the boundedness of $M_n(z)$ in $\Re e\, z > 0$, uniformly in n.

If a *semigroup is holomorphic (and exponentially stable)*, we get more. We observe that, under this assumption, in the previous proof we can replace the real axis with a line $\Re e\, z = -\sigma$, $\sigma > 0$. Hence, in the holomorphic case we get boundedness even in a half-plane $\Re e\, z > -\sigma$. Now we add and subtract $i\omega$ to the Riccati equation, and we replace x and y with $(A - i\omega)^{-1}B$. We thus obtain

$$\begin{aligned} n \cdot B^*(A^* + i\omega I)^{-1} & PBB^*P(A - i\omega)^{-1}B \\ &= B^*(A^* + i\omega)^{-1}PB + B^*P(A - i\omega)^{-1}B \\ &\quad + B^*(A^* + i\omega)^{-1}Q(A - i\omega)^{-1}B. \end{aligned}$$

This equality can be analitically extended from the imaginary axis to a sector

$$\tilde{\mathcal{S}} = \{z : \pi/2 - \theta \leq |\arg z| \leq \pi/2 + \theta\},$$

where $0 < \theta < \pi/2$. On this set the following equality holds:

$$\begin{aligned} n\|B^*P_n & (zI - A)^{-1}B\|^2 \\ &= B^*(A^* + z)^{-1}Q(A - zI)^{-1}B + B^*(A^* + zI)^{-1}PB \\ &\quad + B^*P(A - zI)^{-1}B. \end{aligned}$$

It is well-known that there exists a number c such that $\|P_n\| < c$ for every n. Moreover, $\|(zI - A)^{-1}\| < c$ and $\|(zI + A^*)^{-1}\| < c$ are bounded in $\tilde{\mathcal{S}}$. Hence we have the boundedness of $\sqrt{n}B^*P_n(zI - A)^{-1}$, uniformly in n and in z in $\tilde{\mathcal{S}}$. As we already know the boundedness for $\Re e\, z > -\sigma$, the following result can be formulated:

Theorem 3. *There exists a constant μ such that $\|M(z)\| < \mu$ for each z, $\Re e\, z > 0$. In the holomorphic case the inequality can be extended to the sector $\mathcal{S} = \{z : |\arg z| \leq \pi/2 + \theta\}$.*

Now we use again the fact that the input u is *finite dimensional.* In this case $M_n(z)$ is a matrix: in fact, it is a square matrix. We noted that its inverse exists so that its determinant is non-zero for $\Re e\, z > -\sigma'$, where $\sigma' > 0$ is any number such that

$$\|e^{(A - BB^*P)t}\| < \text{const} \cdot e^{-\sigma' t}$$

(both in the parabolic and hyperbolic cases).

We can apply the Montel theorem to the entries of $M_n(z)$ and obtain the existence of a subsequence, still denoted by $\{M_n(z)\}$, which converges to a certain matrix-valued function $M_0(z)$ uniformly on compact sets of $\Re e\, z > -\sigma$.

This matrix $M_0(z)$ is holomorphic and bounded for $\Re e\, z > 0$ in the general case, and $\Re e\, z > -\sigma$, $\sigma > 0$, (and even in $\mathcal{S}$) in the holomorphic case.

Theorem 4. *Let $M_0(z)$ be the function constructed above. The function* $\det M_0(z)$ *does not have unstable zeros.*

Proof. In fact, we have

$$\Pi(i\omega) = \lim_n \Pi_n(i\omega) = \lim_n M_n^*(i\omega) M_n(i\omega) = M_0^*(i\omega) M_0(i\omega).$$

Hence $M_0(i\omega)$ is a factor of $\Pi(i\omega)$ which is not identically zero, since $\Pi(i\omega)$ is not identically zero. If z_0, with $\Re e\, z_0 > -\sigma$ ($\sigma > 0$ in the parabolic case, $\sigma = 0$ in the hyperbolic case), is a zero of $M_0(z)$, then the Hurwitz theorem implies the existence of a sequence $\{z_n\}$ such that $z_n \to z_0$ and $\det M_n(z_n) = 0$. This is not possible since each matrix $M_n(z)$ is invertible in a neighborhood of z_0. ■

6. A Condition for Spectral Factorization

The factor $M_0(z)$ constructed in the previous section is a candidate for an outer factor. In fact, we prove that it is an outer factor *provided that an outer factor exists!*

Theorem 5. *The factor $M_0(z)$ is outer if the function*

$$\frac{1}{1+\omega^2} \log \big[\det \Pi(i\omega) \big]$$

is integrable.

