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IDENTIFICATION OF A QUASILINEAR PARABOLIC EQUATION FROM FINAL DATA[†]

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We study the identification of the nonlinearities $A, \vec{b}$ and c appearing in the quasilinear parabolic equation

$$y_t - \text{div}\left(A(y)\nabla y + \vec{b}(y)\right) + c(y) = u \quad \text{in } \Omega \times (0, T),$$

assuming that the solution of an associated boundary value problem is known at the terminal time, $y(x, T)$, over a (probably small) subset of Ω, for each source term u. Our work can be divided into two parts. Firstly, the uniqueness of $A, \vec{b}$ and c is proved under appropriate assumptions. Secondly, we consider a finite-dimensional optimization problem that allows for the reconstruction of the nonlinearities. Some numerical results in the one-dimensional case are presented, even in the case of noisy data.

Keywords: quasilinear parabolic equation, inverse problem, parameter estimation

1. Introduction

Let us consider the following boundary value problem associated with the quasilinear parabolic equation:

$$\begin{cases} y_t(x,t) - \text{div}\left(A\big(y(x,t)\big)\nabla y(x,t) + \vec{b}\big(y(x,t)\big)\right) + c\big(y(x,t)\big) = u(x,t) & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $Q = \Omega \times (0, T)$, Ω being a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $T > 0$ and $\Sigma = \partial\Omega \times (0, T)$.

We are concerned with the following type of *inverse problem*: Given an initial datum y_0, a source term u, and having some knowledge about the solution y, we want to determine the nonlinearities $(A, \vec{b}, c)$ of the operator.

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The reconstruction of one nonlinearity of a quasilinear parabolic operator has already been considered in different settings in the literature, among others in (Barbu and Kunisch, 1995; Chavent and Lemonnier, 1974). As more recent, we highlight (Hanke and Scherzer, 1999; Kärkkäinen, 1996). These analyses were carried out while assuming that some distributed observations of the solution are given *a priori*: In (Kärkkäinen, 1996), for a term u, the corresponding solution $y(x,t)$ is known for every $(x,t) \in Q$. In (Hanke and Scherzer, 1999), where $n = 1$, some points $x_1, \ldots, x_m$ are fixed in Ω, and $y(x_i, t)$ is known for all $i \in \{1, \ldots, m\}$ and every $t \in [0, T]$. Our approach is somewhat different: we will assume that for each source term u, the solution at the final time $y(x, T)$ is known over a (probably small) fixed non-empty open subset of Ω. On the other hand, several nonlinearities can be identified at the same time. Exactly, up to $(n+2)(n+1)/2$ nonlinearities can be recovered, not just one as in the cited references.

Our work can be divided into two parts. In the next section, the uniqueness of $(A, \vec{b}, c)$ is proved under reasonable assumptions (see (H1)–(H6) below). This property can be deduced by using the approximate controllability for this type of equations, when the function u is viewed as a control. Section 3 is devoted to the practical reconstruction of the nonlinearities in the one-dimensional space case. We use an output least-squares method with a regularization technique to transform the identification problem to a finite-dimensional optimization problem. In order to illustrate the applicability of our approach, we present some numerical results taking into account the influence of noise on the data.

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2. Uniqueness of the Nonlinearities

Let us introduce the hypotheses that will be assumed regarding the nonlinearities of the quasilinear parabolic operator:

(H1) $A(y) = (a_{ij}(y))_{1 \le i,j \le n}$, where $a_{ij} \in C^{1+\delta}(\mathbb{R})$ for some $\delta \in (0,1)$, $a_{ij}(y) = a_{ji}(y)$ for every $y \in \mathbb{R}$ and all $i, j \in \{1, \ldots, n\}$. Moreover, there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \ge \alpha \|\xi\|^2, \quad \forall y \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^n.$$

(H2) $\vec{b}(y) = (b_1(y), \ldots, b_n(y))$, where $b_i \in C^{1+\delta}(\mathbb{R})$ for each $i \in \{1, \ldots, n\}$ with $\vec{b}(0) = \vec{0}$.

(H3) $c \in C^{1+\delta}(\mathbb{R})$ and there exists $\beta \le 0$ such that

$$\frac{\mathrm{d}c}{\mathrm{d}y}(y) \ge \beta, \quad \forall y \in \mathbb{R}.$$

With this notation, the parabolic operator $y_t - \operatorname{div}\,(A(y)\nabla y + \vec{b}(y)) + c(y)$ is the usual abridged representation of

$$\frac{\partial y}{\partial t}(x,t) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}\big(y(x,t)\big)\frac{\partial y}{\partial x_j}(x,t) + b_i\big(y(x,t)\big)\right) + c\big(y(x,t)\big).$$

Remark 1.

- Clearly, the operator in (1) does not change if $\vec{b}(y)$ is replaced by $\vec{b}(y) + \vec{k}$, where $\vec{k}$ is any element of $\mathbb{R}^n$. Hence, condition $\vec{b}(0) = 0$ is just imposed to fix ideas.
- Needless to say, condition $\beta \leq 0$ in (H3) is not restrictive. Obviously, if it is satisfied with $\beta > 0$, it also holds for $\beta = 0$.
- The regularity conditions on the nonlinearities can be weakened by taking $\delta = 0$ in some particular cases, e.g. to recover $A(y)$ (with $\vec{b}(y) = \vec{0}$ and $c(y) = 0$), or to identify $c(y)$, assuming $a_{ij}(y) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ and $\vec{b}(y) = \vec{0}$.

For brevity, we set

$$\mathcal{C} = \big\{(A, \vec{b}, c) \text{ verifying (H1)–(H3)}\big\}.$$

As usual, given $k \in \mathbb{N}$ and $\delta, \tilde{\delta} \in [0,1)$, we write

$$C^{\delta}(\overline{\Omega}) = \left\{ y \in C(\overline{\Omega}) : \sup_{x,\tilde{x}\in\overline{\Omega}, x\neq\tilde{x}} \frac{|y(x) - y(\tilde{x})|}{|x-\tilde{x}|^{\delta}} < +\infty \right\},$$

$$C^{k+\delta}(\overline{\Omega}) = \left\{ y \in C^{\delta}(\overline{\Omega}) : \frac{\mathrm{d}^j y}{\mathrm{d}x^j} \in C^{\delta}(\overline{\Omega}), \quad \forall j = 1, \ldots, k \right\}.$$

What is more,

$$C^{\delta,\tilde{\delta}}(\overline{Q}) = \left\{ y \in C(\overline{Q}) : \sup_{(x,t),(\tilde{x},\tilde{t})\in\overline{Q},(x,t)\neq(\tilde{x},\tilde{t})} \frac{|y(x,t) - y(\tilde{x},\tilde{t})|}{|x-\tilde{x}|^{\delta} + |t-\tilde{t}|^{\tilde{\delta}}} < +\infty \right\},$$

$$C^{1+\delta,(1+\delta)/2}(\overline{Q}) = \left\{ y \in C^{0,(1+\delta)/2}(\overline{Q}) : \frac{\partial y}{\partial x_i} \in C^{\delta,\delta/2}(\overline{Q}) \quad \forall i = 1, \ldots, n \right\},$$

$$C^{2+\delta,1+\delta/2}(\overline{Q}) = \left\{ y \in C(\overline{Q}) : \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x_i}, \frac{\partial^2 y}{\partial x_i \partial x_j} \in C^{\delta,\delta/2}(\overline{Q}) \quad \forall i, j = 1, \ldots, n \right\}.$$

Given a triplet $(A, \vec{b}, c) \in \mathcal{C}$, it can be proved that problem (1) is well-posed at least when the right-hand term u, the initial datum y_0 and the boundary $\partial\Omega$ are regular enough, in the following sense:

Theorem 1. *Let us assume that* $(A, \vec{b}, c) \in \mathcal{C}$ *and, moreover, the following conditions are fulfilled:*

(H4) $\partial\Omega \in C^{2+\delta}$,

(H5) $u \in C^{\delta}(\overline{Q})$,

(H6) $y_0 \in C^{2+\delta}(\overline{\Omega})$ *and* $y_0(x) = 0$ *for all* $x \in \partial\Omega$.

Then there exists a unique solution y *of problem (1) belonging to the space* $C^{1+\rho,(1+\rho)/2}(\overline{Q}) \cap C^{2+\rho,1+\rho/2}(\overline{\Omega} \times [\epsilon, T])$ *for some* $\rho \in (0,1)$ *and every* $\epsilon > 0$.

