

## REDUCTION OF LARGE CIRCUIT MODELS VIA LOW RANK APPROXIMATE GRAMIANs

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We describe a model reduction algorithm which is well-suited for the reduction of large linear interconnect models. It is an orthogonal projection method which takes as the projection space the sum of the approximate dominant controllable subspace and the approximate dominant observable subspace. These approximate dominant subspaces are obtained using the Cholesky Factor ADI (CF-ADI) algorithm. We describe an improvement upon the existing implementation of CF-ADI which can result in significant savings in computational cost. We show that the new model reduction method matches moments at the negative of the CF-ADI parameters, and that it can be easily adapted to allow for DC matching, as well as for passivity preservation for multi-port RLC circuit models which come from modified nodal analysis.

**Keywords:** model reduction, Lyapunov equations, Cholesky-Factor ADI, moment matching, passivity

### 1. Introduction

If modified nodal analysis (MNA) is used to generate a system of equations for a multi-port, coupled RLC network, the system will have the form

$$\mathcal{L}\dot{x} = -\mathcal{G}x + Nu, \quad (1)$$

$$y = Cx, \quad (2)$$

where

$$\mathcal{L} = \begin{bmatrix} E & 0 \\ 0 & L \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} G & M \\ -M^T & 0 \end{bmatrix}, \quad x = \begin{bmatrix} v \\ i \end{bmatrix}. \quad (3)$$

The vectors  $y$  and  $u$  denote the port currents and voltages, respectively. The square matrices  $E$ ,  $L$ , and  $G$  are all symmetric, and denote the nodal capacitance, inductance, and conductance matrices, respectively. We further assume that  $E$  and  $L$  are non-singular. The rectangular matrix  $M$  is the incidence matrix associated with the

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inductor currents. The matrix  $N$  indicates where the voltage sources are, and the matrix  $C$  specifies where the port currents are measured.

Model reduction methods for such systems usually proceed by projection, which can be either orthogonal, involving only one orthonormal projection matrix, or oblique, involving two bi-orthogonal projection matrices.

The new model reduction algorithm is chosen to be an orthogonal projection method, because orthogonal projection has certain advantages over oblique projection in the context of the reduction of large circuit models. The new method chooses its projection matrix to include columns which form an orthonormal basis for the sum of the approximate dominant eigenspaces of the system controllability and observability gramians.

The approximate dominant gramian eigenspaces are obtained via an efficient Lyapunov equation solver, the Cholesky-Factor ADI (CF-ADI) algorithm (Li *et al.*, 1999). We describe an improvement upon the existing implementation of CF-ADI in (Li *et al.*, 1999), which can result in significant savings in computational cost, in the case when an iterative Krylov subspace method such as GMRES is used to solve shifted linear systems. We show how to convert shifted linear systems with different right-hand sides in CF-ADI to ones with the same right-hand side, and how to solve all these systems with only one Krylov subspace. The solution of shifted systems was discussed in detail in (Freund, 1993b).

We then show that the new model reduction algorithm which uses CF-ADI to obtain the approximate dominant gramian eigenspaces produces a reduced system whose transfer function matches the moments of the original transfer function at the negatives of the CF-ADI parameters.

However, including only the sum of the dominant gramian eigenspaces in the projection matrix will not, in general, produce a reduced model whose transfer function matches moments of the original transfer function around  $s = 0$ . In other words, the steady state response of the reduced system is not guaranteed to be the same as that of the original one. To remedy this, we augment the projection matrix by several Arnoldi vectors to obtain the new projection matrix. Orthogonal projection via the augmented matrix achieves transfer function matching at  $s = 0$ , and appears to have no detrimental effect on the quality of the approximation in the rest of the frequency domain.

Odabasioglu *et al.* (1998) showed that for multi-port RLC circuits from modified nodal analysis, if  $\mathcal{L}$  and  $\mathcal{G}$  are each projected separately via an orthogonal projection matrix, then the reduced system preserves passivity. This result holds for general orthogonal projection matrices, including the augmented projection matrix described in this paper.

This paper is organized in the following way. Section 2 gives background on model reduction. Section 3 describes the CF-ADI algorithm and a more efficient way of implementing it. Section 4 describes the new model reduction algorithm and proves its moment matching property. Section 5 gives numerical results. Section 6 addresses the issues of DC matching and passivity preservation. Section 7 contains the conclusions.

## 2. Model Reduction

We rewrite the system (1, 2) in the standard state space form

$$\dot{x} = Ax + Bu, \quad (4)$$

$$y = Cx, \quad (5)$$

where  $A = -\mathcal{L}^{-1}\mathcal{G}$  and  $B = \mathcal{L}^{-1}N$ . The matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times p}$ ,  $C \in \mathbb{R}^{q \times n}$  are called the system matrix, the input matrix and the output matrix, respectively. In single-input, single-output (SISO) systems,  $p = 1$ ,  $q = 1$ . Even in multiple-input, multiple-output (MIMO) systems,  $p$  and  $q$  are both very small compared with the number of state variables  $n$ .

If  $A$  is stable, the controllability gramian  $P$  and the observability gramian  $Q$  of the system (4), (5) are the unique, symmetric solutions of the following two Lyapunov equations (Chandrasekharan, 1996; Sontag, 1998):

$$AP + PA^T + BB^T = 0, \quad (6)$$

$$A^TQ + QA + C^TC = 0. \quad (7)$$

The gramians provide information about the controllability and observability of the system, and are needed in optimal Hankel-norm or Truncated Balanced Realization-type model reductions (Enns, 1984; Glover, 1984; Moore, 1981; Pernebo and Silverman, 1982). Eigenvectors of  $P$  with the largest eigenvalues are the modes which are the most controllable, and eigenvectors of  $Q$  with the largest eigenvalues are the modes which are the most observable.

The system in (4), (5) is characterized by its transfer function  $G(s)$ , where

$$G(s) = C(sI - A)^{-1}B, \quad Y(s) = G(s)U(s), \quad (8)$$

and  $Y(s)$  and  $U(s)$  are the Laplace transforms of the output  $y$  and the input  $u$ , respectively. Model order reduction seeks to obtain a smaller system

$$\dot{x}_r = A_r x_r + B_r u, \quad (9)$$

$$y_r = C_r x_r \quad (10)$$

such that the number of state variables of this new system is much smaller than  $n$ , and the transfer function of the new system,  $G_r(s)$ ,

$$G_r(s) = C_r (sI - A_r)^{-1} B_r, \quad Y_r(s) = G_r(s)U_r(s) \quad (11)$$

is close to the original.

### 2.1. Projection Methods

Most model reduction methods are projection methods. These methods choose two projection matrices, the right projection matrix,  $U_k \in \mathbb{R}^{n \times k}$ , and the left projection

matrix,  $V_k \in \mathbb{R}^{n \times k}$ . The system (4), (5) is then reduced according to the projection equations,

$$A_k^r = V_k^T A U_k, \quad (12)$$

$$B_k^r = V_k^T B, \quad C_k^r = C U_k. \quad (13)$$

## 2.2. Moment Matching Methods

One main category of model reduction methods comprises moment matching methods, wherein the transfer function of the reduced model matches a number of moments of the original transfer function. Moment matching is usually done implicitly, via Krylov subspaces (Feldmann and Freund, 1995; Freund, 1999; Gallivan *et al.*, 1994; 1996a; 1996b; Grimme *et al.*, 1996). The moments themselves are never calculated.