Proof. We assume that $\det \Pi(i\omega) \not\equiv 0$, so that we also have $\det M_0(i\omega) \not\equiv 0$. Hence we must only prove that $\det M_0(z)$ is a *scalar* outer function, i.e. that it satisfies (2). We prove that (2) holds when $z_0 = 1$. We know that $M_n(z)$ is an outer function, so that the following holds:

$$\log |\det M_n(1)| = \frac{1}{\pi} \int_{-\infty}^{+\infty} \log |\det M_n(it)| \frac{x_0}{1+t^2} \, \mathrm{d}t.$$

The limit of the left-hand side for $n \to +\infty$ is $\log |\det M_0(1)|$ and $M_0(z)$ satisfies condition (2) if we can prove that we can exchange the limit for $n \to +\infty$ and the integral.

The function $M_n(it)$ is bounded from above, but it tends to zero at infinity so that the logarithm is unbounded. Yet we get

$$\big[\det M_n(it) \big]^2 = \det \Pi_n(it) \geq \det \Pi(it).$$

In fact, if $P \geq Q$ and $\det Q \neq 0$, we have $Q^{-1/2} P Q^{-1/2} \geq I$, i.e. $1 \leq \det(Q^{-1/2} P Q^{-1/2}) = \det P / \det Q$. Hence we have the following inequalities:

$$\frac{\log \det \Pi(i\omega)}{2(1+\omega^2)} \leq \frac{\log |\det M_n(i\omega)|}{1+\omega^2} \leq \frac{\mathrm{const}}{1+\omega^2}.$$

The functions on both the sides of the inequalities are integrable on $(-\infty, +\infty)$ so that we can apply the dominated-convergence theorem to the sequence $\big\{ \frac{1}{1+t^2} \log |\det M_n(it)| \big\}$ and we see that we can exchange the limit and the integral. ■

Now we discuss the previous theorem. If $\Pi(i\omega)$ admits an outer factor $N(z) \in H^\infty$ and $\det \Pi(i\omega) \not\equiv 0$, then $\det N(i\omega) \not\equiv 0$ so that $\det N(z)$ is an outer factor of $\det \Pi(i\omega)$. It is known (Rosenblum and Rovnyak, 1985) that an outer factor of $\det \Pi(i\omega)$ exists if and only if

$$\Pi(i\omega) = \Pi^*(i\omega) \geq 0, \qquad \frac{\log \det \Pi(i\omega)}{1+\omega^2} \in L^1(\mathbb{R}). \tag{14}$$

Consequently, if $\det \Pi(i\omega) \not\equiv 0$ and there exists any outer factor, then the assumption in the theorem is satisfied and we can conclude that $M_0(z)$ *is an outer factor!*

An important case in which the condition in the previous theorem is satisfied is the one when $\Pi(z)$ is holomorphic in a neighborhood of the imaginary axis with $\det \Pi(i\omega) \not\equiv 0$, so that $\log \det \Pi(i\omega)$ is locally integrable and the following condition holds:

$$\Pi(i\omega) \geq \frac{c}{1+|\omega|^\alpha}, \quad |\omega| \gg 0,$$

where $c > 0$ and $\alpha \geq 0$. In particular, the case $\alpha = 0$ gives the condition (5) of (Callier and Winkin, 1999).

Remark 1. We note that condition (14) may hold even if $\det \Pi(i\omega)$ has zeros on the imaginary axis.

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