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CIRCLE CRITERION AND BOUNDARY CONTROL SYSTEMS IN FACTOR FORM: INPUT–OUTPUT APPROACH

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A circle criterion is obtained for a SISO Lur'e feedback control system consisting of a nonlinear static sector-type controller and a linear boundary control system in factor form on an infinite-dimensional Hilbert state space H previously introduced by the authors (Grabowski and Callier, 1999). It is assumed for the latter that (a) the observation functional is infinite-time admissible, (b) the factor control vector satisfies a compatibility condition, and (c) the transfer function belongs to $\mathrm{H}^{\infty}(\Pi^{+})$ and satisfies a frequency-domain inequality of the circle criterion type. We also require that the closed-loop system be well-posed, i.e. for any initial state $x_0 \in \mathrm{H}$ the truncated input and output signals u_T, y_T belong to $\mathrm{L}^2(0,T)$ for any $T > 0$. The technique of the proof adapts Desoer-Vidyasagar's circle criterion method (Desoer and Vidyasagar, 1975, Ch. 3, Secs. 1 and 2, pp. 37–43, Ch. 5, Sec. 2, pp. 139–142 and Ch. 6, Secs. 3 and 4, pp. 172–174), and uses the input-output map developed by the authors (Grabowski and Callier, 2001). The results are illustrated by two transmission line examples: (a) that of the loaded distortionless $RLCG$ type, and (b) that of the unloaded RC type. The conclusion contains a discussion on improving the results by the loop-transformation technique.

Keywords: infinite-dimensional control systems, semigroups, input-output relations

1. Introduction

In a Hilbert space H with a scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$ consider the SISO model of boundary control in factor form (Grabowski and Callier, 1999),

$$\begin{cases} \dot{x}(t) = A\big[x(t) + u(t)d\big], \\ y = c^{\#}x. \end{cases} \tag{1}$$

We assume that $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates a linear exponentially stable (**EXS**), C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H, $d \in \mathrm{H}$ is a factor control vector,

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$u \in \mathrm{L}^2(0,\infty)$ is a scalar control function, y is a scalar output defined by an A-bounded linear observation functional $c^{\#}$. The restriction of $c^{\#}$ to $D(A)$ is representable as $c^{\#}|_{D(A)} = h^*A$ for some $h \in \mathrm{H}$ (for $q \in \mathrm{H}$, q^* denotes the bounded linear functional $q^*x := \langle x, q\rangle_{\mathrm{H}}$, $x \in \mathrm{H}$).

Define two operators:

$$V \in \mathbf{L}\big(\mathrm{H}, \mathrm{L}^2(0,\infty)\big), \quad (Vx)(t) := h^*S(t)x$$
$$W \in \mathbf{L}(\mathrm{L}^2(0,\infty), \mathrm{H}), \quad Wu := \int_0^\infty S(t)\,du(t)\,\mathrm{d}t.$$

Recall that L and $R = L^*$,

$$Lf = f', \qquad D(L) = \mathrm{W}^{1,2}(0,\infty),$$
$$Rf = -f', \qquad D(R) = \big\{f \in \mathrm{W}^{1,2}(0,\infty) : \ f(0) = 0\big\},$$

are the generators of the semigoups of *left-* and *right-shifts* on $\mathrm{L}^2(0,\infty)$, respectively.

Definition 1. The observation functional $c^{\#}$ is called *admissible* if the *observability operator*

$$P = VA, \quad D(P) = D(A)$$

is bounded. The factor control vector $d \in \mathrm{H}$ is called *admissible* if

$$\mathrm{Range}(W) \subset D(A).$$

From (Grabowski and Callier, 1999, Thm. 4.1, p. 100; 2001, eqn. (1.8)) the following result may be concluded:

Lemma 1. *If $c^{\#}$ is admissible, then $\overline{P}$, the closure of P, has the form*

$$\mathrm{Range}(V) \subset D(L), \quad \overline{P} = LV,$$

while if d is admissible, then the r e a c h a b i l i t y o p e r a t o r $Q = AW$ is in $\mathbf{L}(L^2(0,\infty), H)$.

Furthermore, from (Grabowski and Callier, 2001, Sec. 3) it follows that if the *compatibility condition*

$$d \in D(c^{\#}) \tag{2}$$

holds, then the function

$$\hat{g}(s) := sc^{\#}(sI - A)^{-1}d - c^{\#}d = sh^*A(sI - A)^{-1}d - c^{\#}d \tag{3}$$

is well-defined and analytic on the complex right half-plane $\Pi^+ = \{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$.

If, apart from (2), $c^{\#}$ is admissible, then:

(i) $\hat{g}(s) = s\big(\widehat{\overline{P}d}\big)(s) - c^{\#}d$ with $\widehat{\overline{P}d} \in \mathrm{H}^{\infty}(\Pi^{+}) \cap \mathrm{H}^{2}(\Pi^{+})$, where $\mathrm{H}^{\infty}(\Pi^{+})$ denotes the Banach space of analytic functions f on Π^{+}, equipped with the norm $\|f\|_{\mathrm{H}^{\infty}(\Pi^{+})} = \sup_{s\in\Pi^{+}} |f(s)|$, and $\mathrm{H}^{2}(\Pi^{+})$ is the Hardy space of functions f analytic on Π^{+} such that $\sup_{\sigma>0} \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^{2}\, \mathrm{d}\omega < \infty$, where $f(j\omega) := \lim_{\sigma\to 0+} f(\sigma + j\omega)$ exists for almost all $\omega \in \mathbb{R}$. The space $\mathrm{H}^{2}(\Pi^{+})$ is unitarily isomorphic with $\mathrm{L}^{2}(0,\infty)$ through the normalized Laplace transform. To be more precise,

$$\langle f, g\rangle_{\mathrm{L}^{2}(0,\infty)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(j\omega)\overline{\hat{g}(j\omega)}\, \mathrm{d}\omega,$$

where $\hat{f}$ and $\hat{g}$ are the Laplace transforms of f and g, respectively. The latter facts are fundamental ingredients of the Paley-Wiener theory (Duren, 1970, Ch. 11).

(ii) The convolution operator K with kernel $\overline{P}d$, i.e. $Ku := \overline{P}d \star u$, belongs to $\mathbf{L}(\mathrm{L}^{2}(0,\infty))$, and it maps the domain of R into itself.