An example of an oblique moment matching method which uses Krylov subspaces is one which chooses the left projection matrix  $V_k$  and the right projection matrix  $U_k$  to be bi-orthogonal, via the Lanczos process, and where

$$\text{span}(U_k) = \text{span} \{A^{-1}B, \dots, A^{-k}B\} = \mathcal{K}_k(A^{-1}, A^{-1}B), \quad (14)$$

$$\text{span}(V_k) = \text{span} \{(A^T)^{-1}C^T, \dots, (A^T)^{-k}C^T\} = \mathcal{K}_k(A^{-T}, A^{-T}C^T). \quad (15)$$

This approach produces a reduced system whose transfer function matches  $2k$  moments of the original at  $s = 0$ . Since Lanczos bi-orthogonalization can have breakdowns, “look-ahead” Lanczos is usually implemented (Freund, 1993a).

Moment matching can also be done via orthogonal projection, in which case the left and right projection matrices are the same. The single projection matrix  $U_k = V_k$  can be chosen to be orthonormal and to satisfy either (14) or (15). The Arnoldi process can be used to produce this projection matrix. The approach which uses the Arnoldi process avoids Lanczos breakdowns, but in general it only matches  $k$  moments around  $s = 0$ .

Moments can also be matched at points other than 0, in which case the projection matrices will span the sum of rational Krylov subspaces. For example,  $U_k$  can be chosen so that

$$\begin{aligned} \text{span}(U_k) &= \sum_{i=1}^m \text{span} \{(A - p_i I)^{-1}B, (A - p_i I)^{-2}B, \dots, (A - p_i I)^{-k_i}B\}, \\ &= \sum_{i=1}^m \mathcal{K}_{k_i}((A - p_i I)^{-1}, (A - p_i I)^{-1}B), \\ k &= k_1 + k_2 + \dots + k_m. \end{aligned}$$

Moments will be matched at the points  $\{p_1, p_2, \dots, p_m\}$ . The left projection matrix  $V_k$  can be the same as  $U_k$ , or it can span the sum of rational Krylov subspaces based on  $A^T$  and  $C^T$ .

Moment matching methods require only matrix-vector products or linear system solves, and hence are very efficient. However, there is no uniform error bound for the reduced system's transfer function. The error will be small at points where moments are matched, but it is difficult to guarantee that the error will also be small elsewhere.

### 2.3. Truncated Balanced Realization

The ‘‘Truncated’’ Balanced Realization (TBR) (Enns, 1984; Moore, 1981; Pernebo and Silverman, 1982) produces a guaranteed stable reduced model, and has a frequency domain  $L^\infty$ -error bound. There is no theoretical result concerning the optimality or near optimality of the TBR reduction in the  $L^\infty$  norm. However, TBR produces in general a reduced model with globally accurate frequency response approximation. This reduced model is usually superior to the models produced by moment matching methods.

The Square Root method of implementing TBR is proposed in (Safonov and Chiang, 1989; Tombs and Postlethwaite, 1987). It has better numerical properties than the implementation in (Glover, 1984). When referring to ‘the TBR algorithm’ in future chapters, the implementation in Algorithm 1 is assumed.

Given a stable system in the standard state space form (4), (5), Algorithm 1 produces the  $k$ -th order TBR reduction.

**Algorithm 1.** The Square Root method to calculate the  $k$ -th order TBR reduction.

1. Find the Cholesky factors  $Z^B$  and  $Z^C$  of the solutions  $P$  and  $Q$  to (6), (7),

$$P = Z^B(Z^B)^T, \quad Q = Z^C(Z^C)^T. \quad (16)$$

2. Calculate the singular value decomposition of  $(Z^C)^T Z^B$ ,

$$U^L \Sigma (U^R)^T = (Z^C)^T Z^B, \quad (17)$$

where

$$U^R = \begin{bmatrix} u_1^R & \cdots & u_n^R \end{bmatrix}, \quad U^L = \begin{bmatrix} u_1^L & \cdots & u_n^L \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{bmatrix}. \quad (18)$$

3. If  $\sigma_k > \sigma_{k+1}$ , let

$$S^B = Z^B \begin{bmatrix} u_1^R, \dots, u_k^R \end{bmatrix} \begin{bmatrix} 1/\sqrt{\sigma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/\sqrt{\sigma_k} \end{bmatrix}, \quad (19)$$

and

$$S^C = Z^C [u_1^L, \dots, u_k^L] \begin{bmatrix} 1/\sqrt{\sigma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/\sqrt{\sigma_k} \end{bmatrix}. \quad (20)$$

4. The  $k$ -th order Truncated Balanced Realization is given by

$$A_k^{\text{tbr}} = (S^C)^T A S^B, \quad B_k^{\text{tbr}} = (S^C)^T B, \quad C_k^{\text{tbr}} = C S^B. \quad (21)$$

The controllability and observability gramians of the  $k$ -th order reduced system  $(A_k^{\text{tbr}}, B_k^{\text{tbr}}, C_k^{\text{tbr}})$  are diagonal and equal,

$$P_k^{\text{tbr}} = Q_k^{\text{tbr}} = \Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_k). \quad (22)$$

The resulting transfer function  $G_k^{\text{tbr}}(s)$  has  $L^\infty$ -error bound,

$$\begin{aligned} \|G(jw) - G_k^{\text{tbr}}(jw)\|_{L^\infty} &:= \sup_w \|G(jw) - G_k^{\text{tbr}}(jw)\|_2 \\ &\leq 2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n). \end{aligned} \quad (23)$$

TBR is a projection method with left projection matrix  $S^C$  and right projection matrix  $S^B$ , such that  $(S^C)^T S^B = I_{k \times k}$  and

$$\text{colsp}(S^B) \subseteq \text{colsp}(Z^B), \quad \text{colsp}(S^C) \subseteq \text{colsp}(Z^C). \quad (24)$$

A merit of the Square Root method is that it relies on the Cholesky factors  $Z^B$  and  $Z^C$  of the gramians  $P$  and  $Q$  rather than the gramians themselves, which has advantages in terms of numerical stability.

The vast majority of the work involved in Algorithm 1 comes from Step 1 to obtain  $Z^B$  and  $Z^C$ , and from Step 2, the balancing singular value decomposition. Both Steps 1 and 2 are  $O(n^3)$  if done exactly, even if the system matrix  $A$  is sparse, which makes Algorithm 1 impractical for problems with more than a few hundred components in the state vector. For this reason, TBR has long been considered too expensive to apply to large problems.

### 3. Cholesky Factor ADI

The Cholesky Factor ADI (CF-ADI) algorithm, shown as Algorithm 2, solves Lyapunov equations only approximately. The CF-ADI approximation to

$$AX + XA^T + BB^T \quad (25)$$

is given by  $X_J^{\text{cfadi}} = Z_J^{\text{cfadi}}(Z_J^{\text{cfadi}})^T$ , and the algorithm iterates on the Cholesky factor  $Z_J^{\text{cfadi}}$  of the approximate solution  $X_J^{\text{cfadi}}$ . The CF-ADI algorithm is closely related to another low rank reformulation of the ADI algorithm which was independently proposed in (Penzl, 1999b). However, in that version, the work required to produce a  $k$ -th rank approximation to  $X$  increases as  $O(k^2)$ , whereas for the CF-ADI algorithm the work increases as  $O(k)$ .