Moreover, there exists a constant K *depending only on* β *and* T *such that*

$$\|y\|_{L^\infty(Q)} \leq K \left(|c(0)| + \|y_0\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(Q)}\right). \tag{2}$$

Proof. The existence of a unique solution y to problem (1) belonging to the space $C^{\rho_1,\rho_1/2}(\overline{Q})$ for some $\rho_1 \in (0,1)$ such that $\nabla y \in L^\infty(Q)$ is a consequence of Theorem 6.3 of (Ladyzhenskaya *et al.*, 1968, p.459). Let us point out that we are assuming the compatibility condition of zeroth order between the initial and boundary conditions (see (H6)), but not the compatibility condition of first order (Ladyzhenskaya *et al.*, 1968, p.449). By seeing the solution of the quasilinear problem as the corresponding one for a linear problem in divergence form with Hölder continuous coefficients, it can be deduced from Theorem 1 of (Lunardi and Vespri, 1991) that $y \in C^{1+\rho_1,(1+\rho_1)/2}(\overline{Q})$.

In order to complete the proof, it is enough to see y as the unique solution of the linear problem

$$\begin{cases} y_t(x,t) - \displaystyle\sum_{i,j=1}^{n} \tilde{a}_{ij}(x,t) \frac{\partial^2 y}{\partial x_i \partial x_j}(x,t) = \tilde{u}(x,t) & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{3}$$

where $\tilde{a}_{ij}(x,t) = a_{ij}(y(x,t))$, and

$$\tilde{u}(x,t) = u(x,t) - c\big(y(x,t)\big) + \sum_{i,j=1}^{n} \frac{\mathrm{d}a_{ij}}{\mathrm{d}y}\big(y(x,t)\big) \frac{\partial y}{\partial x_i}(x,t) \frac{\partial y}{\partial x_j}(x,t)$$
$$+ \sum_{i=1}^{n} \frac{\mathrm{d}b_i}{\mathrm{d}y}\big(y(x,t)\big) \frac{\partial y}{\partial x_i}(x,t).$$

By the known regularity of y and hypotheses (H1)–(H3), (H5), we can guarantee that each coefficient $\tilde{a}_{ij}$ and the right-hand side term $\tilde{u}$ belong to $C^{\rho_1\delta,\rho_1\delta/2}(\overline{Q})$. The lack of the first-order compatibility between the initial datum, the homogeneous boundary condition and the term $\tilde{u}$ does not allow us to conclude that $y \in C^{2+\rho,1+\rho/2}(\overline{Q})$ for $\rho = \rho_1\delta$. Nevertheless, it can be proved that $y \in C^{2+\rho,1+\rho/2}(\overline{\Omega} \times [\epsilon, T])$ for every $\epsilon > 0$ in the following way: Let us decompose problem (3) as the sum of the problems

$$\begin{cases} y_t(x,t) - \displaystyle\sum_{i,j=1}^{n} \tilde{a}_{ij}(x,t)\frac{\partial^2 y}{\partial x_i \partial x_j}(x,t) = \tilde{u}(x,t) & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_1(x) & \text{in } \Omega, \end{cases} \tag{4}$$

and

$$\begin{cases} y_t(x,t) - \displaystyle\sum_{i,j=1}^{n} \tilde{a}_{ij}(x,t)\frac{\partial^2 y}{\partial x_i \partial x_j}(x,t) = 0 & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) - y_1(x) & \text{in } \Omega, \end{cases} \tag{5}$$

where $y_1 \in C^{2+\rho}(\overline{\Omega})$ is the unique solution of the elliptic boundary-value problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \tilde{a}_{ij}(x,0)\frac{\partial^2 y}{\partial x_i \partial x_j}(x) = \tilde{u}(x,0) & \text{in } \Omega, \\ y(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

see (Gilbarg and Trudinger, 1977, Thm. 6.14, p.101).

Function $y_1(x)$ has been chosen in such a way that the compatibility condition of first order is now satisfied for problem (4). Hence it has a unique solution $y_1(x,t) \in C^{2+\rho,1+\rho/2}(\overline{Q})$ as a consequence of Theorem 5.3 of (Ladyzhenskaya *et al.*, 1968, p.320). On the other hand, problem (5) only satisfies the zeroth-order compatibility (i.e. $y_0(x) - y_1(x) = 0$ on $\partial\Omega$). Therefore another classical result of Ladyzhenskaya *et al.* (1968, Thm. 9.1, pp.341–342) implies that the unique solution $y_2(x,t)$ of (5) belongs to $C^{\rho,\rho/2}(\overline{Q})$. Given $\epsilon > 0$, introduce a function $\varphi_\epsilon(t) \in C^\infty([0,T])$ such that $\varphi_\epsilon(t) = 0$ on $[0, \epsilon/2]$ and $\varphi_\epsilon(t) = 1$ on $[\epsilon, T]$. Now, it is easy to show that $y_2(x,t)\varphi_\epsilon(t)$ is the unique solution to the problem (4), replacing $\tilde{u}(x,t)$ by $y_2(x,t)\varphi'_\epsilon(t) \in C^{\rho,\rho/2}(\overline{Q})$ and $y_1(x)$ by 0. For this problem, the compatibility condition of first order is clearly satisfied, and hence, applying once more Theorem 5.3 of (Ladyzhenskaya *et al.*, 1968), we get $y_2(x,t)\varphi_\epsilon(t) \in C^{2+\rho,1+\rho/2}(\overline{Q})$. We conclude by noticing that $y(x,t) = y_1(x,t) + y_2(x,t) = y_1(x,t) + y_2(x,t)\varphi_\epsilon(t)$ in $\overline{\Omega} \times [\epsilon, T]$.

Estimation (2) is a direct consequence of (Ladyzhenskaya *et al.*, 1968, Thm. 2.9, p.23), taking into account that here

$$a(x,t,y,\nabla y) = -\sum_{i,j=1}^{n} \frac{\mathrm{d}a_{ij}}{\mathrm{d}y}(y)\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j} - \sum_{i=1}^{n} \frac{\mathrm{d}b_i}{\mathrm{d}y}(y)\frac{\partial y}{\partial x_i} + c(y) - u(x,t).$$

By (H3) and the Mean-Value Theorem, we have

$$\begin{aligned} a(x,t,y,\vec{0})y &= \big(c(y)-u(x,t)\big)y = \big(c(0)-u(x,t)\big)y + \frac{\mathrm{d}c}{\mathrm{d}y}(\theta y)y^2 \\ &\geq \left(\beta - \frac{1}{2}\right)y^2 - \frac{1}{2}\big(|c(0)| + \|u\|_{L^\infty(Q)}\big)^2 \end{aligned}$$

for some $\theta \in (0,1)$.

Applying Theorem 2.9 of (Ladyzhenskaya *et al.*, 1968), it follows that

$$\begin{aligned} \|y\|_{L^\infty(Q)} &\leq \min_{\mu > 0.5-\beta} \exp(\mu T)\left(\|y_0\|_{L^\infty(\Omega)} + \frac{|c(0)| + \|u\|_{L^\infty(Q)}}{\sqrt{2(\mu+\beta-0.5)}}\right) \\ &\leq K\big(|c(0)| + \|y_0\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(Q)}\big), \end{aligned}$$

where

$$K = \min_{\mu > 0.5-\beta} \exp(\mu T)\left(1 + \frac{1}{\sqrt{2(\mu+\beta-0.5)}}\right).$$

■

Taking into account that the initial datum y_0 will remain fixed in the sequel, we will denote by $y_{u,A,\vec{b},c}$ the unique solution to problem (1).

Let us present some results concerning the approximate controllability for parabolic equations that will be useful later (see (Fernández and Zuazua, 1999; Lions, 1971) for related results).

For that purpose, we recall the space

$$W(0,T) = \left\{ y \in L^2\big(0,T;H_0^1(\Omega)\big) : y_t \in L^2\big(0,T;H^{-1}(\Omega)\big)\right\},$$

where $H_0^1(\Omega)$ denotes the usual Sobolev space and $H^{-1}(\Omega)$ is its dual.

Theorem 2. (Approximate Controllability)

(a) *Assume that* (H6) *is fulfilled and* $(A,\vec{b},c) \in \mathcal{C}$. *Then the set*

$$R(T) = \left\{ y_{u,A,\vec{b},c}(x,T) : u \in C^\delta(\overline{Q}) \right\}$$

is dense in $L^2(\Omega)$.