Finally, by (Grabowski and Callier, 2001, Thm. 4.1) the following result holds:

Lemma 2. *If (2) is satisfied, $c^{\#}$ is admissible and*

$$\hat{g} \in H^{\infty}(\Pi^{+}), \tag{4}$$

then the input-output operator F,

$$F = -KR - c^{\#}dI, \quad D(F) = D(R),$$

is bounded and its closure $\overline{F}$ is given by

$$\mathrm{Range}(K) \subset D(R), \quad \overline{F} = -RK - c^{\#}dI.$$

Moreover, $\hat{g}$ is then the transfer function of the system (1).

2. Additional Properties of the Input-Output Map

Definition 2. The operator $H \in \mathbf{L}(\mathrm{L}^{2}(0,\infty))$ is called *causal* or *nonanticipative* if

$$(Hu_{T})_{T} = (Hu)_{T}, \quad \forall u \in \mathrm{L}^{2}(0,\infty),$$

where u_{T} denotes the *truncation* of u at time $T > 0$,

$$u_{T}(t) = \begin{cases} u(t) & \text{if} \quad t < T, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3. *The closure* $\overline{F}$ *of the input-output map* F *has the following properties:*

(i) $\overline{F}$ *is causal.*

(ii) If $k_1 < 0$ *and* $k_2 > 0$ *are such that*

$$\inf_{\omega\in\mathbb{R}} \Big[k_1 k_2 \left|\hat{g}(j\omega)\right|^2 - (k_1+k_2)\operatorname{Re}\hat{g}(j\omega) + 1\Big] \geq \delta > 0, \tag{5}$$

then for any $u \in L^2(0,\infty)$ *we have*

$$\big\langle (I - k_1\overline{F})u, (I - k_2\overline{F})u \big\rangle_{L^2(0,\infty)} \geq \delta \left\|u\right\|^2_{L^2(0,\infty)}, \tag{6}$$

and the operator $(I - k_1F^*)(I - k_2\overline{F})$ *is strictly passive (Desoer and Vidyasagar, 1975, p. 173), i.e.*

$$\big\langle \big[(I - k_1\overline{F})u_T\big]_T, \big[(I - k_2\overline{F})u_T\big]_T \big\rangle_{L^2(0,T)} \geq \delta \left\|u_T\right\|^2_{L^2(0,T)}, \quad \forall T > 0. \tag{7}$$

Proof. As for (i), observe that

$$\big(\overline{F}u\big)_T = \left\{\begin{array}{ll} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t \overline{P}d(t-\tau)u(\tau)\,\mathrm{d}\tau, & t<T \\ 0, & t>T \end{array}\right\} - c^{\#}du_T,$$

$$\begin{aligned}\overline{F}u_T &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \overline{P}d(t-\tau)\left\{\begin{array}{ll} u(\tau), & \tau<T \\ 0, & \tau>T\end{array}\right\}\mathrm{d}\tau - c^{\#}du_T \\ &= \left\{\begin{array}{ll} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t \overline{P}d(t-\tau)u(\tau)\,\mathrm{d}\tau, & t<T \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^T \overline{P}d(t-\tau)u(\tau)\,\mathrm{d}\tau, & t>T \end{array}\right\} - c^{\#}du_T,\end{aligned}$$

so that $\big(\overline{F}u_T\big)_T = \big(\overline{F}u\big)_T$ and $\overline{F}$ is nonanticipative.

As for (ii), applying the Paley-Wiener theory, for any $u \in \mathrm{L}^2(0,\infty)$ we get

$$\begin{aligned}&\big\langle (k_1\overline{F} - I)u, (k_2\overline{F} - I)u \big\rangle_{\mathrm{L}^2(0,\infty)} \\ &\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty} \big[k_1\hat{g}(j\omega) - 1\big]\hat{u}(j\omega)\overline{\big[k_2\hat{g}(j\omega) - 1\big]\,\hat{u}(j\omega)}\,\mathrm{d}\omega \\ &\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[k_1k_2\left|\hat{g}(j\omega)\right|^2 - (k_1+k_2)\operatorname{Re}\hat{g}(j\omega) + 1\Big]\left|\hat{u}(j\omega)\right|^2\,\mathrm{d}\omega \\ &\quad\geq \inf_{\omega\in\mathbb{R}}\Big[k_1k_2\left|\hat{g}(j\omega)\right|^2 - (k_1+k_2)\operatorname{Re}\hat{g}(j\omega) + 1\Big]\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{u}(j\omega)\right|^2\,\mathrm{d}\omega \\ &\quad\geq \delta\left\|u\right\|^2_{\mathrm{L}^2(0,\infty)}.\end{aligned}$$

By (6) with $u = u_T$ we have

$$\left\langle (I-k_1\overline{F})u_T, (I-k_2\overline{F})u_T \right\rangle_{\mathrm{L}^2(0,\infty)} \geq \delta \left\| u_T \right\|^2_{\mathrm{L}^2(0,\infty)} = \delta \left\| u_T \right\|^2_{\mathrm{L}^2(0,T)}, \quad \forall T > 0, \tag{8}$$

but

$$\begin{aligned}
&\left\langle (I-k_1\overline{F})u_T, (I-k_2\overline{F})u_T \right\rangle_{\mathrm{L}^2(0,\infty)} \\
&\qquad = \left\| u_T \right\|^2_{\mathrm{L}^2(0,\infty)} - k_1 \left\langle \overline{F}u_T, u_T \right\rangle_{\mathrm{L}^2(0,\infty)} \\
&\qquad\quad - k_2 \left\langle u_T, \overline{F}u_T \right\rangle_{\mathrm{L}^2(0,\infty)} + k_1k_2 \left\| \overline{F}u_T \right\|^2_{\mathrm{L}^2(0,\infty)} \\
&\qquad = \left\| u_T \right\|^2_{\mathrm{L}^2(0,T)} - k_1 \left\langle \left(\overline{F}u_T \right)_T, u_T \right\rangle_{\mathrm{L}^2(0,T)} \\
&\qquad\quad - k_2 \left\langle u_T, \left(\overline{F}u_T \right)_T \right\rangle_{\mathrm{L}^2(0,T)} + k_1k_2 \left\| \overline{F}u_T \right\|^2_{\mathrm{L}^2(0,\infty)}.
\end{aligned}$$