**Algorithm 2.** The Cholesky Factor ADI Algorithm.

INPUT:  $A, B$ .

0. Choose CF-ADI parameters,  $\{p_1, \dots, p_{J_{\max}}\}$ ,  $\text{Re}\{p_i\} < 0$ , (real or complex conjugate pairs).

Define:  $P_i = \left( \frac{\sqrt{-2p_{i+1}}}{\sqrt{-2p_i}} \right) [I - (p_{i+1} + p_i)(A + p_{i+1}I)^{-1}]$ .

$$1a. \quad z_1 = \left( \sqrt{-2p_1} \right) (A + p_1I)^{-1}B, \quad (26)$$

$$1b. \quad Z_1^{\text{cfadi}} = \begin{bmatrix} z_1 \end{bmatrix},$$

FOR  $j = 2, 3, \dots, J_{\max}$

$$2a. \quad z_j = P_{j-1}z_{j-1}, \quad (27)$$

2b. If ( $\|z_j\|_2 > \text{tol}_1$  or  $\|z_j\|_2/\|Z_{j-1}\|_2 > \text{tol}_2$ ) and ( $j \leq J_{\max}$ )

$$Z_j^{\text{cfadi}} = \begin{bmatrix} Z_{j-1}^{\text{cfadi}} & z_j \end{bmatrix}. \quad (28)$$

Otherwise,  $J = j - 1$ , stop.

END

OUTPUT:  $Z_j^{\text{cfadi}} \in \mathbb{C}^{n \times Jp}$ ,  $Z_j^{\text{cfadi}}(Z_j^{\text{cfadi}})^T \in \mathbb{R}^{n \times n}$ ,  $X_j^{\text{cfadi}} := Z_j^{\text{cfadi}}(Z_j^{\text{cfadi}})^T \approx X$ .

It can be seen from Algorithm 2 that the columns of  $Z_j^{\text{cfadi}}$  span a rational Krylov subspace,  $\mathcal{K}_J(\mathbf{R}(A), w_1)$ , with successive matrix products by non-identical rational functions of  $A$ .

The dominant eigenspace of the approximate Lyapunov solution  $X_j^{\text{cfadi}}$  can be easily found from the singular value decomposition of  $Z_j^{\text{cfadi}}$ . The SVD,  $Z_j^{\text{cfadi}} = U_J \Lambda_J V_J^T$ ,  $\Lambda_J = \text{diag}(\lambda_1, \dots, \lambda_J)$ , can be obtained cheaply because  $Z_j$  contains only  $Jp$  columns, where  $p$  is the number of columns in  $B$ . The approximate Lyapunov solution  $X_j^{\text{cfadi}} = Z_j^{\text{cfadi}}(Z_j^{\text{cfadi}})^T$  then has eigen-decomposition,  $X_j^{\text{cfadi}} = U_J \Lambda_J^2 U_J^T$ . The dominant eigenvectors of  $X_j^{\text{cfadi}}$  are simply the dominant left singular vectors of  $Z_j^{\text{cfadi}}$ .

The choice of CF-ADI parameters is a rational min-max problem,

$$\min_{p_1, p_2, \dots, p_J} \max_{x \in \text{spec}(A)} \left| \prod_{j=1}^J \frac{(p_j - x)}{(p_j + x)} \right|,$$

and a discussion of parameter selection can be found in (Ellner and Wachspress, 1991; Lu and Wachspress, 1991).

Approximate gramians can be obtained via the CF-ADI algorithm as

$$P \approx P_j^{\text{cfadi}} := Z_j^B (Z_j^B)^T, \quad Q \approx Q_j^{\text{cfadi}} := Z_j^C (Z_j^C)^T,$$

$$Z_j^B = \text{cf-adi}(A, B, J, \{p_1, \dots, p_J\}),$$

$$Z_j^C = \text{cf-adi}(A^T, C^T, J, \{q_1, \dots, q_J\}).$$

Note that  $A = -\mathcal{L}^{-1}\mathcal{G}$  implies  $(A + pI)^{-1}w = (p\mathcal{L} - \mathcal{G})^{-1}\mathcal{L}w$ , so  $A = -\mathcal{L}^{-1}\mathcal{G}$  does not need to be explicitly calculated in the CF-ADI algorithm. Similarly,  $A^T$  does not need to be explicitly calculated either.

### 3.1. Dominant Eigenspace Approximation

Because of cost considerations, CF-ADI is run only a small number of steps,  $J \ll n$ . Hence the CF-ADI approximations  $P_j^{\text{cfadi}} \equiv Z_j^B(Z_j^B)^T$  and  $Q_j^{\text{cfadi}} \equiv Z_j^C(Z_j^C)^T$  are necessarily low rank. Ideally, these low approximations should capture the dominant eigenspaces of the exact gramians. This is, in a sense, the most one could expect out of low rank approximations.

### 3.2. Numerical results

This section provides numerical results on how well CF-ADI approximates the dominant eigenspace of the exact solution  $X$ . The matrix whose columns are the  $j$  exact dominant eigenvectors of  $X$  is denoted by  $U_j^{\text{opt}} = [u_1^{\text{opt}}, \dots, u_j^{\text{opt}}]$ , since the columns of  $U_j^{\text{opt}}$  span the range of  $X_j^{\text{opt}}$ , the optimal rank  $j$  approximation to  $X$ .

Figure 1 illustrates a dominant eigenspace approximation for an example which comes from inductance extraction of an on-chip planar square spiral inductor suspended over a copper plane. The original order 500 system was symmetrized according to (Miguel Silveira *et al.*, 1996). The matrix  $A$  is a symmetric  $500 \times 500$  matrix, and the input coefficient matrix  $B \in \mathbb{R}^{500}$  has only one column.

Figure 1 shows the numerical results after 20 CF-ADI iterations. The relative error after 20 iterations is  $\|X - X_j^{\text{cfadi}}\|_2 / \|X\|_2 = 10^{-8}$ .

