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EXTERNALLY AND INTERNALLY POSITIVE TIME-VARYING LINEAR SYSTEMS

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The notions of externally and internally positive time-varying linear systems are introduced. Necessary and sufficient conditions for the external and internal positivities of time-varying linear systems are established. Moreover, sufficient conditions for the reachability of internally positive time-varying linear systems are presented.

1. Introduction

Roughly speaking, positive systems are systems whose trajectories are entirely contained in the nonnegative orthant $\mathbb{R}_+^n$ whenever the initial state and input are nonnegative. Positive systems arise while modelling systems in engineering, economics, social sciences, biology, medicine and other areas (d'Alessandro and de Santis, 1994; Berman *et al.*, 1989; Berman and Plemmons, 1994; Farina and Rinaldi, 2000; Kaczorek, 2001; Rumchev and James, 1990; 1995). The single-input single-output externally and internally positive linear time-invariant systems were investigated in (Berman *et al.*, 1989; Berman and Plemmons, 1994; Farina and Rinaldi, 2000). The notions of externally and internally positive systems were extended to singular continuous-time, discrete-time and two-dimensional linear systems in (Kaczorek, 2001). The reachability and controllability of standard and singular internally positive linear systems were analysed in (Fanti *et al.*, 1990; Klamka, 1998; Otha *et al.*, 1984; Valcher, 1996). The notions of weakly positive discrete- and continuous-time linear systems were introduced in (Kaczorek, 1998b; 2001). Recently, the positive two-dimensional (2D) linear systems have been extensively investigated by Fornasini and Valcher (Fornasini and Valcher, 1997; Valcher, 1996; 1997) and (Kaczorek 2001).

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2. Preliminaries

Let $\mathbb{R}^{p\times q}$ be the set of real $p \times q$ matrices and $\mathbb{R}^p := \mathbb{R}^{p\times 1}$. The set of $p \times q$ real matrices with nonnegative entries will be denoted by $\mathbb{R}_+^{p\times q}$ and $\mathbb{R}_+^p := \mathbb{R}_+^{p\times 1}$. Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \tag{1a}$$

$$y(t) = C(t)x(t) + D(t)u(t), \tag{1b}$$

where $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ signifies the input vector, $y(t) \in \mathbb{R}^p$ stands for the output vector, and $A(t)$, $B(t)$, $C(t)$, $D(t)$ are real matrices of appropriate dimensions with continuous-time entries. A solution $x(t)$ to the equation satisfying the initial condition $x(t_0) = x_0$ is given by (Gantmacher, 1959)

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(t)u(\tau)\,\mathrm{d}\tau, \tag{2}$$

where $\Phi(t, t_0)$ is the fundamental matrix defined by

$$\Phi(t, t_0) = I_n + \int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t} A(\tau)\int_{t_0}^{\tau} A(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau + \cdots, \tag{3}$$

I_n being the $n \times n$ identity matrix.

If $A(t_1)A(t_2) = A(t_2)A(t_1)$ for $t_1, t_2 \in [t_0, \infty)$, then (3) takes the form (Gantmacher, 1959)

$$\bar{\Phi}(t, t_0) = \exp\Big(\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\Big). \tag{3a}$$

The fundamental matrix $\Phi(t, t_0)$ satisfies the matrix differential equation

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \tag{4}$$

and the initial condition $\Phi(t_0, t_0) = I_n$.

3. Externally Positive Systems

Definition 1. The system (1) is called *externally positive* if for all $u(t) \in \mathbb{R}_+^m$, $t \geq t_0$ and zero initial conditions ($x_0 = 0$) the output vector $y(t) \in \mathbb{R}_+^p$ for $t \geq t_0$.

Let $g(t) \in \mathbb{R}^{p\times m}$ be the matrix impulse response of the system (1). It is well-known that the output vector $y(t)$ of the system (1) with zero initial conditions for an input vector $u(t)$ is given by the formula

$$y(t) = \int_{t_0}^{t} g(t, \tau)u(\tau)\,\mathrm{d}\tau, \quad t \geq t_0, \tag{5}$$

where

$$g(t,\tau) = C(t)\Phi(t,\tau)B(t) + D(t)\delta(t-\tau), \tag{6}$$

for $t \geq \tau$, and $\delta(t)$ is the Dirac impulse.