(b) *Suppose that* $\hat{a}_{ij}, \hat{b}_i \in L^\infty(Q)$ *for all* $i,j \in \{1,\dots,n\}$, $\hat{c} \in L^\infty(Q)$ *and there exists* $\alpha > 0$ *such that*

$$\sum_{i,j=1}^{n} \hat{a}_{ij}(x,t)\xi_i\xi_j \geq \alpha\|\xi\|^2 \qquad a.e.\ (x,t) \in Q, \quad \forall \xi \in \mathbb{R}^n.$$

Then the set

$$\mathcal{S} = \left\{ z_v(x,t) : v \in L^2(Q) \right\}$$

is dense in $L^2(0,T;H_0^1(\Omega))$, *where* z_v *denotes the unique solution in* $W(0,T)$ *of the linear problem*

$$\begin{cases} z_t(x,t) - \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n \hat{a}_{ij}(x,t)\frac{\partial z}{\partial x_j}(x,t) + \hat{b}_i(x,t)z(x,t)\right) & \\ \qquad + \hat{c}(x,t)z(x,t) = v(x,t) & \textit{in } Q, \\ z(x,t) = 0 & \textit{on } \Sigma, \\ z(x,0) = 0 & \textit{in } \Omega. \end{cases} \tag{7}$$

Proof. Part (a) is a consequence of the following argumentation containing an exact controllability result: Given $y_d \in L^2(\Omega)$, for each $\epsilon > 0$, there exists $\hat{y} \in \mathcal{D}(\Omega)$ (i.e. a function in $C^\infty(\overline{\Omega})$ with compact support strictly contained in Ω) such that $\|y_d - \hat{y}\|_{L^2(\Omega)} < \epsilon$. So the function

$$y(x,t) = y_0(x) + \frac{t}{T}\big(\hat{y}(x) - y_0(x)\big) \quad x \in \Omega, \quad t \in (0,T)$$

can be viewed as the unique solution of (1) with $u = y_t - \operatorname{div}\big(A(y)\nabla y + \vec{b}(y)\big) + c(y)$ and $y(T) = \hat{y}$. By using hypotheses (H1)–(H3), it is straightforward to verify that $u \in C^\delta(\overline{Q})$.

In Part (b), for each $v \in L^2(Q)$, it is well-known (Lions, 1971) that (7) has a unique solution z_v in $W(0,T)$. Suppose that there exists an element $\varphi \in L^2(0,T;H^{-1}(\Omega))$ such that

$$\langle \varphi, z_v \rangle = 0, \quad \forall v \in L^2(Q),$$

where $\langle \cdot,\cdot \rangle$ denotes the duality product between the spaces $L^2(0,T;H^{-1}(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$. By the Hahn-Banach Theorem, the density property holds if and only if $\varphi = 0$. By introducing the adjoint state $p \in W(0,T)$ as the unique solution of the problem

$$\begin{cases} -p_t(x,t) - \displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\hat{a}_{ji}(x,t)\frac{\partial p}{\partial x_j}(x,t)\right) + \sum_{i=1}^n \hat{b}_i(x,t)\frac{\partial p}{\partial x_i}(x,t) + & \\ \qquad + \hat{c}(x,t)p(x,t) = \varphi(x,t) & \text{in } Q, \\ p(x,t) = 0 & \text{on } \Sigma, \\ p(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Multiplying this equation by z_v and integrating it by parts, it follows that

$$0 = \langle \varphi, z_v \rangle = \int_Q p(x,t) v(x,t)\,\mathrm{d}x\,\mathrm{d}t, \quad \forall v \in L^2(Q).$$

Therefore, $p = 0$ in Q and, consequently, $\varphi = 0$. ∎

Based on previous results, we can establish the main theorem of this section, which reads as follows:

Theorem 3. (Uniqueness) *Given a (probably small) non-empty open set* $\omega \subset \Omega$, *assume that* (H4) *and* (H6) *are fulfilled, and that there exist two triplets* $(A, \vec{b}, c)$, $(A^\star, \vec{b}^\star, c^\star)$ *in* $\mathcal{C}$ *such that*

$$y_{u,A,\vec{b},c}(x,T) = y_{u,A^\star,\vec{b}^\star,c^\star}(x,T), \quad \forall\, x \in \omega, \quad \forall\, u \in C^\delta(\overline{Q}).$$

Then $(A(y), \vec{b}(y), c(y)) = (A^\star(y), \vec{b}^\star(y), c^\star(y)), \quad \forall y \in \mathbb{R}$.

Proof. Given $u, v \in C^\delta(\overline{Q})$ and $\lambda \in (0,1)$, in order to simplify the notation, set

$$y_\lambda = y_{u+\lambda v, A, \vec{b}, c}, \qquad y_u = y_{u,A,\vec{b},c}, \qquad z_\lambda = \frac{y_\lambda - y_u}{\lambda}$$

$$y_\lambda^\star = y_{u+\lambda v, A^\star, \vec{b}^\star, c^\star}, \qquad y_u^\star = y_{u,A^\star,\vec{b}^\star,c^\star}, \qquad z_\lambda^\star = \frac{y_\lambda^\star - y_u^\star}{\lambda}.$$

By hypothesis, it is known that $y_u(x,T) = y_u^\star(x,T)$, $y_\lambda(x,T) = y_\lambda^\star(x,T)$ for all $x \in \omega$ and, consequently, $z_\lambda(T) = z_\lambda^\star(T)$ in ω for each λ.

It is not difficult to show that z_λ is the unique solution in $W(0,T)$ of the following linear problem:

$$\begin{cases} z_t - \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n a_{ij}\big(y_\lambda(x,t)\big)\frac{\partial z}{\partial x_j} + b_{\lambda i}(x,t) z\right) & \\ \quad + c_\lambda(x,t) z = v(x,t) & \text{in } Q, \\ z(x,t) = 0 & \text{on } \Sigma, \\ z(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{8}$$

where

$$b_{\lambda i}(x,t) = \sum_{j=1}^n \left(\int_0^1 \frac{\mathrm{d}a_{ij}}{\mathrm{d}y}\big(s y_\lambda(x,t) + (1-s) y_u(x,t)\big)\,\mathrm{d}s\right) \frac{\partial y_u}{\partial x_j}(x,t)$$
$$+ \int_0^1 \frac{\mathrm{d}b_i}{\mathrm{d}y}\big(s y_\lambda(x,t) + (1-s) y_u(x,t)\big)\,\mathrm{d}s$$

and

$$c_\lambda(x,t) = \int_0^1 \frac{\mathrm{d}c}{\mathrm{d}y}\big(s y_\lambda(x,t) + (1-s) y_u(x,t)\big)\,\mathrm{d}s.$$

On the other hand, owing to the estimate (2) of Theorem 1, we know that there exists a constant K depending on β and T, but independent of λ, such that

$$\|y_\lambda\|_{L^\infty(Q)} \leq K\big(|c(0)| + \|y_0\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(Q)} + \|v\|_{L^\infty(Q)}\big)$$

for all $\lambda \in (0,1)$. Moreover, by Theorem 10.1 of (Ladyzhenskaya *et al.*, 1968, p.204) and the Arzelá-Ascoli Theorem, we can verify that $y_\lambda \to y_u$ in $C(\overline{Q})$ as $\lambda \to 0$.