Figure 1(a) measures the closeness of the twenty-dimensional dominant eigenspaces of  $X$  and  $X_{20}^{\text{cfadi}}$ . This measure is provided by the concept of principle angles between subspaces (Golub and van Loan, 1996). Let  $S^1$  and  $S^2$  be two subspaces, of dimensions  $d_1$  and  $d_2$ , respectively, and assume that  $d_1 \geq d_2$ . Then the  $d_2$  principle angles are defined as  $\theta_1, \dots, \theta_{d_2}$  such that

$$\cos(\theta_j) = \max_{u^1 \in S^1, \|u^1\|=1} \max_{u^2 \in S^2, \|u^2\|=1} (u^1)^T u^2 = (u_j^1)^T u_j^2 \quad (29)$$

under the constraints

$$(u^1)^T u_i^1 = 0, \quad (u^2)^T u_i^2 = 0, \quad i = 1, \dots, j-1. \quad (30)$$

If the columns of  $U^1$  are an orthonormal basis for  $S^1$ , and the columns of  $U^2$  an orthonormal basis for  $S^2$ , and  $(U^2)^T U^1$  has a singular value decomposition,

$$(U^1)^T U^2 = U \Sigma V^T, \quad (31)$$

then

$$\cos(\theta_j) = \Sigma(j, j), \quad u_j^1 = U^1 U(:, j), \quad u_j^2 = U^2 V(:, j). \quad (32)$$

Thus these two bases,  $\{u_1^1, \dots, u_{d_2}^1\}$  and  $\{u_1^2, \dots, u_{d_2}^2\}$ , are mutually orthogonal,  $(u_i^1)^T u_j^2 = 0$ , if  $i \neq j$ . Here  $(u_i^1)^T u_i^2 = \cos(\theta_i)$  indicates the closeness of  $u_i^1$  and



$u_i^2$ . If  $S^1 = S^2$ , then  $\cos(\theta_j) = 1$ ,  $j = 1, \dots, d_1 = d_2$ . If  $S^1 \perp S^2$ , then  $\cos(\theta_j) = 0$ ,  $j = 1, \dots, d_2$ . A basis for the intersection of  $S^1$  and  $S^2$  is given by those basis vectors whose principle angle is 0,

$$\text{range}(S^1) \cap \text{range}(S^2) = \text{span}\{u_1^1, \dots, u_s^1\} = \text{span}\{u_1^2, \dots, u_s^2\}, \quad (33)$$

$$1 = \cos(\theta_1) = \dots = \cos(\theta_s) > \cos(\theta_{s+1}). \quad (34)$$

Thus the closeness of two subspaces is measured by how many of their principle angles are close to 0.

In Fig. 1(a), the cosines of the principle angles between  $U_{20}^{\text{cfadi}}$  and  $U_{20}^{\text{opt}}$  are plotted. The cosines of 18 of the principle angles are 1, and the cosines of the last two are above 0.85, indicating a close match for all dominant eigenvectors. This is not surprising since  $\|X - X_{20}^{\text{cfadi}}\|/\|X\|$  is less than  $10^{-8}$ .

Because the eigenvectors of  $X_{20}^{\text{opt}}$  associated with larger eigenvalues are more important than the eigenvectors of  $X_{20}^{\text{opt}}$  associated with smaller (non-zero) eigenvalues in view of a later application to model reduction, as they indicate the more controllable or observable modes among the top twenty, it is worthwhile to see how well each eigenvector of  $X_{20}^{\text{opt}}$  is individually matched by  $U_{20}^{\text{cfadi}}$ . This is measured by the norm of the projection of the exact dominant eigenvector,  $u_j^{\text{opt}}$ , onto  $U_{20}^{\text{cfadi}}$ . The direction  $u_j^{\text{opt}}$  is contained in the column span of  $U_{20}^{\text{cfadi}}$  if  $\|(u_j^{\text{opt}})^T U_{20}^{\text{cfadi}}\|_2 = 1$ . This is a different criterion than the one based on principle angles, as  $u_j^{\text{opt}}$  may not be one of the vectors in the orthogonal basis in (32).

As can be seen in Fig. 1(b), and not from Fig. 1(a),  $u_{20}^{\text{opt}}$  is better represented by the vectors in  $U_{20}^{\text{cfadi}}$  than  $u_{19}^{\text{opt}}$ . Everything being equal, it is preferable for  $u_{19}^{\text{opt}}$  to be better represented than  $u_{20}^{\text{opt}}$ , because  $u_{19}^{\text{opt}}$  is more important in terms of controllability or observability.

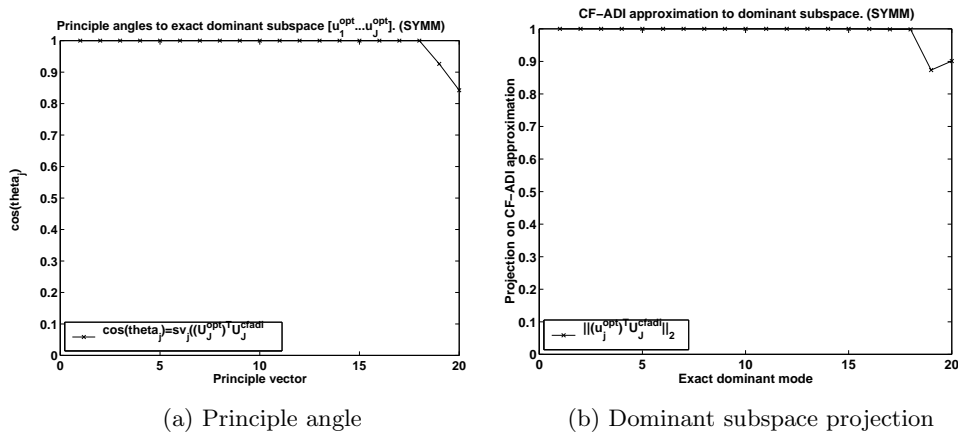


Fig. 1. Results for a symmetric matrix,  $n = 500$ , 20 CF-ADI iterations, convergence.

In contrast to Figs. 1, Figs. 2 and 3 demonstrate a dominant subspace approximation when CF-ADI is not run to convergence.

Figure 2 is the same spiral inductor example as in Figure 1, but CF-ADI is only run for 7 steps. In Fig. 2(a),  $\|z_7^{\text{cfadi}}\|$  is small but  $\|X - X_j^{\text{cfadi}}\|_2$  has stagnated. The relative error  $\|X - X_7^{\text{cfadi}}\|_2 / \|X\|_2$  is between  $10^{-2}$  and  $10^{-3}$ , whereas the relative error of the optimal 7-th rank approximation is  $10^{-5}$ . However, it can be seen from Fig. 2(b) that the intersection of the column span of  $U_7^{\text{cfadi}}$  and the column span of  $U_7^{\text{opt}}$  has dimension 6, since the cosines of 6 principle angles are 1. In Fig. 2(c), it can be seen that the top five dominant eigenvectors of  $X$ , the five most important modes, are contained entirely in the column span of  $U_7^{\text{cfadi}}$ . The norm of the projection of  $u_6^{\text{opt}}$  onto  $U_7^{\text{cfadi}}$  is around 0.9, while that of  $u_7^{\text{opt}}$  is around 0.5.

Thus dominant eigenspace information about  $X$  can emerge, even when CF-ADI has not converged.