Since

$$\left\| \overline{F}u_T \right\|^2_{\mathrm{L}^2(0,\infty)} \geq \left\| \left(\overline{F}u_T \right)_T \right\|^2_{\mathrm{L}^2(0,T)}$$

and $k_1k_2 < 0$, we obtain the estimate

$$k_1k_2 \left\| \overline{F}u_T \right\|^2_{\mathrm{L}^2(0,\infty)} \leq k_1k_2 \left\| \left(\overline{F}u_T \right)_T \right\|^2_{\mathrm{L}^2(0,T)}.$$

Finally,

$$\left\langle (I-k_1\overline{F})u_T, (I-k_2\overline{F})u_T \right\rangle_{\mathrm{L}^2(0,\infty)} \leq \left\langle \left[(I-k_1\overline{F})u_T \right]_T, \left[(I-k_2\overline{F})u_T \right]_T \right\rangle_{\mathrm{L}^2(0,T)},$$

and (7) follows from (8). ∎

Remark 1. A similar result is found in (Desoer and Vidyasagar, 1975, Ex. 1, p. 174). Since $\overline{F}$ is causal, its adjoint operator F^* is *anticausal* (Desoer and Vidyasagar, 1975, Lemma 9.1.8, p. 201).

Remark 2. The frequency-domain inequality (5) means geometrically that the plot of the transfer function $\hat{g}(j\omega)$ is located strictly inside the circle with centre at $\left(k_1^{-1} + k_2^{-1}\right)/2$ and radius $\left(k_2^{-1} - k_1^{-1}\right)/2$.

3. The Circle Criterion

For the feedback system given in Fig. 1, we assume the following:

(A1) The linear part of the feedback system from u to y is our boundary control system in factor form, where A generates a linear **EXS** semigroup $\{S(t)\}_{t\geq 0}$ on H, $c^\#$ is admissible, $d \in D(c^\#)$ and $\hat{g} \in \mathrm{H}^\infty(\Pi^+)$. Hence, for any $x_0 \in \mathrm{H}$, its input-output equation in $\mathrm{L}^2(0,T)$ for any $T > 0$ is given by

$$y_T = \left(\overline{P}x_0 \right)_T + \left(\overline{F}u \right)_T = \left(\overline{P}x_0 \right)_T + \left(\overline{F}u_T \right)_T. \tag{9}$$

The last equality is due to the causality of $\overline{F}$.

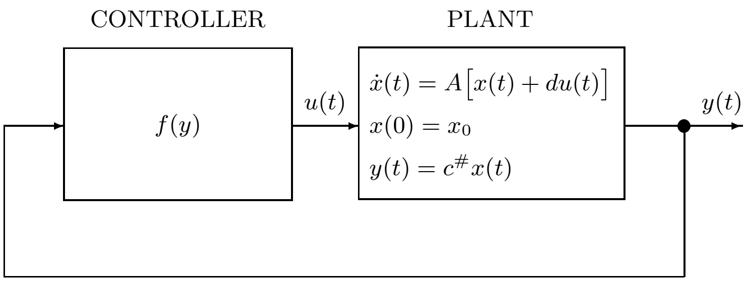

Fig. 1. The Lur'e control system.

(A2) The nonlinearity $f(y)$ is assumed to lie in the sector $[k_1, k_2]$, i.e. with y and $f(y)$ in $\mathbb{R}$,

$$\big[f(y) - k_1 y\big]\big[f(y) - k_2 y\big] \le 0.$$

Moreover, f is supposed to be piecewise continuous with $f(0) = 0$ and f continuous at 0.

(A3) The extended input-output map $\overline{F}$ of the boundary control system satisfies the frequency-domain inequality (5).

(A4) The feedback system given in Fig. 1 is *well-posed*, i.e. for any $x_0 \in \mathrm{H}$ the truncated loop signals u_T and y_T belong to $\mathrm{L}^2(0, T)$ for any $T > 0$.

Remark 3. For reasons of mathematical elegance, the usual sign inversion is absent in the feedback loop of Fig. 1. The standard setting of the circle criterion as in, e.g. (Vidyasagar, 1993, Sec. 5.6, Thm. 37, Case (iii), p. 227) is recovered by replacing $f(y)$ by $-f(y)$, and k_1 and k_2 by $-k_2$ and $-k_1$, respectively.

Theorem 1. (Version of the circle criterion) *Consider the feedback system of Fig. 1, where assumptions (A1)÷(A4) hold. Then*

(i) u *and* $y \in L^2(0, \infty)$.

(ii) If, in addition, $d \in H$ *is an admissible factor control vector, then the null equilibrium point of* H *is* *globally strongly asymptotically stable*, *i.e. it is* *Lyapunov-stable* *and* *globally strongly attracting.*

Proof. By assumptions (A4) and (A2) and from $u = f(y)$, it follows that y_T and u_T satisfy

$$\langle u_T - k_1 y_T, u_T - k_2 y_T \rangle_{\mathrm{L}^2(0,T)} \le 0, \quad \forall T > 0. \tag{10}$$

As regards (i), the substitution of (9) into (10) gives

$$\left\langle u_T - k_1(\overline{P}x_0)_T - k_1(\overline{F}u_T)_T, u_T - k_2(\overline{P}x_0)_T - k_2(\overline{F}u_T)_T\right\rangle_{L^2(0,T)}$$

$$= \left\langle \left[(I - k_1\overline{F})u_T\right]_T, \left[(I - k_2\overline{F})u_T\right]_T\right\rangle_{L^2(0,T)} - k_1\left\langle(\overline{P}x_0)_T, \left[(I - k_2\overline{F})u_T\right]_T\right\rangle_{L^2(0,T)}$$

$$- k_2\left\langle \left[(I - k_1\overline{F})u_T)\right]_T, (\overline{P}x_0)_T\right\rangle_{L^2(0,T)} + k_1k_2\left\|(\overline{P}x_0)_T\right\|^2_{L^2(0,T)} \le 0, \quad \forall T > 0.$$