Theorem 1. *The system (1) is externally positive if and only if*

$$g(t) \in \mathbb{R}_+^{p\times m} \quad \textit{for} \quad t \geq t_0. \tag{7}$$

Proof. The necessity follows immediately from Definition 1 and the definition of the impulse response. To show the sufficiency, assume that (7) holds. Then from (5), for $u(t) \in \mathbb{R}_+^m$, $t \geq t_0$ we have $y(t) \in \mathbb{R}_+^p$ for $t \geq t_0$. ■

4. Internally Positive Systems

Definition 2. System (1) is called *internally positive* if for every $x_0 \in \mathbb{R}_+^n$ and all $u(t) \in \mathbb{R}_+^m$ the state vector $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$ for $t \geq t_0$.

From the comparison of Definitions 1 and 2 it follows that every internally positive system (1) is always externally positive.

Lemma 1. The fundamental matrix satisfies

$$\Phi(t,t_0) \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq t_0, \tag{8}$$

if and only if the off-diagonal entries a_{ij}, $i \neq j$, $i,j = 1,\dots,n$ of the matrix $A(t)$ satisfy the condition

$$\int_{t_0}^{t} a_{ij}(\tau)\,\mathrm{d}\tau \geq 0 \quad \text{for} \quad i \neq j, \quad i,j = 1,\dots,n. \tag{9}$$

Proof. First, we shall show that (9) implies (8). Let $x_i(t)$ (resp. $z_i(t)$) be the i-th component of the vector $x(t)$ (resp. $z(t)$) and

$$x_i(t) = z_i(t)\exp\left(\int_{t_0}^{t} a_{ii}(\tau)\,\mathrm{d}\tau\right), \quad i = 1,\dots,n. \tag{10}$$

Substitution of (10) into (1a) for $u(t) = 0$, $t \geq t_0$ yields (Ratajczak, 1967)

$$\dot{z}(t) = \bar{A}(t)z(t), \tag{11}$$

where $\bar{A}(t) = [\bar{a}_{ij}(t)] \in \mathbb{R}^{n\times n}$

$$\bar{a}_{ij}(t) = \begin{cases} a_{ij}(t)\exp\left(\displaystyle\int_{t_0}^{t} \big[a_{jj}(\tau) - a_{ii}(\tau)\big]\,\mathrm{d}\tau\right) & \text{for} \quad i \neq j, \\ 0 & \text{for} \quad i = j, \end{cases} \tag{12}$$

From (10) it follows that

$$z_i(t_0) = x_i(t_0) \geq 0 \quad \text{for} \quad i = 1, \dots, n \quad \text{if} \quad x_0 \in \mathbb{R}_+^n. \tag{13}$$

Using (2) for $u(t) = 0,\ t \geq t_0$ and (3) for (11), we obtain

$$z(t) = \bar{\Phi}(t, t_0) z_0, \tag{14}$$

where

$$\bar{\Phi}(t, t_0) = I_n + \int_{t_0}^{t} \bar{A}(\tau)\, \mathrm{d}\tau + \int_{t_0}^{t} \bar{A}(\tau) \int_{t_0}^{\tau} \bar{A}(\tau_1)\, \mathrm{d}\tau_1\, \mathrm{d}\tau + \cdots . \tag{15}$$

From (12) it follows that if (9) holds, then $\bar{A}(t) \in \mathbb{R}_+^{n \times n}$, and by (15) this implies $\bar{\Phi}(t, t_0) \in \mathbb{R}_+^{n \times n}$ and $z(t) \in \mathbb{R}_+^n,\ t \geq t_0$ for any $z_0 \in \mathbb{R}_+^n$. Hence, by (10) and (13) we have $x(t) \in \mathbb{R}_+^n,\ t \geq t_0$ for any $x_0 \in \mathbb{R}_+^n$. Therefore (9) implies (8). The necessity follows immediately from (3a) and the fact that $\bar{\Phi}(t, t_0) \in \mathbb{R}_+^{n \times n}$ only if $\int_{t_0}^{t} \bar{A}(\tau)\, \mathrm{d}\tau$ is a Metzler matrix for any $t \geq t_0$ (Kaczorek, 1998a). ■

Remark 1. If the matrix $A(t)$ is independent of t, i.e. $A(t) = A = [a_{ij}]$ and $a_{ij} \geq 0$ for $i \neq j$, then A is the Metzler matrix (Farina and Rinaldi, 2000; Kaczorek, 2001) and $\Phi(t, t_0) = \exp(A(t - t_0))$.