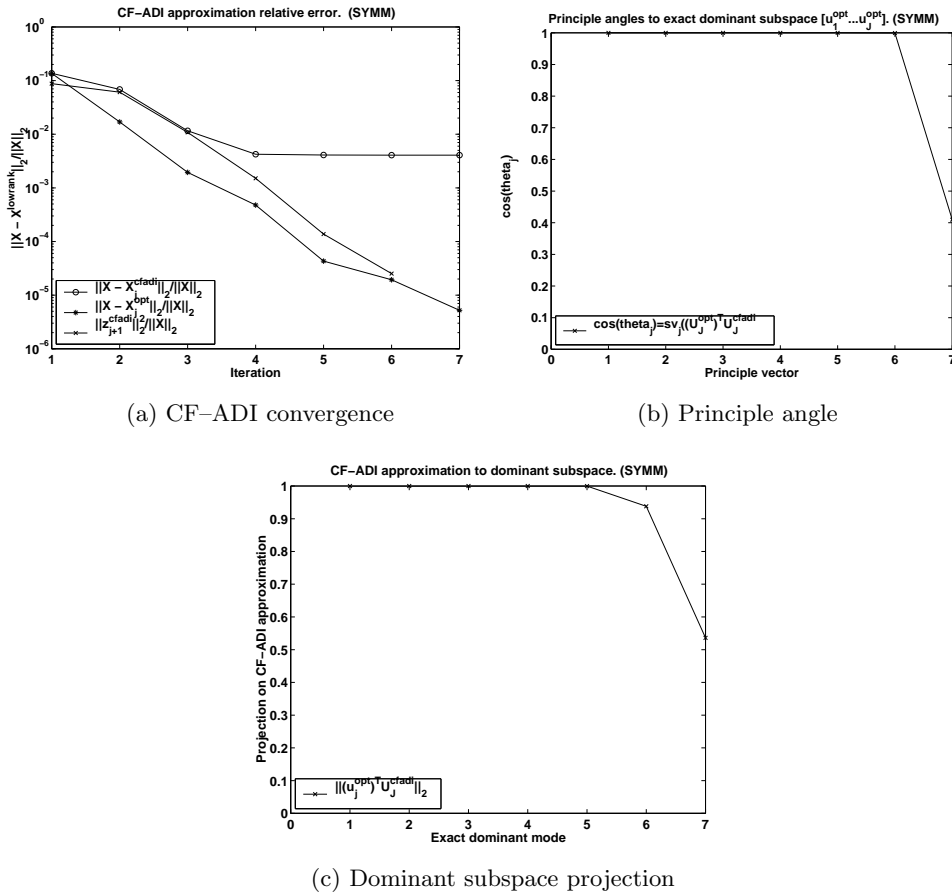
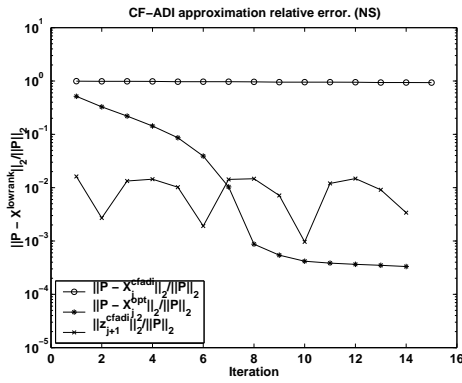


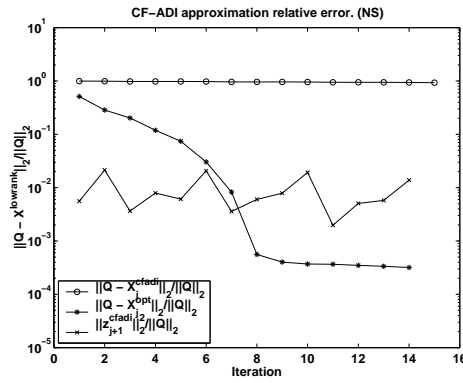
Fig. 2. Results for a symmetric matrix,  $n = 500$ , 7 CF-ADI iterations, without convergence.

Figure 3 shows another example of running CF-ADI only a small number of steps, before convergence occurs. It comes from the discretization of a transmission line using the formulation in (Marques *et al.*, 1998). The system matrix  $A$  is  $256 \times 256$ , the input matrix  $B$  has one column, the output matrix  $C$  has one row.

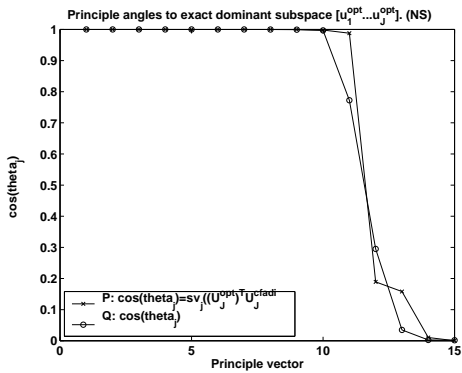
Figure 3 contains results for the solutions to the two Lyapunov equations (6), (7). The solution to (6) is denoted by  $P$ , and the solution to (7) is denoted by  $Q$ .



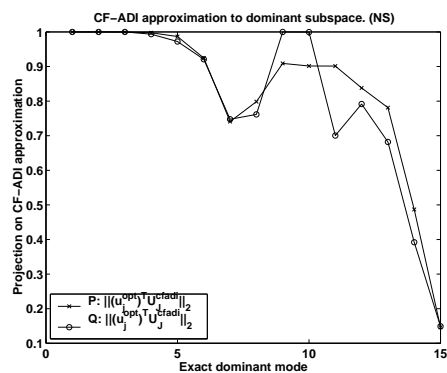
(a) CF-ADI convergence,  $AP + PA^T = -BB^T$



(b) CF-ADI convergence,  $A^T Q + QA = -C^T C$



(c) Principle angle



(d) Dominant subspace projection

Fig. 3. Results for a non-symmetric matrix,  $n = 256$ , 15 CF-ADI iterations, without convergence.

When compared with the Lyapunov solution associated with the spiral inductor example, whose system matrix is symmetric, the two Lyapunov solutions associated with the non-symmetric matrix  $A$  in this example have slower eigenvalue decay.

Since the eigenvalues of a non-symmetric matrix can be in an arbitrary region in the open left half plane, the problem of parameter selection is also more difficult for this example than for the symmetric example. The selection procedure in (Wachspress, 1995) was followed, and the resulting parameters are complex.

Figures 3(a) and 3(b) show that the CF-ADI error is not decreasing at all during 15 iterations. The relative error stagnates at 1. However, Fig. 3(c) shows that the intersection of the span of the fifteen dominant eigenvectors of  $P$  and the span of the fifteen dominant eigenvectors of the CF-ADI approximation has dimension 10 (almost 11). Similarly, the intersection of the span of the fifteen dominant eigenvectors of  $Q$  and the span of the fifteen dominant eigenvectors of the CF-ADI approximation has dimension 10.

Figure 3(d) provides an interesting picture. Recall that eigenvectors of  $P$  or  $Q$  associated with larger eigenvalues are more important than the eigenvectors associated with smaller eigenvalues. In Fig. 3(d), a lower index indicates a more important eigenvector. It can be seen that the five most important eigenvectors of  $P$  ( $Q$ ) are represented almost completely in  $\text{span}(U_{15}^{\text{cfadi}-P(Q)})$ . What is interesting is that the 9-th and 10-th eigenvectors of  $Q$  are also completely represented, even though eigenvectors 7 and 8 are not. The eigenvectors of  $P$  display similar, if not as dramatic, behavior, whereby some middle eigenvectors are not as well captured as the eigenvectors to their left and right.

This example demonstrates that even if the CF-ADI error is large, some information about the dominant eigenspace can still emerge, although there may also be missing information.