Note now that (7) holds by (A3) and Lemma 3. Hence

$$\begin{aligned}\delta\left\|u_T\right\|^2_{L^2(0,T)} &\le k_1\left\langle(\overline{P}x_0)_T, \left[(I - k_2\overline{F})u_T\right]_T\right\rangle_{L^2(0,T)} - k_1k_2\left\|(\overline{P}x_0)_T\right\|^2_{L^2(0,\infty)} \\ &\quad + k_2\left\langle\left[(I - k_1\overline{F})u_T\right]_T, (\overline{P}x_0)_T\right\rangle_{L^2(0,T)} \\ &\le |k_1|\left\|\overline{P}\right\|\left\|x_0\right\|\left\|I - k_2\overline{F}\right\|\left\|u_T\right\|_{L^2(0,T)} \\ &\quad + k_2\left\|\overline{P}\right\|\left\|x_0\right\|\left\|I - k_1\overline{F}\right\|\left\|u_T\right\|_{L^2(0,T)} + |k_1|\,k_2\left\|\overline{P}\right\|^2\left\|x_0\right\|^2.\end{aligned}$$

This is a trinomial inequality with respect to $\|u_T\|$. Examining its properties, we conclude that there exists a constant γ which depends on $\|\overline{P}\|$, $\|\overline{F}\|$, $|k_1|$, k_2 but does not depend on T such that for all $T > 0$

$$\left\|u_T\right\|_{L^2(0,T)} \le \gamma\left\|x_0\right\|_{H}, \quad \forall x_0 \in H.$$

Since $\|u_T\|$ is an increasing function of T, we finally get

$$\left\|u\right\|_{L^2(0,\infty)} \le \gamma\left\|x_0\right\|, \quad \forall x_0 \in H. \tag{11}$$

But

$$\left\|y\right\|_{L^2(0,\infty)} \le \left\|\overline{P}\right\|\left\|x_0\right\| + \left\|\overline{F}\right\|\left\|u\right\|_{L^2(0,\infty)}$$

and therefore by (11) there exists a constant $\rho > 0$, again independent of T, such that

$$\left\|y\right\|_{L^2(0,\infty)} \le \rho\left\|x_0\right\|, \quad \forall x_0 \in H. \tag{12}$$

As for (ii), if, in addition, $d \in H$ is an admissible factor control vector, then

$$x(t) = S(t)x_0 + QR_tu, \quad t \ge 0,$$

where $\{S(t)\}_{t\ge 0} \subset \mathbf{L}(H)$ is a C_0-semigroup which is **EXS**, $Q \in \mathbf{L}(L^2(0,\infty), H)$ denotes the reachability map and R_t denotes the reflection operator at $t > 0$,

$$(R_tu)(\tau) := \begin{cases} u(t - \tau), & \tau \in [0, t), \\ 0, & \tau \ge t. \end{cases}$$

The mapping $0 \le t \longmapsto x(t) \in H$ is strongly continuous. Employing (11), we conclude that there exists a constant $\varepsilon > 0$, independent of T, such that

$$\left\|x(t)\right\|_{H} \le \varepsilon\left\|x_0\right\|_{H}, \quad \forall x_0 \in H, \quad \forall t \ge 0. \tag{13}$$

The stability of the null equilibrium follows easily from (13).

As regards state attraction to zero, $\|S(t)\|$ tends to zero exponentially fast as $t \to \infty$. Hence we may without loss of generality consider $x(t) = QR_t u$. For any fixed $u \in \mathrm{L}^2(0,\infty)$, the function $t \longmapsto R_t u$ is in $\mathrm{BUC}\,[0,\infty;\mathrm{L}^2(0,\infty))$, where $\mathrm{BUC}\,[0,\infty;\mathrm{Z})$ stands for the Banach space of bounded, uniformly continuous functions defined on $[0,\infty)$ and taking values in a Hilbert space Z, equipped with standard supremum norm $\|f\|_{\mathrm{BUC}[0,\infty;\mathrm{Z})} := \sup_{t\geq 0}\|f(t)\|_{\mathrm{Z}}$, $f \in \mathrm{BUC}[0,\infty;\mathrm{Z})$. Indeed,

$$\|R_t u - R_s u\|^2_{\mathrm{L}^2(0,\infty)} = \int_0^\infty \left[\left\{\begin{array}{cr} u(t-\tau), & 0\leq\tau<t \\ 0, & \tau\geq t\end{array}\right\} - \left\{\begin{array}{cr} u(s-\tau), & 0\leq\tau<s \\ 0, & \tau\geq s\end{array}\right\}\right]^2 \mathrm{d}\tau.$$

Let $s > t$. Then

$$\begin{aligned}\|R_t u - R_s u\|^2_{\mathrm{L}^2(0,\infty)} &= \int_0^\infty \left[\left\{\begin{array}{ccr} u(t-\tau)-u(s-\tau) & \text{if} & 0\leq\tau<t \\ -u(s-\tau) & \text{if} & t\leq\tau<s \\ 0 & \text{if} & \tau\geq s\end{array}\right\}\right]^2 \mathrm{d}\tau \\ &= \int_0^t \big[u(t-\tau)-u(s-\tau)\big]^2\,\mathrm{d}\tau + \int_t^s u^2(s-\tau)\,\mathrm{d}\tau \\ &= \int_0^t \big[u(\xi)-u(s-t+\xi)\big]^2\,\mathrm{d}\xi + \int_0^{s-t} u^2(\xi)\,\mathrm{d}\xi \\ &\leq \|u - T(s-t)u\|^2_{\mathrm{L}^2(0,\infty)} + \int_0^{s-t} u^2(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where $\{T(t)\}_{t\geq 0}$ stands for the semigroup of left-shifts on $\mathrm{L}^2(0,\infty)$ with infinitesimal generator L. Similarly, for $t > s$ we get

$$\|R_t u - R_s u\|^2_{\mathrm{L}^2(0,\infty)} \leq \|T(t-s)u - u\|^2_{\mathrm{L}^2(0,\infty)} + \int_0^{t-s} u^2(\xi)\,\mathrm{d}\xi.$$

Both these estimates yield

$$\begin{aligned}&\|R_t u - R_s u\|^2_{\mathrm{L}^2(0,\infty)} \leq \varepsilon(|t-s|), \quad \forall t,s\geq 0, \\ &\varepsilon(\delta) := \|T(\delta)u - u\|^2_{\mathrm{L}^2(0,\infty)} + \int_0^\delta u^2(\xi)\,\mathrm{d}\xi.\end{aligned}$$

The uniform continuity and boundedness hold as the function ε is continuous, non-negative and bounded on $[0,\infty)$ with the upper bound $5\|u\|^2_{\mathrm{L}^2(0,\infty)}$, and $\varepsilon(0) = 0$. The sharpest upper bound for the function $t \longmapsto R_t u$ directly follows from the observation that the reflection operator is a contraction on $\mathrm{L}^2(0,\infty)$.