Now, a standard argumentation (see Theorem 4.5, p.166, in (Ladyzhenskaya *et al.*, 1968)) makes it possible to prove that $z_\lambda \to z_v$ in $L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ as $\lambda \to 0$, where z_v is the unique solution in $W(0,T)$ of the problem

$$\begin{cases} z_t - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(y_u)\frac{\partial z}{\partial x_j} + \frac{\mathrm{d}a_{ij}}{\mathrm{d}y}(y_u)\frac{\partial y_u}{\partial x_j} z\right) & \\ \quad - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\mathrm{d}b_i}{\mathrm{d}y}(y_u) z\right) + \frac{\mathrm{d}c}{\mathrm{d}y}(y_u) z = v & \text{in } Q, \\ z(x,t) = 0 & \text{on } \Sigma, \\ z(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{9}$$

The same argument shows that $z_\lambda^\star \to z_v^\star$ in $L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ as $\lambda \to 0$, where $z_v^\star$ is the unique solution in $W(0,T)$ of the problem

$$\begin{cases} z_t - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\star(y_u^\star)\frac{\partial z}{\partial x_j} + \frac{\mathrm{d}a_{ij}^\star}{\mathrm{d}y}(y_u^\star)\frac{\partial y_u^\star}{\partial x_j} z\right) & \\ \quad - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\mathrm{d}b_i^\star}{\mathrm{d}y}(y_u^\star) z\right) + \frac{\mathrm{d}c^\star}{\mathrm{d}y}(y_u^\star) z = v & \text{in } Q, \\ z(x,t) = 0 & \text{on } \Sigma, \\ z(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Furthermore, $z_v(T) = z_v^\star(T)$ in ω. Let us point out that all the coefficients appearing in previous linear problems are Hölder continuous in $\overline{Q}$, due to the regularity of $y_u, y_u^\star$ (see Theorem 1) and hypotheses (H1)–(H3). Now, we introduce $\chi = z_v - z_v^\star$ that satisfies $\chi(T) = 0$ in ω and can be viewed as the unique solution in $W(0,T)$ of the problem (9) where the right-hand side term v has been replaced by $\tilde{v}$, with

$$\begin{aligned} \tilde{v} \;=\; & \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\big(a_{ij}(y_u) - a_{ij}^\star(y_u^\star)\big)\frac{\partial z_v^\star}{\partial x_j} + \left(\frac{\mathrm{d}a_{ij}}{\mathrm{d}y}(y_u)\frac{\partial y_u}{\partial x_j} - \frac{\mathrm{d}a_{ij}^\star}{\mathrm{d}y}(y_u^\star)\frac{\partial y_u^\star}{\partial x_j}\right) z_v^\star\right) \\ & + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left(\frac{\mathrm{d}b_i}{\mathrm{d}y}(y_u) - \frac{\mathrm{d}b_i^\star}{\mathrm{d}y}(y_u^\star)\right) z_v^\star\right) + \left(\frac{\mathrm{d}c^\star}{\mathrm{d}y}(y_u^\star) - \frac{\mathrm{d}c}{\mathrm{d}y}(y_u)\right) z_v^\star. \end{aligned} \tag{10}$$

Now, for each $\Phi \in \mathcal{D}(\omega)$ (i.e. a function in $C^\infty(\overline{\omega})$ with compact support strictly contained in ω), we consider the adjoint problem

$$\begin{cases} -p_t - \sum_{i,j=1}^n a_{ij}(y_u)\dfrac{\partial^2 p}{\partial x_i \partial x_j} + \sum_{i=1}^n \dfrac{\mathrm{d}b_i}{\mathrm{d}y}(y_u)\dfrac{\partial p}{\partial x_i} + \dfrac{\mathrm{d}c}{\mathrm{d}y}(y_u)p = 0 & \text{in } Q, \\ p(x,t) = 0 & \text{on } \Sigma, \\ p(x,T) = \Phi(x) & \text{in } \Omega. \end{cases} \tag{11}$$

The unique solution p of (11) belongs to $C^{2+\rho,1+\rho/2}(\overline{Q})$.

Multiplying by p the equation that satisfies χ, integrating it by parts and using the fact that $p(T)\chi(T) = 0$ in Ω, we obtain $\langle \tilde{v}, p\rangle = 0$ or, equivalently,

$$\sum_{i,j=1}^n \int_Q \left(\left(a_{ij}(y_u) - a^\star_{ij}(y^\star_u)\right)\frac{\partial z^\star_v}{\partial x_j} + \left(\frac{\mathrm{d}a_{ij}}{\mathrm{d}y}(y_u)\frac{\partial y_u}{\partial x_j} - \frac{\mathrm{d}a^\star_{ij}}{\mathrm{d}y}(y^\star_u)\frac{\partial y^\star_u}{\partial x_j}\right) z^\star_v \right) \frac{\partial p}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t$$

$$= -\sum_{i=1}^n \int_Q \left(\frac{\mathrm{d}b_i}{\mathrm{d}y}(y_u) - \frac{\mathrm{d}b^\star_i}{\mathrm{d}y}(y^\star_u)\right) z^\star_v \frac{\partial p}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \left(\frac{\mathrm{d}c^\star}{\mathrm{d}y}(y^\star_u) - \frac{\mathrm{d}c}{\mathrm{d}y}(y_u)\right) z^\star_v p\,\mathrm{d}x\,\mathrm{d}t,$$

for all $v \in C^\delta(\overline{Q})$.

Since $\{z^\star_v : v \in L^2(Q)\}$ is dense in $L^2(0,T;H^1_0(\Omega))$ (cf. Theorem 2(b)) and $C^\delta(\overline{Q})$ is dense in $L^2(Q)$, we deduce that

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(\left(a_{ij}(y_u) - a^\star_{ij}(y^\star_u)\right)\frac{\partial p}{\partial x_i}\right) + \sum_{i,j=1}^n \left(\frac{\mathrm{d}a_{ij}}{\mathrm{d}y}(y_u)\frac{\partial y_u}{\partial x_j} - \frac{\mathrm{d}a^\star_{ij}}{\mathrm{d}y}(y^\star_u)\frac{\partial y^\star_u}{\partial x_j}\right)\frac{\partial p}{\partial x_i}$$

$$+ \sum_{i=1}^n \left(\frac{\mathrm{d}b_i}{\mathrm{d}y}(y_u) - \frac{\mathrm{d}b^\star_i}{\mathrm{d}y}(y^\star_u)\right)\frac{\partial p}{\partial x_i} + \left(\frac{\mathrm{d}c}{\mathrm{d}y}(y_u) - \frac{\mathrm{d}c^\star}{\mathrm{d}y}(y^\star_u)\right) p = 0 \quad \text{in } Q. \tag{12}$$

Taking into account the regularity properties of the functions involved, (12) can be rewritten as follows:

$$-\sum_{i,j=1}^n \left(a_{ij}\big(y_u(x,t)\big) - a^\star_{ij}\big(y^\star_u(x,t)\big)\right)\frac{\partial^2 p}{\partial x_i \partial x_j}(x,t)$$

$$+ \sum_{i=1}^n \left(\frac{\mathrm{d}b_i}{\mathrm{d}y}\big(y_u(x,t)\big) - \frac{\mathrm{d}b^\star_i}{\mathrm{d}y}\big(y^\star_u(x,t)\big)\right)\frac{\partial p}{\partial x_i}(x,t)$$

$$+ \left(\frac{\mathrm{d}c}{\mathrm{d}y}\big(y_u(x,t)\big) - \frac{\mathrm{d}c^\star}{\mathrm{d}y}\big(y^\star_u(x,t)\big)\right)p(x,t) = 0, \quad \forall\,(x,t) \in Q. \tag{13}$$

Taking $t = T$ in (13) and employing $p(x,T) = \Phi(x)$ and $y_u(T) = y_u^\star(T)$ in ω, we obtain

$$- \sum_{i,j=1}^{n} \big(a_{ij}\big(y_u(x,T)\big) - a_{ij}^\star\big(y_u(x,T)\big)\big) \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} \left(\frac{\mathrm{d}b_i}{\mathrm{d}y}\big(y_u(x,T)\big) - \frac{\mathrm{d}b_i^\star}{\mathrm{d}y}\big(y_u(x,T)\big)\right) \frac{\partial \Phi}{\partial x_i}(x) + \left(\frac{\mathrm{d}c}{\mathrm{d}y}\big(y_u(x,T)\big) - \frac{\mathrm{d}c^\star}{\mathrm{d}y}\big(y_u(x,T)\big)\right) \Phi(x) = 0 \quad \text{in } \omega, \tag{14}$$

for all $\Phi \in \mathcal{D}(\omega)$.

