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FUZZY VERSUS PROBABILISTIC BENEFIT/COST RATIO ANALYSIS FOR PUBLIC WORK PROJECTS

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The benefit/cost (B/C) ratio method is utilized in many government and public work projects to determine if the expected benefits provide an acceptable return on the estimated investment and costs. Many authors have studied probabilistic cash flows in recent years. They introduced some analytical methods which determine the probability distribution function of the net present value and internal rate of return of a series of random discrete cash flows. They considered serially correlated cash flows and the uncertainty of future capital investment and reinvestment rates and they presented some formulae for the B/C ratio for probabilistic cash flows. In the paper, the expected value and the variance of a probabilistic cash flow are obtained by means of moments. Then a probabilistic B/C ratio is given. Fuzzy set theory has the capability of representing vague knowledge and allows mathematical operators and programming to be applied to the fuzzy domain. The theory is primarily concerned with quantifying the vagueness in human thoughts and perceptions. The fuzzy B/C ratios are developed for a single investment project and for multiple projects having equal or different lives.

Keywords: economic justification, risk, fuzzy set theory, B/C ratio

1. Introduction

There are many types of public projects and many agencies involved. Four classes reasonably cover the spectrum of projects entered into by a government. They include cultural development, protection, economic services and natural resources. Government projects have a number of interesting characteristics that set them apart from projects in the private sector. Many government projects are huge, having first costs of tens of millions of dollars. They tend to have extremely long lives, such as 50 years for a bridge or a dam. Further, public sector projects are not easily evaluated, since it can take many years before their benefits are realized. Therefore, public sector projects should especially be evaluated under risky conditions.

A risk is defined as an exposure to injury, loss or other undesirable consequences. Thus the risk has two dimensions: (a) exposure, and (b) the nature of undesirable consequences. The exposure can be measured by the probability of encountering the undesired consequences. The measurement of the second dimension depends on the

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nature of the consequences and the metric or metrics used. An economic risk enters economic decision making when there exists a possibility of undesirable consequences and an uncertainty about the factors which cause various decision consequences to occur (Buck and Askin, 1986).

In recent years, several authors have studied the evaluation of the expected net present value and the variance of the net present value of probabilistic cash flows under random timings. Buck and Askin (1986) define partial means and related measures, and show their relationships to several less general risk measures. Schlaifer (1961) and Morris (1968) developed Gaussian linear loss integrals. These integrals are measures over a partial domain of X in contrast to full domain measures given by traditional statistical moments. Hillier (1963) introduced an analytical method which determines the probability distribution function of the net present value and the internal rate of return of a series of random discrete cash flows which occur at constant times. Wagle (1967) introduced a method similar to Hillier's. This method did not require the means and variances of cash flows to be supplied by the user. These two measures are computed from the data. Young and Contreras (1975) considered random lump-sum cash flows occuring at random times and uniform cash flows with random starting and cessation times. Spahr (1982) considers the uncertainty of future capital investment and reinvestment rates. Necessary formulae for the variance of the future worth are presented. He assumes a constant timing of cash flows. Giacotto (1984) considers serially correlated cash flows occuring at constant times. He presents necessary formulae for the mean and variance of the net present value. Park (1984) presents necessary formulae for the benefit/cost ratios when the cash flows are probabilistic. The formulae also assume constant timings of cash flows. Zinn *et al.* (1977) introduced formulae for the expected net present value, variance and semivariance of net present values for different cash flow profiles with random time. Tufekci and Young (1987) present the moments of the net present value of probabilistic investment alternatives. Benzion and Yagil (1987) compare two discounting methods for evaluating multi-period stochastic income streams that are identical and independent over time.