Since $Q \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})$, the function $t \longmapsto QR_t u$ is in $\mathrm{BUC}[0,\infty;\mathrm{H})$. Thus the linear operator given by $(\mathcal{G}u)(t) := QR_t u$ belongs to $\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{BUC}[0,\infty;\mathrm{H}))$ as

$$\|\mathcal{G}u\|_{\mathrm{BUC}[0,\infty;\mathrm{H})} = \sup_{t\geq 0}\|QR_t u\|_{\mathrm{H}} \leq \|Q\|_{\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})}\|u\|_{\mathrm{L}^2(0,\infty)}, \quad \forall u \in \mathrm{L}^2(0,\infty).$$

Consider now the subspace $\mathrm{BUC}_0[0,\infty;\mathrm{H})$ of those functions of $\mathrm{BUC}[0,\infty;\mathrm{H})$ that have zero limit at infinity. If $u \in D(L)^1$, then by (Pazy, 1993, Cor. 2.10, p. 109):

$$\begin{aligned} QR_t u \;&=\; A\int_0^t S(t-\tau)du(\tau)\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-\tau)du(\tau)\,\mathrm{d}\tau - du(t) \\ &=\; \int_0^t S(t-\tau)d\dot{u}(\tau)\,\mathrm{d}\tau + S(t)du(0) - du(t). \end{aligned} \tag{14}$$

Since $\|S(\cdot)d\|_{\mathrm{H}}$ and $\dot{u} \in \mathrm{L}^2(0,\infty)$, the first term in (14) tends to 0 as $t \to \infty$ (Desoer and Vidyasagar, 1975, Ex. 4, p. 242). The second term in (14) decays exponentially, while the third term tends to 0 as $t \to \infty$ because $u \in D(L)$. Thus we have proved that $\mathcal{G}u \in \mathrm{BUC}_0[0,\infty;\mathrm{H})$ for u belonging to a dense subspace $D(L)$ of $\mathrm{L}^2(0,\infty)$, where $\mathrm{BUC}_0[0,\infty;\mathrm{H})$ is a closed subspace of $\mathrm{BUC}[0,\infty;\mathrm{H})$. By the continuity of $\mathcal{G}$, the inverse image of $\mathrm{BUC}_0[0,\infty;\mathrm{H})$ is a closed subspace of $\mathrm{L}^2(0,\infty)$. But it contains a dense subspace which proves that the image of $\mathrm{L}^2(0,\infty)$ under $\mathcal{G}$ equals $\mathrm{BUC}_0[0,\infty;\mathrm{H})$, whence $\lim_{t\to\infty} x(t) = 0$. ■

4. Examples

In this section we discuss two examples of electrical transmission lines illustrating the results of Section 3.

4.1. Distortionless RLCG–Transmission Line

The *distortionless* transmission line is an $RLCG$ transmission line for which $\alpha := R/L = G/C$. Following (Grabowski and Callier, 2001, Sec. 5.1), consider such a line loaded by a resistance R_0. On the Hilbert space $\mathrm{H} = \mathrm{L}^2(-r,0) \oplus \mathrm{L}^2(-r,0)$ with $r = \sqrt{LC}$, equipped with the standard scalar product, its dynamics is governed by the abstract model in the factor form (1). To be more precise,

- The state space operator A takes the form

$$\begin{cases} \;\;Ax \;=\; x', \\ D(A) \;=\; \big\{x \in \mathrm{W}^{1,2}(-r,0) \oplus \mathrm{W}^{1,2}(-r,0) :\; x(0) = C_S x(-r)\big\}, \end{cases} \tag{15}$$

  where

$$C_S = \begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix}, \quad b = \frac{\kappa}{\rho^2}, \quad \kappa = \frac{R_0 - z}{R_0 + z}, \quad z = \sqrt{\frac{L}{C}}, \quad \rho = e^{\alpha r}.$$

  The operator A generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H (or even a C_0-group if $\det C_S \neq 0$). This semigroup is **EXS** iff $|\lambda(C_S)| < 1$ or, equivalently, $|b| < 1$ (Górecki *et al.*, 1989, pp. 148–154), which is the case.

[1] The proof goes through with $D(L)$ replaced by $D(R)$. In this case $u(0) = 0$.

- The observation functional $c^{\#}$ is given by

$$c^{\#}x = c_0^T x(-r), \quad D(c^{\#}) = \{x \in \mathrm{H} : \ c_0^T x \text{ is right-continuous at } -r\}, \tag{16}$$

where

$$c_0 = \begin{bmatrix} 0 \\ a \end{bmatrix}, \quad a = \frac{1+\kappa}{\rho} \geq 0.$$

It is representable on $D(A)$ as

$$c^{\#}\big|_{D(A)} = h^* A, \quad h = \vartheta \begin{bmatrix} b\mathbf{1} \\ -\mathbf{1} \end{bmatrix} \in \mathrm{H}, \quad \vartheta := \frac{a}{1+b},$$

where $\mathbf{1}$ denotes the constant function taking the value 1 on $[-r, 0]$. The admissibility of $c^{\#}$ was implicitly discussed in (Grabowski, 1994, p. 363). The Lyapunov proof of this fact is presented in (Grabowski and Callier, 2001).

- The factor control vector is identified as

$$d = \frac{-1}{1+b} d_0, \quad d_0 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \in \mathrm{H}, \tag{17}$$

and d is admissible (Grabowski and Callier, 2001).

The system dynamics can also be described by

$$\left\{ \begin{array}{rcl} w(t) & = & C_S w(t-r) + u(t) b_0 \\ y(t) & = & c_0^T w(t-r) \end{array} \right\}, \quad b_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{18}$$

The compatibility condition (2) holds with $c^{\#}d = -\vartheta$, and by (3) the transfer function of the system is obtained as

$$\hat{g}(s) = \frac{a e^{-sr}}{1 + b e^{-2sr}}. \tag{19}$$

This can be confirmed by applying the Laplace transform directly to (18). Moreover,

$$\|\hat{g}\|_{\mathrm{H}^\infty(\Pi^+)} = \frac{a}{1-|b|},$$

and thus (4) is satisfied. The situation is even better in that g is in the Callier-Desoer algebra $\mathcal{A}_-(0)$. All these results and many others can be found in (Grabowski and Callier, 2001).