Let us select some specific functions Φ. Firstly, we choose $\Phi = \Psi$, where $\Psi \in \mathcal{D}(\omega)$ such that $\Psi(x) = 1$ for all $x \in \tilde{\omega}$, where $\tilde{\omega}$ is an open set strictly contained in ω. In this case, relation (14) gives

$$\frac{\mathrm{d}c}{\mathrm{d}y}\big(y_u(x,T)\big) = \frac{\mathrm{d}c^\star}{\mathrm{d}y}\big(y_u(x,T)\big) \tag{15}$$

in $\tilde{\omega}$ and, therefore, in ω by moving $\tilde{\omega}$. Hence the last term in (14) can be removed. Secondly, for each $i \in \{1,\dots,n\}$, we take $\Phi(x) = x_i \Psi(x)$, with Ψ as before. By arguing again as in the previous case, equality (14) implies

$$\frac{\mathrm{d}b_i}{\mathrm{d}y}\big(y_u(x,T)\big) = \frac{\mathrm{d}b_i^\star}{\mathrm{d}y}\big(y_u(x,T)\big) \quad \text{in } \omega, \quad \forall i \in \{1,\dots,n\}. \tag{16}$$

This means that (14) becomes

$$\sum_{i,j=1}^{n} \big(a_{ij}\big(y_u(x,T)\big) - a_{ij}^\star\big(y_u(x,T)\big)\big) \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x) = 0.$$

Here, by choosing $\Phi(x) = x_i x_j \Psi(x)$, with Ψ as before, the same argumentation together with the symmetry of the matrices A and $A^\star$ (see (H1)) yields

$$a_{ij}\big(y_u(x,T)\big) = a_{ij}^\star\big(y_u(x,T)\big) \quad \text{in } \omega, \quad \forall i,j \in \{1,\dots,n\}. \tag{17}$$

Using Theorem 2(a), we deduce that $\{y_u(T)|_\omega : u \in C^\delta(\overline{Q})\}$ is dense in $L^2(\omega)$. Hence, for each $y \in \mathbb{R}$, there exists a sequence $\{u_k\}_k \subset C^\delta(\overline{Q})$ such that

$$y_{u_k}(T)|_\omega \longrightarrow y \quad \text{in } L^2(\omega).$$

Applying relations (15)–(17) with $u = u_k$ and passing to the limit in k, we get

$$A(y) = A^\star(y), \quad \frac{\mathrm{d}\vec{b}}{\mathrm{d}y}(y) = \frac{\mathrm{d}\vec{b}^\star}{\mathrm{d}y}(y), \quad \frac{\mathrm{d}c}{\mathrm{d}y}(y) = \frac{\mathrm{d}c^\star}{\mathrm{d}y}(y), \quad \forall y \in \mathbb{R}.$$

Hypotheses $\vec{b}(0) = \vec{b}^\star(0) = \vec{0}$ (see (H2)) allow us to conclude that

$$\vec{b}(y) = \vec{b}^\star(y), \quad \forall y \in \mathbb{R}.$$

Moreover, there exists a real number $\tilde{k}$ such that

$$c^\star(y) = c(y) + \tilde{k}, \quad \forall y \in \mathbb{R}.$$

To show that $\tilde{k} = 0$, we fix $u \in C^\delta(\overline{Q})$ and introduce $\zeta = y_u - y_{u-\tilde{k}}$. Arguing as at the beginning of the proof, we derive that ζ is the unique solution of the following linear problem:

$$\begin{cases} \zeta_t - \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n a_{ij}\big(y_u(x,t)\big)\frac{\partial \zeta}{\partial x_j} + \tilde{b}_i(x,t)\zeta\right) + \tilde{c}(x,t)\zeta = \tilde{k} & \text{in } Q, \\ \zeta(x,t) = 0 & \text{on } \Sigma, \\ \zeta(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{18}$$

where

$$\begin{aligned} \tilde{b}_i(x,t) \;=\; & \sum_{j=1}^n \left(\int_0^1 \frac{\mathrm{d}a_{ij}}{\mathrm{d}y}\big(sy_u(x,t) + (1-s)y_{u-\tilde{k}}(x,t)\big)\,\mathrm{d}s\right)\frac{\partial y_{u-\tilde{k}}}{\partial x_j}(x,t) \\ & + \int_0^1 \frac{\mathrm{d}b_i}{\mathrm{d}y}\big(sy_u(x,t) + (1-s)y_{u-\tilde{k}}(x,t)\big)\,\mathrm{d}s \end{aligned}$$

and

$$\tilde{c}(x,t) = \int_0^1 \frac{\mathrm{d}c}{\mathrm{d}y}\big(sy_u(x,t) + (1-s)y_{u-\tilde{k}}(x,t)\big)\,\mathrm{d}s.$$

Moreover, by hypothesis, we know that $\zeta(T) = 0$ in ω. Once more, given $\Phi \in \mathcal{D}(\omega)$, we consider p the unique solution of the adjoint problem

$$\begin{cases} -p_t - \displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\big(y_u(x,t)\big)\frac{\partial p}{\partial x_j}\right) + \sum_{i=1}^n \tilde{b}_i(x,t)\frac{\partial p}{\partial x_i} + \tilde{c}(x,t)p = 0 & \text{in } Q, \\ p(x,t) = 0 & \text{on } \Sigma, \\ p(x,T) = \Phi(x) & \text{in } \Omega. \end{cases} \tag{19}$$

Taking into account the regularity hypotheses (H1)–(H4), it follows that p belongs to $C^{2+\rho,1+\rho/2}(\overline{Q})$ for some $\rho \in (0,1)$. Multiplying by p the equation satisfied by ζ, integrating it by parts and using the fact that $p(T)\zeta(T) = 0$ in Ω, we have

$$\tilde{k}\int_Q p(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{20}$$

Selecting $\Phi(x) \geq 0$ for all $x \in \omega$, $\Phi \not\equiv 0$, the Maximum Principle implies $p(x,t) \geq 0$ for all $(x,t) \in \overline{Q}$, $p \not\equiv 0$. Together with equality (20), this yields $\tilde{k} = 0$. ■

Remark 2. When $\overline{\omega} \cap \partial\Omega \neq \emptyset$, it can be shown that $\tilde{k} = 0$ by remarking that y_u and $y_u^\star$ are classical solutions of the corresponding PDE in a neighborhood of $t = T$ (in fact, they belong to $C^{2+\rho,1+\rho/2}(\overline{\Omega} \times [\epsilon, T])$ for every $\epsilon > 0$, see Theorem 1) and $y_u(T) = y_{u-\tilde{k}}(T)$ in ω. Then it can be verified that for any $x \in \overline{\omega} \cap \partial\Omega$, we have

$$-\mathrm{div}\,\Big(A\big(y_u(x,T)\big)\nabla y_u(x,T) + \vec{b}\big(y_u(x,T)\big)\Big) + c\big(y_u(x,T)\big) = u(x,T),$$

$$-\mathrm{div}\,\Big(A\big(y_u(x,T)\big)\nabla y_u(x,T) + \vec{b}\big(y_u(x,T)\big)\Big) + c\big(y_u(x,T)\big) + \tilde{k} = u(x,T),$$

which implies $\tilde{k} = 0$.

3. Identification Process. Numerical Experiments

In this section, our analysis is restricted to the problem (1) with $n = 1, \vec{b} = 0$ and $y_0 = 0$, just for simplicity. We are concerned with getting numerical approximations, $\tilde{A}$ and $\tilde{c}$, of the functions A and c from a finite number of observations η_{ij}. Consider (1) for a finite set of source terms $\{u_j\}_{j=1}^{n_u}$. For each u_j, we have measurements η_{ij} of $y_{u_j,A,0,c}(x_i,T)$ at some points $x_i \in \overline{\omega}$, $i = 0,\dots,n_x$.

We use the output least-squares method with a regularization technique to transform the identification problem into a minimization one. Due to errors in the data, it may happen that there is no $(\tilde{A}, \tilde{c})$ satisfying $y_{u_j,\tilde{A},0,\tilde{c}}(x_i,T) = \eta_{ij}$ $\forall i, j$. Hence the standard approach to solving parameter identification problems is based on introducing the nonlinear least-squares formulation which, in our case (with more than one source term), would try to minimize the functional

$$\frac{1}{2}\sum_{j=1}^{n_u}\int_\omega \big(y_{u_j,\tilde{A},0,\tilde{c}}(x,T) - \eta_j(x)\big)^2\,\mathrm{d}x$$

over some set of feasible parameters, $\eta_j(x)$ being a continuous observation which interpolates the data η_{ij}, $i = 1,\dots,n_x$.

To guarantee the existence of a solution, we add the regularization term

$$\gamma\left\{\int_{\tilde{I}}\left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}y}(y)\right)^2 dy + \int_{\tilde{I}}\left(\frac{\mathrm{d}\tilde{c}}{\mathrm{d}y}(y)\right)^2 dy\right\},$$

where γ is a positive scalar and $\tilde{I}$ stands for the working interval where the nonlinearities will be recovered. Moreover, from the practical point of view, the regularization term has the effect of damping oscillations in the numerical solutions. Further discussions on regularization for parameter estimation problems are given in (Banks and Kunisch, 1989) and references therein.

For our experiments we have chosen $\Omega = (0,2)$, $\omega = (1,2)$, $T = 1$, the following source terms:

$$u_j(x,t) = \sin(i_j\pi t)\sin(k_j\pi x/2) + i_j - k_j \quad \text{for} \quad i_j, k_j \in \{1,2,\dots,5\}, \tag{21}$$

and the points $\{x_i = 1 + i/10\}_{i=0}^{10} \subset \overline{\omega}$.