Theorem 2. *System (1) is internally positive if and only if*

(i) *the off-diagonal entries of* $A(t)$ *satisfy (9),*

(ii) $B(t) \in \mathbb{R}_+^{n \times m},\ C(t) \in \mathbb{R}_+^{p \times n},\ D(t) \in \mathbb{R}_+^{p \times m}$ *for* $t \geq 0$.

Proof. (Necessity) Let $u(t) = 0$ for $t \geq t_0$ and $x_0 = e_j$. The trajectory does not leave the orthant $\mathbb{R}_+^n$ only if $\dot{x}(t_0) = A(t_0) e_j \geq 0$, and this implies (9). For the same reasons, for $x_0 = 0$ we have $\dot{x}(t_0) = Bu(t_0) \geq 0$, and this implies $B(t) \in \mathbb{R}^{p \times m}$, $t \geq t_0$ since $u(t_0) \in \mathbb{R}_+^m$ may be arbitrary. From (1b), for $u(t_0) = 0$ we have $y(t_0) = C(t_0) x_0 \in \mathbb{R}_+^p$ and $C(t) \in \mathbb{R}_+^{p \times n},\ t \geq 0$ since $x_0 \in \mathbb{R}_+^n$ may be arbitrary. Similarly, from (1b), for $x_0 = 0$ we obtain $y(t_0) = D(t_0) u(t_0) \in \mathbb{R}_+^p$ and $D(t) \in \mathbb{R}_+^{p \times m}$ for $t \geq 0$ since $u(t_0) \in \mathbb{R}_+^m$ may be arbitrary.

(Sufficiency) If the condition (9) is satisfied, then, by Lemma, (8) holds and from (2) we obtain $x(t) \in \mathbb{R}_+^n$ for any $x_0 \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+^m,\ t \geq t_0$, since $B(t) \in \mathbb{R}_+^{n \times m}$. If $C(t) \in \mathbb{R}_+^{p \times n}$ and $D(t) \in \mathbb{R}+^{p \times m}$ for $t \geq 0$, then from (1b) we obtain $y(t) \in \mathbb{R}_+^p$ since $x(t) \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+^m$ for $t \geq t_0$. ■

5. Reachability

Definition 3. The state $x_f(t) \in \mathbb{R}_+^n$ of the system (1) is called reachable in time $t_f - t_0$ if there exists an input vector $u(t) \in \mathbb{R}_+^m$ for $[t_0, t_f]$ which steers the state of the system from $x_0 = 0$ to x_f.

Definition 4. If every state $x_f(t) \in \mathbb{R}_+^n$ of the system (1) is reachable in time $t_f - t_0$, then the system is called reachable in time $t_f - t_0$.

Definition 5. If for every state $x_f(t) \in \mathbb{R}_+^n$ there exists $t_f > t_0$ such that the state is reachable in time $t_f - t_0$, then the system (1) is called reachable.

A matrix $A \in \mathbb{R}_+^{n\times n}$ is called monomial (or the generalised permutation matrix) if in each row and in each column only one entry is positive and the remaining entries are zero.

Theorem 3. *The internally positive system (1) is reachable in time* $t_f - t_0$ *if*

$$R(t_f, t_0) := \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)B^T(\tau)\Phi^T(t_f, \tau)\,\mathrm{d}\tau \quad (T \text{ denotes the transpose}) \tag{16}$$

is a monomial matrix. The input vector which steers the state vector of (1) from $x_0 = 0$ *to* x_f *is given by*

$$u(t) = B^T(t)\Phi^T(t_f, t)R^{-1}(t_f, t)x_f, \tag{17}$$

for $t \in [t_0, t_f]$.