The closed-loop linear system operator corresponding to the linear feedback law $f(y) = \mu y$ takes the form

$$A_\mu x = x',$$
$$D(A_\mu) = \Big\{ x \in \mathrm{W}^{1,2}(-r,0) \oplus \mathrm{W}^{1,2}(-r,0) : \; x(0) = \big[C_S + \mu b_0 c_0^T\big]\, x(-r) \Big\}.$$

Indeed, $D(A_\mu)$ consists of these x for which $x + \mu d c^{\#} x \in D(A)$. This holds if $x \in \mathrm{W}^{1,2}(-r,0) \oplus \mathrm{W}^{1,2}(-r,0)$ and $x(0) + \mu d c^{\#} x = C_S \big[x(-r) + \mu d c^{\#} x\big]$ or, equivalently, if $x(0) = \big[C_S + \mu b_0 c_0^T\big]\, x(-r)$. The semigroup generated on $\mathrm{H} = \mathrm{L}^2(-r,0) \oplus \mathrm{L}^2(-r,0)$ by A_μ is **EXS** iff all the eigenvalues of the matrix $C_S + \mu b_0 c_0^T$ are in the open unit disk (Górecki *et al.*, 1989). This is the case if

$$|\mu| < \frac{1+b}{a}. \tag{20}$$

The linear stability condition (20) yields the *Hurwitz sector* which has to be compared with a sector (k_1, k_2) generated by the frequency-domain inequality

$$1 - (k_1 + k_2)\,\mathrm{Re}\,\big[\hat{g}(j\omega)\big] + k_1 k_2\, |\hat{g}(j\omega)|^2 \geq 0, \quad \forall \omega \in \mathbb{R}. \tag{21}$$

By (20) it is clear that the upper limit for k_2 is $(1+b)/a$ and the lower limit for k_1 is $-(1+b)/a$.

We have the following possibilities:

1. If $b \leq 0$, then by substituting $k_2 = -k_1 = (1+b)/a$ into (21) we obtain

$$1 - \left(\frac{1+b}{a}\right)^2 |\hat{g}(j\omega)|^2 = \frac{-4b \sin^2 \omega r}{(1-b)^2 + 4b \cos^2 \omega r} \geq 0, \quad \forall \omega \in \mathbb{R},$$

   and therefore the Hurwitz sector (20) coincides with that implied by (21).

2. If $b > 0$, then for $k_1 = -(1+b)/a$ we cannot take $k_2 = (1+b)/a$. Here the Hurwitz sector (20) is essentially larger than the sector implied by (21) and another choice of k_1 and k_2 has to be proposed. Assuming $k_1 = -(1+b)/a$, we look for a maximal allowed value of k_2 for which (21) is satisfied. Since

$$1 - (k_1 + k_2)\,\mathrm{Re}\,\big[\hat{g}(j\omega)\big] + k_1 k_2\, |\hat{g}(j\omega)|^2$$
$$= \frac{(1+b)^2 \cos^2 \omega r + (1-b)^2 \sin^2 \omega r + [(1+b)^2 - k_2 a(1+b)] \cos \omega r - k_2 a(1+b)}{(1+b)^2 \cos^2 \omega r + (1-b)^2 \sin^2 \omega r},$$

   treating the numerator as a polynomial in $\cos \omega r$, we find the maximal allowed value of k_2 for which the frequency domain inequality (21) holds,

$$k_2 = \frac{1+b}{a} - \frac{8b}{a(1+b)}. \tag{22}$$

   Then

$$1 - (k_1 + k_2)\,\mathrm{Re}[\hat{g}(j\omega)] + k_1 k_2\, |\hat{g}(j\omega)|^2 = \frac{4b(1 + \cos \omega r)^2}{(1+b)^2 \cos^2 \omega r + (1-b)^2 \sin^2 \omega r} \geq 0.$$

Replacing k_1 and k_2 by $k_1+\varepsilon$ and $k_2+\varepsilon$, respectively, where $\varepsilon>0$ is sufficiently small, we obtain (5) from (21). Indeed, from (21) and (19) we get for $\varepsilon\in(0,k_2-k_1)$

$$\begin{aligned}1-(k_1+\varepsilon+k_2-\varepsilon)\,\mathrm{Re}\,\big[\hat g(j\omega)\big]+(k_1+\varepsilon)(k_2-\varepsilon)\,|\hat g(j\omega)|^2\\ =1-(k_1+k_2)\,\mathrm{Re}\,\big[\hat g(j\omega)\big]+k_1k_2\,|\hat g(j\omega)|^2+\big[\varepsilon(k_2-k_1)-\varepsilon^2\big]\,|\hat g(j\omega)|^2\\ \geq\big[\varepsilon(k_2-k_1)-\varepsilon^2\big]\inf_{\omega\in\mathbb{R}}|\hat g(j\omega)|^2=\big[\varepsilon(k_2-k_1)-\varepsilon^2\big]\frac{a^2}{(1+|b|)^2}>0.\end{aligned}$$

Suppose now that condition (A4) of Theorem 1 holds. Then by the same theorem u and $y\in\mathrm{L}^2(0,\infty)$, and the null equilibrium point is globally strongly asymptotically stable for any continuous nonlinearity f vanishing at 0 and lying in the sector $[k_1+\varepsilon,k_2-\varepsilon]$, where

$$k_1=-\frac{1+b}{a},\quad k_2=\begin{cases}\dfrac{1+b}{a} & \text{if}\quad b\leq 0,\\[2mm] \dfrac{1+b}{a}-\dfrac{8b}{a(1+b)} & \text{if}\quad b\geq 0,\end{cases}$$

and $\varepsilon>0$ is sufficiently small. It should be stressed that Lemma 3 and Theorem 1 are proved under the assumption that $k_1<0$, $k_2>0$, which is not the case for $b\in[3-2\sqrt{2},1)$, $3-2\sqrt{2}\approx 0.1716$. The application of the circle criterion for finite-dimensional systems as well as some results on the Lyapunov approach to the circle criterion for boundary control systems in factor form (Grabowski and Callier, 2000) show that the restriction imposed on the signs of k_1 and k_2 is artificial and can be removed. This can be done by applying the *loop transformation technique*, which will be presented in the last section. A simple application of this technique shows that our stability result is actually valid for all $b\in(0,1)$.