The first step of the identification process is related to the choice of the interval where the nonlinearities will be recovered. Our working interval is an enlargement of the interval defined by the data, $I = [\eta_{\min} = \min \eta_{ij}, \eta_{\max} = \max \eta_{ij}]$, by taking a safety barrier, $M > 0$, at both the extremities: $\tilde{I} = [\eta_{\min} - M, \eta_{\max} + M]$. For the calculations, we took an equidistant grid $\{y_i\}_{i=1}^{n_y}$ in $\tilde{I}$ with a mesh size $h = (\eta_{\max} - \eta_{\min})/10$ and the safety barrier $M = 3h$. Hence there are 11 nodes on I, and $n_y = 17$ nodes on $\tilde{I}$.

Reconstruction of A and c consists in determining two vectors $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_{n_y})$ and $\hat{c} = (\hat{c}_1, \ldots, \hat{c}_{n_y})$ in $\mathbb{R}^{n_y}$, where $\hat{a}_i + \alpha$ and $\hat{c}_i$ are the coefficients of the approximations

$$\tilde{A}(y) = \sum_{i=1}^{n_y} \hat{a}_i B_i(y) + \alpha \quad \text{and} \quad \tilde{c}(y) = \sum_{i=1}^{n_y} \hat{c}_i B_i(y), \tag{22}$$

B_i's being piecewise linear B-splines satisfying $B_i(y_j) = \delta_{ij}$ for $i, j = 1, \ldots, n_y$.

Referring to α (see (H1)), our choice is $\alpha = 10^{-12}$. This assures the strict positivity of A, but it does not assume *a priori* precise knowledge about the value of this lower bound. Let us remark that this could be inaccessible in real applications.

Following the aforementioned output least-squares method with regularization, we consider the following constrained minimization problem:

$$\min_{(\hat{a},\hat{c}) \in U_{\mathrm{ad}}} J(\hat{a}, \hat{c}), \tag{23}$$

where

$$\begin{aligned} J(\hat{a}, \hat{c}) \; = \; & \frac{1}{2} \sum_{j=1}^{n_u} \int_\omega \left(y_{u_j, \tilde{A}, 0, \tilde{c}}(x, T) - \eta_j(x) \right)^2 \mathrm{d}x \\ & + \gamma \left\{ \int_{\tilde{I}} \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}y}(y) \right)^2 \mathrm{d}y + \int_{\tilde{I}} \left(\frac{\mathrm{d}\tilde{c}}{\mathrm{d}y}(y) \right)^2 \mathrm{d}y \right\}, \end{aligned} \tag{24}$$

γ being a positive parameter and U_{ad} the set of feasible vectors given by

$$U_{\mathrm{ad}} = \left\{ (\hat{a}, \hat{c}) \in \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} : \hat{a}_i \geq 0, \quad i = 1, \ldots, n_y \right\}. \tag{25}$$

We can rewrite J by expressing the second term as the following norm:

$$\int_{\tilde{I}} \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}y}(y) \right)^2 \mathrm{d}y + \int_{\tilde{I}} \left(\frac{\mathrm{d}\tilde{c}}{\mathrm{d}y}(y) \right)^2 \mathrm{d}y = \left\| R \begin{pmatrix} \hat{a}^T \\ \hat{c}^T \end{pmatrix} \right\|_2^2,$$

where R is the $2(n_y + 1) \times 2n_y$ matrix given by

$$R = \frac{1}{\sqrt{h}} \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix}$$

with

$$R_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}.$$

The following result states the existence of minimizers:

Theorem 4. *There exists at least one solution to the optimization problem (23).*

Proof. Since R has full column range, J is coercive. Moreover, J is continuous (see the Appendix), and the feasible set U_{ad} is non-empty and closed. Hence the result follows. ■

Problem (23) was solved by using the subroutine E04UCF from NAG Library. E04UCF implements a sequential quadratic programming (SQP) algorithm. Each evaluation of the cost functional was computed by taking (for each j) a linear spline which interpolates the values $\eta_{ij} - y_{u_j,\tilde{A},0,\tilde{c}}(x_i, T)$, where $y_{u_j,\tilde{A},0,\tilde{c}}(x_i, T)$ were obtained by solving (1) with a linearized Crank-Nicholson-Galerkin method. To this end, we took a semidiscrete approach with 19 piecewise linear finite elements for the discretization of the spatial domain and the nodes $\{iT/10\}_{i=0}^{10}$ for the discretization of the time variable. The derivatives of the objective function were computed by finite differences.

As for the stopping test, just let us mention that the optimization algorithm terminates successfully if the following conditions are satisfied:

$$\frac{\|(\hat{a}^k - \hat{a}^{k-1}, \hat{c}^k - \hat{c}^{k-1})\|}{1 + \|(\hat{a}^k, \hat{c}^k)\|} < \sqrt{\varepsilon}, \tag{26}$$

$$\frac{\|(\nabla J(\hat{a}^k, \hat{c}^k))_{FR}\|}{1 + \max\left\{1 + |J(\hat{a}^k, \hat{c}^k)|, \left\|(\nabla J(\hat{a}^k, \hat{c}^k))_{FR}\right\|\right\}} \le \sqrt{\varepsilon}, \tag{27}$$

where $(\nabla J(\hat{a}^k, \hat{c}^k))_{FR}$ is the vector with the components of $\nabla J(\hat{a}^k, \hat{c}^k)$ corresponding to the free variables (i.e. not fixed at the bound 0).

As regards the bound conditions described in $\mathcal{C}$, let us point out that the lower bounds in (23) ensure $\tilde{A}^k(y) \ge \alpha$ for all k and all $y \in \mathbb{R}$. On the other hand, from $J(\hat{a}^k, \hat{c}^k) \le J(\hat{a}^0, \hat{c}^0)$ it follows that

$$\frac{\mathrm{d}\tilde{c}^k}{\mathrm{d}y}(y) = \sum_{i=1}^{n_y} \hat{c}_i \frac{\mathrm{d}B_i}{\mathrm{d}y}(y) \ge -\sqrt{\frac{J(\hat{a}^0, \hat{c}^0)}{\gamma h}}, \quad \forall y \notin \{y_1, \ldots, y_{n_y}\}, \quad \forall k.$$

Now we present some numerical results corresponding to the following examples:

Example 1. $A(y) = 0.5\exp(y) + 1, \vec{b}(y) = 0$ and $c(y) = y^3 - y$.

Example 2. $A(y) = \arctan(y) + 2, \vec{b}(y) = 0$ and $c(y) = \exp(-y^2)$.

Table 1. Results for Example 1 with undisturbed data.

nodes y_i	$\tilde{A}(y_i)$	$A(y_i)$	Error	$\tilde{c}(y_i)$	$c(y_i)$	Error
-1.1325	1.1493	1.1611	1.0200e-2	-0.3308	-0.3200	1.0766e-2
-0.9300	1.1950	1.1973	1.8933e-3	0.1738	0.1256	4.8144e-2
-0.7275	1.2402	1.2416	1.1163e-3	0.3466	0.3425	4.1416e-3
-0.5250	1.2944	1.2958	1.0400e-3	0.3998	0.3803	1.9517e-2
-0.3225	1.3609	1.3622	9.2433e-4	0.2935	0.2890	4.4962e-3
-0.1200	1.4414	1.4435	1.3939e-3	0.1228	0.1183	4.5672e-3
0.0825	1.5435	1.5430	3.2381e-4	-0.0800	-0.0819	1.9638e-3
0.2850	1.6649	1.6649	3.0448e-5	-0.2718	-0.2619	9.9009e-3
0.4875	1.8112	1.8141	1.6240e-3	-0.3831	-0.3716	1.1415e-2
0.6900	2.0000	1.9969	1.5560e-3	-0.3760	-0.3615	1.4477e-2
0.8925	2.2197	2.2206	4.1744e-4	-0.2010	-0.1816	1.9413e-2
	$\gamma = 10^{-12}$			ITER=247		

Table 2. Results for Example 1 with noisy data.