### 3.3. Shifted Linear Systems with the Same RHS

In this section we exploit the structure of the CF-ADI approximation  $Z_J$  to obtain a more efficient algorithm when the shifted solves in (27) are obtained iteratively using a Krylov subspace method such as GMRES.

Recall that

$$z_1 = \sqrt{-2p_1}(A + p_1I)^{-1}B, \quad (35)$$

$$M_i = \frac{\sqrt{-2p_{i+1}}}{\sqrt{-2p_i}} [I - (p_{i+1} + p_i)(A + p_{i+1}I)^{-1}]. \quad (36)$$

Then

$$Z_J = [z_1, M_1z_1, M_2M_1z_1, \dots, M_{J-1}M_{J-2} \dots M_1z_1]. \quad (37)$$

$Z_J$  in (37) is obtained after  $J$  matrix-vector solves, where the right-hand side of each solve is the solution of the previous solve.

Now we show how to convert (37) to  $J$  linear systems with the same right-hand side  $B$ . Writing  $Z_J$  out explicitly, we get

$$\begin{aligned} Z_J &= [\sqrt{-2p_1}(A + p_1I)^{-1}B, \\ &\quad \sqrt{-2p_2}[I - (p_2 + p_1)(A + p_2I)^{-1}](A + p_1I)^{-1}B, \\ &\quad \vdots \\ &\quad \sqrt{-2p_J}[I - (p_J + p_{J-1})(A + p_JI)^{-1}] \cdots \\ &\quad \cdots [I - (p_2 + p_1)(A + p_2I)^{-1}](A + p_1I)^{-1}B], \end{aligned} \quad (38)$$

and for simplicity, assume that the ADI parameters,  $p_1, \dots, p_J$  are not repeated,  $p_1 \neq p_2 \neq \dots \neq p_J$ . Taking out scalar multiplications and additions by the identity matrix from (3.3), and expanding  $\prod_{i=1}^j (A + p_iI)^{-1}$  into partial fractions, we get

$$\prod_{i=1}^j (A + p_iI)^{-1} = \sum_{i=1}^j \left( \prod_{k \neq i} \frac{1}{p_k - p_i} \right) (A + p_iI)^{-1}. \quad (39)$$

$Z_J$  then becomes

$$Z_J = V_J M_{J \times J} D_{J \times J}, \quad (40)$$

where

$$V_J \equiv [(A + p_1I)^{-1}B, (A + p_2I)^{-1}B, \dots, (A + p_JI)^{-1}B], \quad (41)$$

$$M_{J \times J} \equiv \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1J} \\ 0 & m_{22} & \cdots & m_{2J} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & m_{JJ} \end{bmatrix}, \quad (42)$$

$$m_{11} = 1, \quad (43)$$

$$m_{ii} = - \sum_{j=1}^{i-1} m_{i-1,j} \frac{p_{i-1} + p_i}{p_j - p_i}, \quad (44)$$

$$m_{j,i} = m_{j,i-1} \frac{p_j + p_{i-1}}{p_j - p_i}, \quad j \neq i, \quad (45)$$

and

$$D_{J \times J} = \begin{bmatrix} \sqrt{-2p_1} & 0 & \cdots & 0 \\ 0 & \sqrt{-2p_2} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \sqrt{-2p_J} \end{bmatrix}. \quad (46)$$

The matrices  $M_{J \times J}$  and  $D_{J \times J}$  are determined completely by the parameters  $p_1, \dots, p_J$ , and cost very little to compute. Thus the cost of calculating  $Z_J$  via (40) comes almost entirely from the calculation of  $V_J$  in (41).

### 3.4. Sharing of Krylov Vectors

Columns of  $V_J$  in (41) can be obtained either exactly, using  $J$  LU factorizations, or approximately, using  $J$  iterative linear system solves,

$$V_J = [v_1, v_2, \dots, v_J], \quad (47)$$

$$(A + p_i I)v_i = B, \quad i = 1, \dots, J. \quad (48)$$

If an iterative Krylov subspace method such as GMRES is used, and if none of the solves in (48) is too difficult, the columns of  $V_J$  can be obtained in a much more efficient way than by doing  $J$  separate solves. The solution of shifted systems is discussed in detail in (Freund, 1993b).

An iterative Krylov subspace method such as GMRES solves the system  $Ax = B$  by finding an approximate solution  $x_m$  in the  $m$ -dimensional Krylov subspace

$$x_m \in \mathcal{K}_m(A, r_0) \equiv \text{span} \{r_0, Ar_0, \dots, A^{m-1}r_0\}. \quad (49)$$

GMRES chooses  $r_0 = B - Ax_0$ . The difficulty of solving a system in (48), or in other words, the dimension of the Krylov subspace required to find a satisfactory solution, depends on the shift  $p_i$ .

If zero is used as the initial guess for all system solves in (48), the Krylov subspace associated with each system is the same, namely,  $\mathcal{K}_m(A, B)$ , since shifts of  $A$  do not affect the Krylov subspace,

$$\mathcal{K}_m(A + p_i I, B) = \text{span} \{B, (A + p_i I)B, \dots, (A + p_i I)^{m-1}B\} \equiv \mathcal{K}_m(A, B).$$

Hence one needs only one set of Krylov vectors for all solves in (48), which can be stored from solve to solve. When a more difficult shift is encountered, one simply adds to the list of stored Krylov vectors.

What is different for each solve in (48) is that decompositions of different Hessenberg matrices are needed. If  $\tilde{H}_m$  denotes the Hessenberg matrix which comes from  $m$  steps of the Arnoldi process with the matrix  $A$ , for the system  $Ax = B$ , then  $\tilde{H}_m + [-p_i I_{kk}]$  is the Hessenberg matrix associated with the shifted system  $(A + p_i I)x_i = B$ . For each system in (48), the decomposition of a different Hessenberg matrix is needed. But if none of the systems in (48) is too difficult, or in other words, if they all can be solved in Krylov subspaces whose dimension is small when compared with the size of  $A$ , then the cost of decomposing small Hessenberg matrices will be low compared with the cost of generating Krylov vectors, and hence the cost of solving  $J$  shifted systems would be only marginally higher than the cost of doing one most difficult solve.

Figure 4 shows the speed-up in the calculation of the CF-ADI approximation  $Z_J$ , which comes from storing the Krylov vectors between solves. The matrix  $A$

is  $500 \times 500$ , and its eigenvalues are well-distributed for fast GMRES convergence. The flops required to generate  $Z_J$ , as a function of  $J$ , are plotted. Doing  $J$  solves separately and not storing the Krylov vectors is denoted by  $+$ , generating  $V_J$  by storing Krylov vectors is represented as  $\times$ , the total cost of obtaining  $Z_J$  from  $V_J$ , including the generation of the matrices  $M_{J \times J}$  and  $D_{J \times J}$  in (40), is shown as  $\circ$ .