4.2. RC–Transmission Line

Following (Grabowski and Callier, 2001, Sec. 5.2), in the Hilbert space $\mathrm{H}=\mathrm{L}^2(0,1)$ with standard scalar product the dynamics of the unloaded RC transmission line can be modelled by (1) with the following choices:

- The state-space operator is

$$Ax=x'',\quad D(A)=\big\{x\in\mathrm{H}^2(0,1):\ x'(1)=0,\ x(0)=0\big\}\tag{23}$$

  which generates an **EXS** analytic self-adjoint semigroup on H. This is because $A=A^*<0$.

- The observation functional

$$c^{\#}x=x(1),\quad D(c^{\#})=\big\{x\in\mathrm{L}^2(0,1):x\text{ is left-continuous at }1\big\}\supset\mathrm{C}[0,1],\tag{24}$$

  whose restriction to $D(A)$ reads as $c^{\#}|_{D(A)}=h^*A$ with $h(\theta)=-\theta$, $0\leq\theta\leq 1$. It was proved in (Grabowski, 1994) that $c^{\#}$ is admissible.

- The factor control vector d is identified as

$$d = -\mathbf{1} \in \mathrm{L}^2(0,1), \quad \mathbf{1}(\theta) = 1, \quad 0 \le \theta \le 1 \tag{25}$$

  and is not admissible. For a proof see (Grabowski and Callier, 1999, Sec. 3.3) or, more briefly, (Grabowski and Callier, 2001, App. B).

It is easy to see that (2) holds with $c^{\#}d = -1$, and using (3) we find the transfer function

$$\hat{g}(s) = \frac{1}{\cosh\sqrt{s}}, \quad s \in \Pi^+. \tag{26}$$

Moreover,

$$\|\hat{g}\|_{\mathrm{H}^\infty(\Pi^+)} = 1, \tag{27}$$

where the norm is attained at $s = 0$. For a more exhaustive discussion of these facts and many others consult again (Grabowski and Callier, 2001).

From (27) it follows that (21) is satisfied for $k_2 = -k_1 = 1$. The closed-loop linear system operator corresponding to the linear feedback $f(y) = \mu y$ is given by

$$A_\mu x = x'', \quad D(A_\mu) = \big\{x \in \mathrm{H}^2(0,1) : \; x'(1) = 0, \; x(0) = \mu x(1)\big\}.$$

It is proved in (Grabowski, 1990) that A_μ generates an analytic semigroup on $\mathrm{L}^2(0,1)$ which is **EXS** for $\mu \in (-\cosh\pi, 1)$ with $\cosh\pi \approx 11.592$. Hence the Hurwitz sector is essentially larger than the sector mentioned above obtained by (21).

The method of converting (21) into (5) by reducing the width of the sector used in the first example is a general one if

$$\inf_{\omega\in\mathbb{R}} |\hat{g}(j\omega)|^2 > 0,$$

which is not the case here, because $\lim_{|\omega|\to\infty} \hat{g}(j\omega) = 0$. However, we can use the fact that for $k_2 = 1$ we have

$$|\hat{g}(j\omega)|^2 \le \|\hat{g}\|^2_{\mathrm{H}^\infty(\Pi^+)} = \frac{1}{k_2^2}.$$

By this estimate, for a sufficiently small $\varepsilon > 0$ we obtain

$$1 - (k_2 - \varepsilon)^2 |\hat{g}(j\omega)|^2 \ge \frac{\varepsilon(2k_2 - \varepsilon)}{k_2^2} > 0,$$

which means that replacing $k_2 = -k_1$ by $k_2 - \varepsilon$ yields the desired conversion of (21) into (5).

Suppose now that condition (A4) of Theorem 1 holds. Then by the same theorem u and $y \in \mathrm{L}^2(0,\infty)$ for any continuous nonlinearity f vanishing at 0 and lying in the sector $[-1+\varepsilon, 1-\varepsilon]$, where $\varepsilon > 0$ is sufficiently small.

5. Discussion and Conclusions

Theorem 1 is valid under the restrictive assumption that $k_1 < 0$ and $k_2 > 0$. However, this restriction is not essential because any boundary control system in factor form satisfying (21) can be reduced to a standard form with $k_1 = -k_2 < 0$ by the loop transformation technique, see, e.g., (Desoer and Vidyasagar, 1975, Ch. 3, Sec. 6.3, p. 51). The main ideas are presented below. One starts with rewriting the closed-loop system equation as[2]

$$\begin{cases} A^{-1}\dot{x} = x + df(y), \\ \quad y = c^{\#}x = c^{\#}(A^{-1}\dot{x} - df) = c^{\#}A^{-1}\dot{x} - c^{\#}df = h^*\dot{x} - c^{\#}df. \end{cases} \tag{28}$$

Let $\mu_0 := (k_1 + k_2)/2$, i.e. μ_0 is the slope of the straight line being the centre of the sector (k_1, k_2). Thus if the function f is in the sector (k_1, k_2), then the function f_0,

$$f_0(y) := f(y) - \mu_0 y, \quad y \in \mathbb{R}, \tag{29}$$

is in the sector $(-(k_2 - k_1)/2, (k_2 - k_1)/2)$. Substituting (29) into (28), we get

$$\begin{cases} A^{-1}\dot{x} = x + \mu_0 dc^{\#}x + df_0(y), \\ \quad y = c^{\#}x = \dfrac{1}{1 + \mu_0 c^{\#}d} h^*\dot{x} - \dfrac{c^{\#}d}{1 + \mu_0 c^{\#}d} f_0(y). \end{cases}$$

For any $z \in D(c^{\#})$ the equation $(I + \mu_0 dc^{\#})x = z$ has a unique solution $x \in D(c^{\#})$ given by

$$x = \Big(I - \frac{\mu_0}{1 + \mu_0 c^{\#}d} dc^{\#}\Big) z.$$

Hence, since $A^{-1}\dot{x}$ and d are both in $D(c^{\#})$, we can solve the first system equation with respect to x and get

$$\begin{cases} \Big[A^{-1} - \dfrac{\mu_0}{1 + \mu_0 c^{\#}d} dh^*\Big]\dot{x} = x + \dfrac{1}{1 + \mu_0 c^{\#}d} df_0(y), \\ \qquad\qquad y = c^{\#}x = \dfrac{1}{1 + \mu_0 c^{\#}d} h^*\dot{x} - \dfrac{c^{\#}d}{1 + \mu_0 c^{\#}d} f_0(y). \end{cases} \tag{30}$$

The transfer function of the linear part of (28) obtained by replacing $f(y)$ by a control u is given by (3) as we have

$$sh^*A(sI - A)^{-1}d = sh^*(sA^{-1} - I)^{-1}d.$$

[2] Since for $w \in D(A^*)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle w, x\rangle_{\mathrm{H}} = \frac{\mathrm{d}}{\mathrm{d}t}\langle A^*w, A^{-1}x\rangle_{\mathrm{H}} = \langle A^*w, A^{-1}\dot{x}\rangle_{\mathrm{H}} = \langle A^*w, x + df(y)\rangle_{\mathrm{H}},$$

any weak solution of the original closed-loop system $\dot{x} = A[x + df(y)]$, $y = c^{\#}x$ satisfies (28) in the classical sense.