nodes y_i	$\tilde{A}(y_i)$	$A(y_i)$	Error	$\tilde{c}(y_i)$	$c(y_i)$	Error
-1.1613	0.9558	1.1565	1.7362e-1	-0.2866	-0.4048	1.1808e-1
-0.9550	1.2529	1.1924	5.0742e-2	-0.1410	0.0840	2.2504e-1
-0.7488	1.2550	1.2365	1.5011e-2	0.3740	0.3289	4.4989e-2
-0.5425	1.2831	1.2906	5.8557e-3	0.4367	0.3828	5.3835e-2
-0.3363	1.3607	1.3572	2.5740e-3	0.2944	0.2983	3.8742e-3
-0.1300	1.4478	1.4390	6.0674e-3	0.1140	0.1278	1.3846e-2
0.0763	1.3601	1.5396	1.1662e-1	-0.0458	-0.0759	2.9991e-2
0.2825	1.3824	1.6632	1.6885e-1	-0.1509	-0.2600	1.0904e-1
0.4888	1.5311	1.8152	1.5648e-1	-0.0870	-0.3720	2.8498e-1
0.6950	1.6818	2.0019	1.5989e-1	0.1239	-0.3593	4.8321e-1
0.9013	1.4268	2.2314	3.6059e-1	0.3099	-0.1691	4.7907e-1
	$\gamma = 10^{-4}$			ITER=164		

For our experiments we took eight source terms u_j given by (21) with (i_j, k_j) taking values $(1,2),(1,3),(1,4),(2,3),(3,2),(4,1),(4,2),(5,2)$ in Example 1 and $(1,2),(1,3),(1,4),(1,5),(4,2),(4,4),(5,1),(5,2)$ in Example 2.

Table 3. Results for Example 2 with undisturbed data.

nodes y_i	$\tilde{A}(y_i)$	$A(y_i)$	Error	$\tilde{c}(y_i)$	$c(y_i)$	Error
-1.4688	1.0108	1.0269	1.5758e-2	0.0377	0.1156	7.7993e-2
-1.2450	1.0939	1.1059	1.0823e-2	0.2296	0.2122	1.7385e-2
-1.0213	1.1989	1.2041	4.2987e-3	0.3306	0.3524	2.1861e-2
-0.7975	1.3184	1.3268	6.3123e-3	0.5171	0.5294	1.2338e-2
-0.5737	1.4712	1.4791	5.3486e-3	0.7152	0.7195	4.3117e-3
-0.3500	1.6552	1.6633	4.8944e-3	0.8851	0.8847	3.9076e-4
-0.1263	1.8666	1.8744	4.1571e-3	0.9938	0.9842	9.6144e-3
0.0975	2.1028	2.0972	2.6952e-3	1.0018	0.9905	1.1267e-2
0.3213	2.3184	2.3109	3.2817e-3	0.9038	0.9019	1.8137e-3
0.5450	2.5069	2.4990	3.1517e-3	0.7415	0.7430	1.5362e-3
0.7687	2.6706	2.6554	5.7284e-3	0.5380	0.5538	1.5781e-2
	$\gamma = 10^{-12}$			ITER=204		

Table 4. Results for Example 2 with noisy data.

nodes y_i	$\tilde{A}(y_i)$	$A(y_i)$	Error	$\tilde{c}(y_i)$	$c(y_i)$	Error
-1.5319	0.8284	1.0075	1.7780e-1	0.2106	0.0957	1.1486e-1
-1.3025	1.0394	1.0840	4.1153e-2	0.2529	0.1833	6.9545e-2
-1.0731	1.1447	1.1794	2.9386e-2	0.1618	0.3161	1.5430e-1
-0.8438	1.3545	1.2991	4.2632e-2	0.4657	0.4907	2.5004e-2
-0.6144	1.4246	1.4491	1.6900e-2	0.7489	0.6856	6.3329e-2
-0.3850	1.6548	1.6325	1.3666e-2	0.8579	0.8622	4.3331e-3
-0.1556	1.8461	1.8456	2.5203e-4	0.9838	0.9761	7.7688e-3
0.0737	1.9462	2.0736	6.1466e-2	1.0583	0.9946	6.3731e-2
0.3031	2.0655	2.2943	9.9730e-2	0.9991	0.9122	8.6929e-2
0.5325	2.2061	2.4893	1.1378e-1	1.0147	0.7531	2.6155e-1
0.7619	1.9562	2.6511	2.6210e-1	1.0783	0.5596	5.1869e-1
	$\gamma = 10^{-4}$			ITER=143		

Our numerical experiments were carried out in MATLAB on a personal computer, taking $\varepsilon = 10^{-10}$ for the stopping test and $(\hat{a}^0, \hat{c}^0) = (0.5, \ldots, 0.5)$ as the starting point. To generate the observations η_{ij}, we solve (1) and take

$$\eta_{ij} = (1 + \hat{\delta}_{ij}) y_{u_j, A, 0, c}(x_i, T),$$

where $\hat{\delta}_{ij}$ are uniformly distributed random numbers in $[-\hat{\delta}, \hat{\delta}]$, with $\hat{\delta}$ denoting the noise level.

For each example we present two tables and one figure. Tables 1 and 3 summarize numerical results corresponding to $\hat{\delta} = 0$ (without explicit data perturbation), and Tables 2 and 4 show numerical results obtained from data with $\hat{\delta} = 0.05$ (noisy data). For each value of $\hat{\delta}$, the nodes of I are listed in the first column. For each node y_i, there are six columns, three for each nonlinearity. Hence, for the first nonlinearity we report the value of the numerical solution, $\tilde{A}(y_i)$, and the value of the exact solution, $A(y_i)$, in the second and third columns, respectively, and the relative error, measured as follows:

$$\frac{|\tilde{A}(y_i) - A(y_i)|}{\max(1, |A(y_i)|)},$$

in the fourth column. The last three columns of each table correspond to the nonlinearity c. At the bottom of the tables, the number of iterations required by the optimization routine is shown together with the current value of the penalty parameter γ.

In the last decades, the choice of the appropriate value for γ has stimulated much research. In most cases this research is somehow theoretical: it assumes *a priori* knowledge of the error level, which is not satisfactory from a practical standpoint. This may be the reason why, in practice, the regularization parameters are most often chosen heuristically. For linear inverse problems, Kunisch and Zou (1998) proposed practical strategies for finding reasonable regularization parameters. As regards this topic, we have made numerous computer runs with different values for γ. It is our experience that the regularization term does not play an important role in the case of the undisturbed data, i.e. good results can be obtained with very small values of γ. However, in the case of noisy data, regularization clearly improves the results. Here we present outcomes obtained with $\gamma = 10^{-4}$ for the noise level $\hat{\delta} = 0.05$. Taking $\gamma < 10^{-5}$, more oscillating solutions are obtained; for $\gamma > 10^{-3}$, the solutions become too smooth.

For both examples, the computed numerical solutions with $\hat{\delta} = 0$ did not satisfy the stopping test (26), but they satisfied (27), i.e. the sequences of iterates did not converge, but the final iterates satisfied the first-order Kuhn-Tucker conditions to the accuracy requested. In these cases, the optimization routine was terminated because no further improvement could be made during the line search.

These preliminary results provide some practical insight into our approach, but more efficient issues regarding solving the minimization problem could be considered. Each evaluation of the cost functional requires solving problem (1) as many times as functions u are considered. To reduce this high computational cost, a parallel version of the algorithm could be used for solving those problems simultaneously.

For each example we show a figure containing four plots. Nonlinearities obtained from data with $\hat{\delta} = 0$ and $\hat{\delta} = 0.05$ appear on the left and on the right, respectively. Nonlinearities A and $\tilde{A}$ are at the top, and c and $\tilde{c}$ are shown at the bottom. Each graph shows the exact solution, A or c, and the numerically identified solution $\tilde{A}$ or $\tilde{c}$. We add asterisks to indicate the values $\tilde{A}(y_i)$ and $\tilde{c}(y_i)$. In each graph the observations η_{ij} are shown on the axis. Note that these data depend on the source terms u_j. The selection of these u_j must be performed while trying to cover (as best