Proof. If $R(t_f, t_0)$ is a monomial matrix, then $R^{-1}(t_f, t_0) \in \mathbb{R}_+^{n\times n}$ and $u(t) \in \mathbb{R}_+^m$ for $[t_0, t_f]$. We shall show that (17) steers the state of (1) from $x_0 = 0$ to x_f. Substituting (17) into (2) for $t = t_f$ and $x_0 = 0$, we obtain

$$x(t_f) = \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)B^T(\tau)\Phi^T(t_f, \tau)R^{-1}(t_f, t_0)x_f\,\mathrm{d}\tau$$

$$= \left[\int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)B^T(\tau)\Phi^T(t_f, \tau)\,\mathrm{d}\tau\right] R^{-1}(t_f, t_0)x_f = x_f.$$

Therefore, if (16) is a monomial matrix, then the positive system (1) is reachable in time $t_f - t_0$. ■

Theorem 4. *The internally positive system (1) is reachable in time* $t_f - t_0$ *if*

$$A(t) = \mathrm{diag}\,[a_1(t), a_2(t), \ldots, a_n(t)], \tag{18}$$

($a_i(t)$, $i = 1, \ldots, n$ *is continuous-time function) and* $B(t) \in \mathbb{R}_+^{n\times n}$ *is a monomial continuous-time matrix.*

Proof. It is well known (Gantmacher, 1959) that if $A(t)$ has the form (18), then $A(t_1)A(t_2) = A(t_2)A(t_1)$ for $t_1, t_2 \in [t_0, \infty)$ and $\Phi(t, t_0) = \exp(\int_{t_0}^{t} A(\tau)\, \mathrm{d}\tau)$ is also a diagonal nonnegative matrix for $t \geq t_0$. Hence the matrix $\Phi(t, t_0)B(t) \in \mathbb{R}_+^{n\times n}$ is a monomial matrix and so is the matrix

$$R(t_f, t_0) = \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)B^T(\tau)\Phi^T(t_f, \tau)\, \mathrm{d}\tau$$

$$= \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)[\Phi(t_f, \tau)B(\tau)]^T\, \mathrm{d}\tau.$$

Then, by Theorem 3, the system (1) is reachable in time $t_f - t_0$. ■

Remark 2. If the diagonal matrix (18) and $B(t)$ are independent of t, then from Theorems 3 and 4 we obtain the corresponding theorems 3.10 and 3.11 in (Kaczorek, 2001).

Similar results can be obtained for the controllability of time-varying linear systems.

6. Example

Consider system (1) with $t_0 = 0$ and

$$A(t) = \begin{bmatrix} 2 & 0 \\ 0 & t \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 & e^t \\ \sqrt{t} & 0 \end{bmatrix}. \tag{19}$$

By Theorem 4, the system is reachable in time $t_f - t_0$. Therefore there exists an input $u(t)$ which steers the state of the system from $x_0 = 0$ to $x_f = [2\ 1]^T$ in time $t_f = 1$. Using (3a), (16) and (17), we obtain

$$\Phi(1, \tau) = \exp\left(\int_{\tau}^{1} A(\tau)\, \mathrm{d}\tau\right) = \begin{bmatrix} \exp\big(2(1-\tau)\big) & 0 \\ 0 & \exp\left(\frac{1}{2}\big(1-\tau^2\big)\right) \end{bmatrix},$$

$$R(t_f, t_0) = R(1, 0) = \int_0^1 \Phi(1, \tau)B(\tau)B^T(\tau)\Phi^T(1, \tau)\, \mathrm{d}\tau = \begin{bmatrix} \frac{e^4}{2}\big(1-e^{-2}\big) & 0 \\ 0 & \frac{1}{2}(e-1) \end{bmatrix},$$

$$u(t) = B^T(t)\Phi^T(1, t)R^{-1}(1, 0)x_f = \begin{bmatrix} 0 & \frac{2t}{e-1}\exp\left(\frac{1}{2}\big(1-t^2\big)\right) \\ \frac{4\exp(-t)}{e^2-1} & 0 \end{bmatrix}.$$

7. Concluding Remarks

The notions of externally and internally positive time-varying linear systems were introduced. Necessary and sufficient conditions for the external and internal positivities of time-varying linear systems were established. The concept of reachability was extended to internally positive time-varying linear systems, and sufficient conditions for the reachability of internally positive time-varying linear systems were established. With minor modifications, the consideration can be extended to discrete time-varying linear systems. A generalization to 2D linear systems with variable coefficients is also possible. An open problem is an extension of the consideration to singular time-varying linear systems.

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