Observe that the linear part of the system (30) has the same form as in (28) upon replacing A^{-1} by $A^{-1} - \mu_0 dh^*/(1+\mu_0 c^{\#} d)$ and multiplying h and d by the scaling factor $(1+\mu_0 c^{\#} d)^{-1}$. Hence by (3) the transfer function of the linear part of (30) is given by

$$\hat{g}_0(s) = \frac{s}{(1+\mu_0 c^{\#} d)^2} h^* \Big[sA^{-1} - \frac{s\mu_0}{1+\mu_0 c^{\#} d} dh^* - I \Big]^{-1} d - \frac{c^{\#} d}{1+\mu_0 c^{\#} d}.$$

But

$$\begin{aligned}\Big[sA^{-1} - \frac{s\mu_0}{1+\mu_0 c^{\#} d} dh^* - I \Big]^{-1} &= \Big\{ (sA^{-1} - I) \Big[I - \frac{s\mu_0}{1+\mu_0 c^{\#} d} (sA^{-1} - I)^{-1} dh^* \Big] \Big\}^{-1} \\ &= \Big[I - \frac{s\mu_0}{1+\mu_0 c^{\#} d} (sA^{-1} - I)^{-1} dh^* \Big]^{-1} (sA^{-1} - I)^{-1} \\ &= \Big[I + \frac{s\mu_0}{1-\mu_0 \hat{g}(s)} (sA^{-1} - I)^{-1} dh^* \Big] (sA^{-1} - I)^{-1},\end{aligned}$$

and therefore

$$\hat{g}_0(s) = \frac{\hat{g}(s) + c^{\#} d}{(1+\mu_0 c^{\#} d)^2} \left[1 + \frac{\mu_0 \hat{g}(s) + \mu_0 c^{\#} d}{1 - \mu_0 \hat{g}(s)} \right] - \frac{c^{\#} d}{1+\mu_0 c^{\#} d} = \frac{\hat{g}(s)}{1-\mu_0 \hat{g}(s)},$$

as was to be expected.

The last step of the loop transformation technique implies that the frequency-domain inequality (21) holds iff

$$1 - \frac{1}{4}(k_2 - k_1)^2 |\hat{g}_0(j\omega)|^2 \geq 0, \quad \forall \omega \in \mathbb{R}. \tag{31}$$

Indeed, we have

$$1 - \frac{1}{4}(k_2 - k_1)^2 |\hat{g}_0(j\omega)|^2 = \frac{[1-\mu_0 \hat{g}(j\omega)][1-\mu_0 \overline{\hat{g}(j\omega)}] - \frac{1}{4}(k_2 - k_1)^2 |\hat{g}(j\omega)|^2}{|1-\mu_0 \hat{g}(j\omega)|^2},$$

and thus (31) is satisfied iff the numerator is non-negative for all $\omega \in \mathbb{R}$. This is valid iff (21) holds.

The considerations above show that any boundary control system in factor form satisfying (21) with arbitrary constants k_1 and k_2 can be reduced to a standard form with k_1 and k_2 replaced by $(k_1 - k_2)/2 < 0$ and $(k_2 - k_1)/2 > 0$, respectively.

Another method for getting a circle criterion for boundary control systems in factor form is the Lyapunov approach, which relies on the construction of a Lyapunov functional in quadratic form. This is studied in detail in (Grabowski and Callier, 2000). In (Bucci, 2000), a version of Popov's criterion was successfully obtained, using the Lyapunov method (and improved in (Bucci, 1999) with the aid of former Popov's approach combined with regularity results for the solution to the closed loop), for the infinite-dimensional Lur'e system of *indirect control*,

$$\begin{cases} \dot{x}(t) = A\{x(t) + df[\sigma(t)]\}, \\ \dot{\sigma}(t) = \langle q, x(t) \rangle_{\mathrm{H}} - \rho f[\sigma(t)]. \end{cases}$$

Regarding the variable σ as the system output, one can readily notice that here the output is differentiable. This is in contrast to (1) with $u(t) = f[y(t)]$, where the output y is generally not differentiable. In (Logemann, 1991), the circle criterion was derived for the Pritchard-Salamon class of control systems. It was proved in (Grabowski, 1994) that the system described in our first example does not belong to that class. This is also the case of the second system. In (Logemann and Curtain, 2000), a circle criterion was derived, using the Lyapunov method, for a nonlinear feedback system having an integrator in its feedback loop and a sector nonlinearity in front of an infinite-dimensional Salamon-Weiss linear plant. Due to the smoothing action of the integrator, the results of (Logemann and Curtain, 2000) are not comparable with those of the present paper.

The relation of boundary control systems in factor form to the Salamon-Weiss class of control systems is studied in (Grabowski and Callier, 1999, Sec. 4.5; 2001, Sec. 7 and App. C). Roughly speaking, up to some conditions, when A is boundedly invertible there is a one-to-one correspondence between the two classes. In particular, one may expect that the results of the present paper have counterparts for the Salamon-Weiss class of control systems. The dynamic models of the Salamon-Weiss class are a natural generalization of the additive finite-dimensional state space equations, but they involve an additional space known as H_{-1}. Its characterization may turn out to be quite complicated and, as a rule, it makes use of distribution theory. That can make this class less attractive for control engineers. On the other hand, boundary control systems in factor form do not consider the space H_{-1} and are also a natural generalization of finite-dimensional systems. It is thereby hoped that such systems can also be useful for control engineers to better understand the complicated tools and methods of infinite-dimensional systems control theory.

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