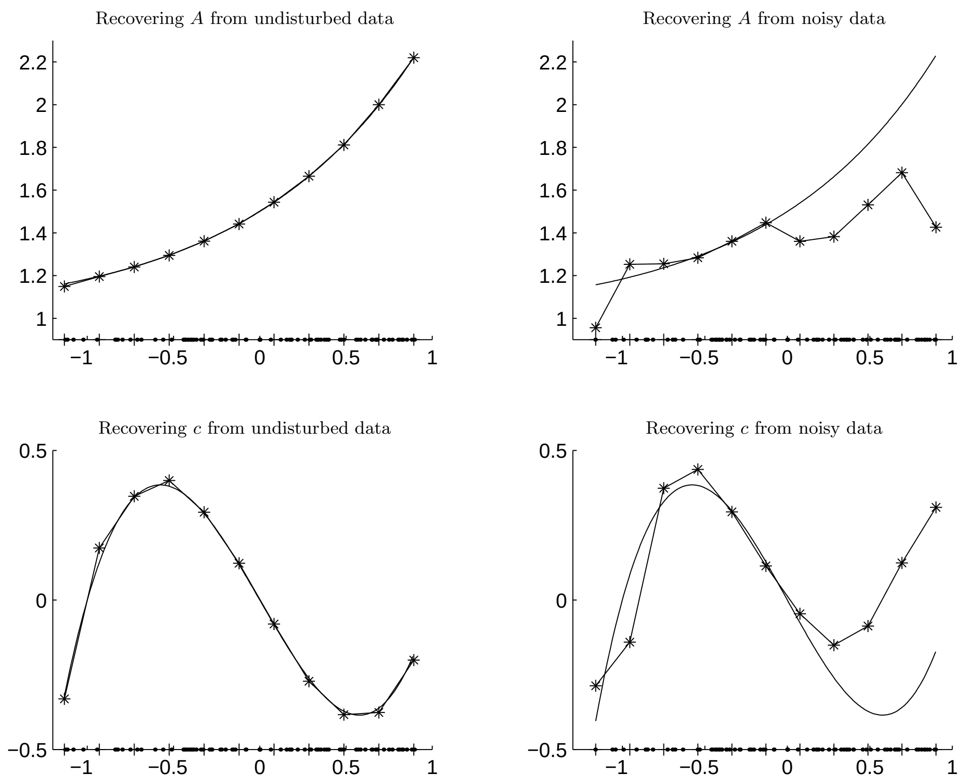

Fig. 1. Results for Example 1: $A(y) = 0.5\exp(y) + 1$, $c(y) = y^3 - y$.

as possible) the interval I with the corresponding observations. Hence, with a good choice, a small number of functions can be enough to recover the nonlinearities. From the practical viewpoint, it is clear for us that the size of ω with respect to Ω has no influence on the quality of the reconstruction.

4. Conclusions

We have studied the identification of the nonlinearities of a quasilinear parabolic operator by using final observations $y(x, T)$ over a (probably small) open subset of Ω. From a theoretical point of view, the uniqueness of the nonlinearities is proved under appropriate assumptions. From a practical standpoint, we present a finite-dimensional optimization problem that allows for the reconstruction of the nonlinearities. In the absence of noise, the numerical results are satisfactory, taking into account that they have been obtained with a small number of nodes. As one could predict, the presence of a relatively high noise level (up to 5%) deteriorates the results, but our approach still provides some useful qualitative information.

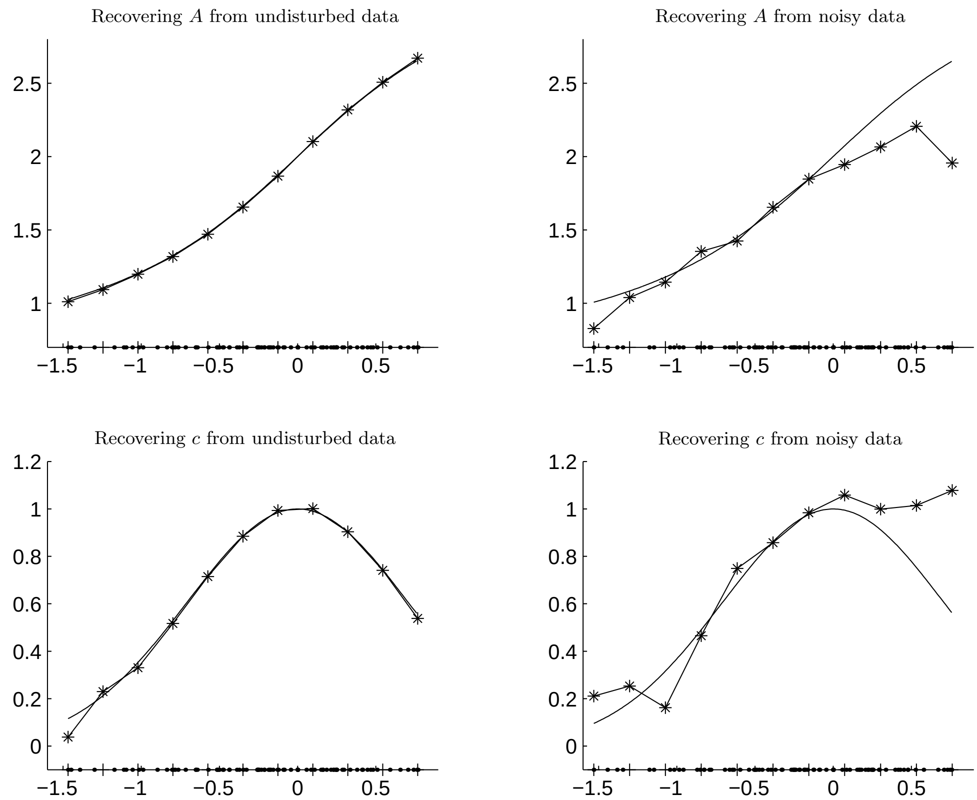

Fig. 2. Results for Example 2: $A(y) = \arctan(y) + 2,\ c(y) = \exp(-y^2)$.

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Appendix

Let us fix an element $u \in L^2(Q)$ and assume that $y_0 = 0$. In the working interval $\tilde{I}$, we consider an equidistant grid $\{y_i\}_{i=1}^{n_y}$ with a mesh size h and B_i piecewise linear B-splines satisfying $B_i(y_j) = \delta_{ij}$ for $i, j = 1, \dots, n_y$.

The continuity of J is the main point in the proof of Theorem 4. Taking into account its definition (cf. (24)), this will be a consequence of the continuity of the mapping

$$F : U_{\mathrm{ad}} \subset \mathbb{R}^{2n_y} \longrightarrow L^2(\Omega)$$

given by

$$F(\hat{a}, \hat{c}) = y_{u,\tilde{A},0,\tilde{c}}(x, T),$$

where $\tilde{A}$ and $\tilde{c}$ are the nonlinear coefficients defined by (22) through $(\hat{a}, \hat{c})$, and U_{ad} is defined in (25).

Firstly, let us show that F is well-defined. Since $\tilde{A}$ and $\tilde{c}$ are bounded Lipschitz continuous functions in $\mathbb{R}$, with $\tilde{A}(s) \geq \alpha$ for every $s \in \mathbb{R}$, the existence of $y_{u,\tilde{A},0,\tilde{c}}$ in $C^{\rho,\rho/2}(\overline{Q}) \cap W(0,T)$ for some $\rho \in (0,1)$ is guaranteed by Theorem 6.6 of (Ladyzhenskaya *et al.*, 1968, p.462). The uniqueness can be deduced by seeing the difference of two solutions as the corresponding one for a linear parabolic problem with homogeneous data (see a similar argumentation in the proof of Theorem 3), and applying (Ladyzhenskaya *et al.*, 1968, Thm. 3.4, p.150).

Now, let us suppose that $(\hat{a}_m, \hat{c}_m) \longrightarrow (\hat{a}, \hat{c})$ in U_{ad} as $m \to +\infty$. Also, let us denote by $\tilde{A}_m$ and $\tilde{c}_m$ the nonlinearities associated with $(\hat{a}_m, \hat{c}_m)$.

The boundedness of $\{(\hat{a}_m, \hat{c}_m)\}_m$ in U_{ad} together with Theorem 7.1 (p.181) and Theorem 10.1 (p.204) of (Ladyzhenskaya *et al.*, 1968) allow us to prove that $\{y_{u,\tilde{A}_m,0,\tilde{c}_m}\}_m$ is bounded in $C^{\rho,\rho/2}(\overline{Q})$ independently of m. Finally, the convergence

$$y_{u,\tilde{A}_m,0,\tilde{c}_m} \longrightarrow y_{u,\tilde{A},0,\tilde{c}} \quad \text{in} \quad L^2\big(0,T;H_0^1(\Omega)\big) \cap C\big([0,T];L^2(\Omega)\big),$$

as $m \to +\infty$ follows from Theorem 4.5, p.166, of (Ladyzhenskaya *et al.*, 1968). In particular, this implies $F(\hat{a}_m, \hat{c}_m) \longrightarrow F(\hat{a}, \hat{c})$ in $L^2(\Omega)$. ■