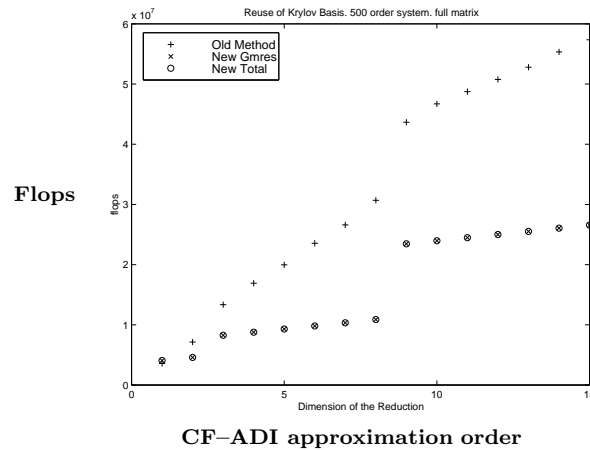


Fig. 4. Negligible cost of additional solves.

Disregarding the jumps at  $J = 3$  and  $J = 9$  for the moment, it can be seen that when Krylov vectors are not stored, the cost of generating  $Z_J$  grows linearly with  $J$ , whereas if the Krylov vectors are stored, the cost of generating  $Z_J$  increases very little as  $J$  increases. The cost of generating  $Z_8$  is only slightly higher than the cost of generating  $Z_3$ . The jump at  $J = 3$  occurred because  $p_3$  is a more difficult shift, so the solution of  $(A + p_3 I)v_i = B$  required more Krylov vectors than the previous two solves. But after the extra Krylov vectors were generated, more solves after  $J = 3$  cost very little, until the next difficult shift at  $J = 9$ .

#### 4. Reduction via the Sum of Dominant Gramian Eigenspaces

Because balancing the gramians requires knowledge of the entire eigenspaces of both gramians, it is in general not possible to approximate a Truncated Balanced Realization without good approximations to the full eigenspaces of both the gramians.

If the CF-ADI errors from solving both (6) and (7) are small after a small number of iterations, then both the gramians are low rank and have been well approximated by CF-ADI. In this case, a low rank implementation of the Square Root method to produce the TBR reduction can be used. This approach was proposed in (Penzl, 1999a).

However, if the CF-ADI errors from solving (6) and (7) are not small, then we only have approximations to the dominant eigenspaces of the two gramians. We have no information on the remaining gramian eigenspaces.

Since, in general, only the dominant eigenspaces of the controllability and observability gramians are obtainable cheaply, we seek a model reduction method which utilizes all the available information. We propose projecting the original system onto the sum of the two dominant eigenspaces. A preliminary version of this approach can be found in (Li and White, 1999).

**Algorithm 3.** ADI-CTOB

1. Set  $Z_J^B = \text{CF-ADI}(A, B, J, \{p_1, \dots, p_J\})$ .
2. Set  $Z_J^C = \text{CF-ADI}(A^T, C^T, J, \{q_1, \dots, q_J\})$ .
3. Calculate SVD:  $Z_J^B = U_{n \times J}^B D_{J \times J}^B (V_{J \times J}^B)^T$ ,  $Z_J^C = U_{n \times J}^C D_{J \times J}^C (V_{J \times J}^C)^T$ .
4. Choose  $k \leq J$ ,  $2k$  being the desired reduction order, and find

$$U_m^{\text{ctob}} = \text{qr}([U_{n \times J}^B(:, 1:k), U_{n \times J}^C(:, 1:k)]).$$

Note that  $k \leq m = \text{rank}(U_m^{\text{ctob}}) \leq 2k$ .

5. Reduce the system

$$A_m^r = (U_m^{\text{ctob}})^T A U_m^{\text{ctob}}, B_m^r = (U_m^{\text{ctob}})^T B, C_m^r = C U_m^{\text{ctob}}. \quad (50)$$

#### 4.1. Moment Matching

If  $k = J$  in the fourth step of the ADI-CTOB algorithm, then the projection matrix  $U_m^{\text{ctob}}$  contains the sum of the column spans of  $Z_J^B$  and  $Z_J^C$ . From (40) and (41), it can be seen that

$$\begin{aligned} \text{colsp}(Z_J^B) &= \sum_{i=1}^j \mathcal{K}_{k_i} ((A + p_i I)^{-1}, (A + p_i I)^{-1} B), \\ \text{colsp}(Z_J^C) &= \sum_{i=1}^l \mathcal{K}_{n_i} ((A^T + q_i I)^{-1}, (A^T + q_i I)^{-1} C^T), \end{aligned}$$

where  $\{p_1, \dots, p_j\}$  is a list of distinct CF-ADI parameters used in generating  $Z_J^B$ ,  $k_i$  being the number of times  $p_i$  appears in the list, and  $\{q_1, \dots, q_l\}$  is the list of distinct CF-ADI parameters used in generating  $Z_J^C$ ,  $n_i$  being the number of times  $q_i$  appears.

Then by Theorem 3.1 in (Grimme, 1997), the transfer function of the reduced system in (50) will match moments of the original transfer function, at the negatives of the CF-ADI parameters,  $\{-p_1, -p_2, \dots, -p_j\}$  and  $\{-q_1, -q_2, \dots, -q_l\}$ . The number



of moments matched at  $p_i$  is  $k_i$ , and the number matched at  $q_i$  is  $n_i$ , unless there are  $i$  and  $j$  such that  $p_i = q_j$ , as then the number of moments matched at  $p_i = q_j$  is  $k_i + n_j$ .

Thus, in trying to approximate the Truncated Balanced Realization, we have returned to moment matching. The advantages of the ADI-CTOB algorithm over a purely moment matching approach include the existence of methods for picking good CF-ADI parameters (Ellner and Wachspress, 1991; Lu and Wachspress, 1991) and the fact that  $Z_J^B$  and  $Z_J^C$  contain more information than the projection matrices from moment matching. The singular values of  $Z_J^B$  ( $Z_J^C$ ) indicate how controllable (observable) a mode is. This information can be used in error estimation and compact model generation.

## 5. Numerical Results

The new model reduction method was tested and compared with TBR and moment matching around  $s = 0$  via Lanczos. The figures in this section come from the same discretized transmission line example as shown in Fig. 3, and illustrate the general case when the dominant eigenspaces of the two gramians are different.

Figure 5(a) shows that projection via the union of the most controllable modes and the most observable modes is a good idea. It compares projection by the sum of the exact dominant eigenspaces (CT U OB) with Truncated Balanced Realization. Both the reductions are of order 10. CT5 U OB5 is obtained by calculating the exact solutions to Lyapunov equations (6), (7), and by using the sum of the two five-dimensional exact dominant eigenspaces. It does not use any CF-ADI approximations. It can be seen that, for this transmission line example, projection by the sum of the exact dominant eigenspaces produces a reduced model that is almost indistinguishable from TBR.

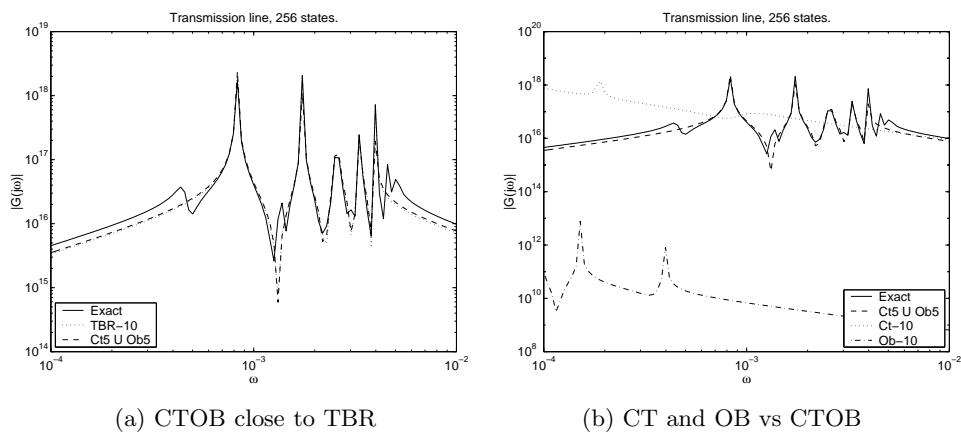
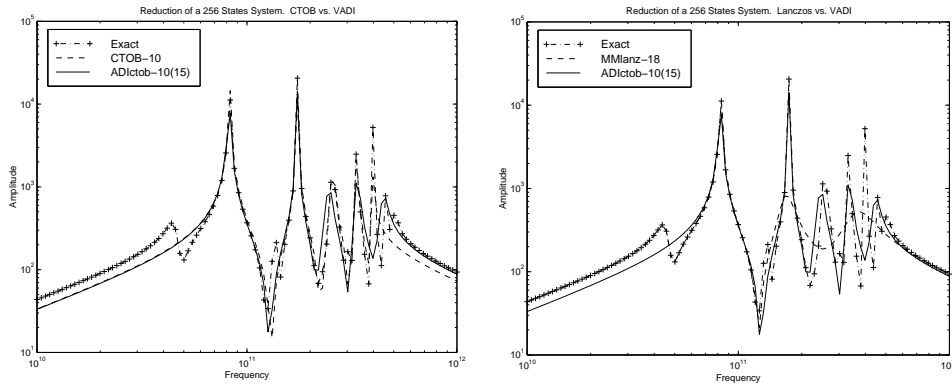


Fig. 5. Dominant Gramian Eigenspaces method.

Figure 5(b) compares CT5 U OB5 with projection by either the ten-dimensional dominant controllable subspace only (CT-10), or the ten-dimensional dominant observable subspace only (OB-10). Neither CT-10 nor OB-10 alone comes close to capturing the frequency response behavior of the original system as well as using the sum of 5 and 5.

Figure 6 makes a comparison using CF-ADI to calculate the dominant gramian eigen-spaces (ADictob), with finding them exactly (CT5 U OB5). Both are of order 10. The two five-dimensional dominant eigenspaces used in ADictob-10(15) are each obtained by 15 iterations of CF-ADI. The frequency responses are close except at the last two peaks. CT5 U OB5 follows the next to last peak and then flattens out, whereas ADictob-10(15) misses the next to last peak and finds the last one.



(a) Dominant Gramian Eigenspaces via CF-ADI (b) Comparison with moment matching

Fig. 6. Dominant Gramian Eigenspaces via CF-ADI.

Figure 6 compares moment matching via Lanczos (MMLanz) with ADictob. The MMLanz reduction is of order 18, requiring 34 matrix-vector solves, while ADictob is of order 10, requiring 30 matrix-vector solves. ADictob clearly captures the global frequency response behavior much better than MMLanz. It captured all but the next to last sharp peak and averages the first tiny peak and a couple of small bumps between sharp peaks. This keeps the  $L^\infty$  error small without having to follow every topographical feature exactly. MMLanz matches the exact frequency response extremely well near  $s = 0$ , where the moments were matched, but completely loses accuracy after the first sharp peak.

## 6. DC Matching and Passivity

Simply projecting the original system via the congruence transformation in (50) will not, in general, guarantee moment matching at  $s = 0$ . But it is very important in the reduction of large circuit models that DC response is preserved in the reduced system.

We propose augmenting  $U_m^{\text{ctob}}$  with several Arnoldi vectors to achieve matching at  $s = 0$  (DC). Let  $U_l^{\text{dc}}$  be a matrix whose columns form an orthonormal basis for  $\mathcal{K}_l(A^{-1}, A^{-1}B)$ . We then augment the projection matrix  $U_m^{\text{ctob}}$  by  $U_l^{\text{dc}}$ . Let  $U^{\text{red}}$  be a matrix whose columns form an orthonormal basis for  $[U_l^{\text{dc}}, U_m^{\text{ctob}}]$ , we then project the system using the augmented projection matrix,  $U^{\text{red}}$ .

Figures 7(a) and 7(b) show the effect of including three Arnoldi vectors in the ten-dimensional projection matrix (ADIctob-10+3). It can be seen that including three Arnoldi vectors resulted in good matching near  $s = 0$ , without detrimental effect on the quality of the approximation in the rest of the frequency domain.

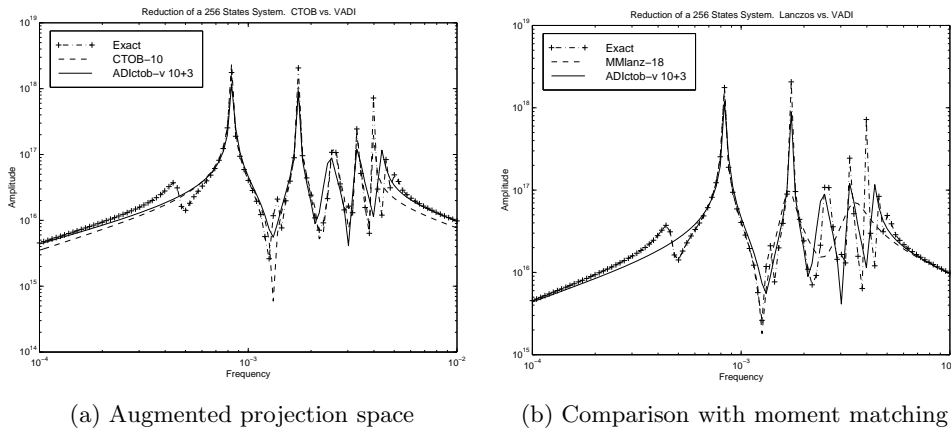


Fig. 7. DC match with no detrimental effect on the rest of the frequency domain.

Odabasioglu *et al.* (1998) showed that for multi-port RLC circuits which come from modified nodal analysis, where  $C = N^T$  in (1) and (2), if  $\mathcal{L}$  and  $\mathcal{G}$  are each projected separately via an orthogonal projection matrix,  $U^{\text{red}}$ ,

$$\begin{aligned} \mathcal{L}^{\text{red}} &= (U^{\text{red}})^T \mathcal{L} U^{\text{red}}, \\ \mathcal{G}^{\text{red}} &= (U^{\text{red}})^T \mathcal{G} U^{\text{red}}, \\ B^{\text{red}} &= (U^{\text{red}})^T B, \quad C^{\text{red}} = C U^{\text{red}}, \end{aligned} \quad (51)$$

then the reduced system preserves passivity. This result holds for general orthogonal projection matrices, including the augmented projection matrix described in the previous section.

## 7. Conclusions

We have presented a model reduction algorithm that is well-suited for the reduction of large interconnect models. It is efficient and utilizes all the gramian information available from performing cheap operations, such as linear system solves. It is an

orthogonal projection method which can easily be adapted to allow for DC matching and passivity preservation.

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