Fuzzy set theory presents an alternative to having to use exact numbers or to have a probability distribution of the cash flow. Many papers on fuzzy B/C ratio analysis can be found in the recent literature. The potential of many new manufacturing systems is being roadblocked in many cases by the wrongful use of traditional justification methods. These new systems can offer not only financial benefits but also longer-term strategic benefits. Thus, traditional piece-by-piece justification approaches, overemphasizing short-term savings in direct manufacturing costs, rather than longer term company strategic benefits, need to be modified to be able to evaluate those benefits which are long-term and perhaps intangible. Chiadamrong (1999) presents a further step by introducing the concept of fuzzy set theory in overcoming the precision-based evaluation. Results from this approach present a better understanding of each alternative and an easier approach to decision makers for evaluation issues which cannot be precisely defined. Momoh and Zhu (1999) develop an integrated approach for reactive power price and control. The reactive power price is divided into fixed and variable parts. The fixed part is the operational cost of the reactive power service. The variable part of the reactive power price is determined based on the capability and contributions to the improvement of the system performance, such as security, reliability and economics. These contributions can be evaluated by computing the sensitivity of the objective function with respect to the reactive power support. The optimal power flow (OPF) approach is used to accomplish this purpose. For VAr planning (or control) purposes, three parallel indices are first presented to determine the sites of new VAr sources. They are the benefit-to-cost ratio index, the voltage reactive sensitivity index, and the bus voltage security index. The analytic hierarchical process is then used to comprehensively consider the effect of the three indices and the network topology for each candidate VAr source site.

Boussabaine and Elhag (1999) present an alternative approach to cash flow analysis for construction projects. Construction managers are interested in the direction of the cash flow movement at valuation periods rather than in its forecast value, and fuzzy set theory applied to decision making might help in this process. Fuzzy models are particularly suited to making decisions involving new technologies where uncertainties inherent in the situation are complex. The problem of a healthy cash flow at valuation periods relates to the proper estimation of cash in and out flows and project progress. This project is based on the assumption that the cash flow at particular valuation stages of a project is ambiguous. The weaknesses of the existing methods for cash flow are discussed and the need for an alternative approach is established.

Temponi *et al.* (1999) propose an objective model to reduce the amount of rework due to subjective evaluation of colour reproduction. A colour quality appraisal refers to a visual evaluation of an original colour image and an assessment of the degree to which that image is matched in photomechanical or electronic reproduction. The evaluation of reproduction quality is quite subjective since it relies on the individual judgments of colour technicians who have different perceptions of colour and its component qualities. In the model, a fuzzy set of quality factors is introduced and a numerical quality value calculated. This measure is then compared with an acceptable quality value experimentally established for particular customers and printing processes. Finally, the installed model is tested in actual production, and compared with the conventional evaluation method in a cost analysis. Buckley (1987), Ward (1985), Chiu and Park (1994), Wang and Liang (1995), Kahraman *et al.* (1995; 2000) are among the authors who deal with the fuzzy present worth analysis, the fuzzy benefit/cost ratio analysis, the fuzzy future value analysis, the fuzzy payback period analysis, and the fuzzy capitalized value analysis.

In the following, first, statistical terms related to the risk analysis and then fuzzy set theory and fuzzy numbers are shortly explained.

1.1. The Expected Value and the Variance of a Probabilistic Cash Flow

Since the expected value of a sum of random variables equals the sum of the expected values of the random variables, the expected present value (PV) is given by

$$E(\mathrm{PV}) = \sum_{j=0}^{N} (1+i)^{-j} E[A_j], \tag{1}$$

where A_j's are statistically independent net cash flows and N is the life of the project. Then the variance of PV is given by

$$\sigma^2[\mathrm{PV}] = V(\mathrm{PV}) = \sum_{j=0}^{N} (1+i)^{-2j} V(A_j). \tag{2}$$

The central limit theorem establishes that the sum of independently distributed random variables tends to be normally distributed as the number of terms in the summation increases. Hence, as N increases, PV tends to be normally distributed with a mean value of $E[\mathrm{PV}]$ and a variance of $V(\mathrm{PV})$ (Canada and White, 1980).

In the case of a set of correlated cash flows (A_j's are not statistically independent), the variance calculation is modified as follows:

$$V[\mathrm{PV}] = \sum_{j=0}^{N} V(A_j)(1+i)^{-2j} + 2\sum_{j=0}^{N-1}\sum_{k=j+1}^{N} \mathrm{Cov}\,[A_j, A_k](1+i)^{-(j+k)}, \tag{3}$$

where $\mathrm{Cov}\,[A_j, A_k]$ is the covariance between A_j and A_k. $\mathrm{Cov}\,[A_j, A_k]$ equals $\rho_{jk}\sigma[A_j]\sigma[A_k]$, where ρ_{jk} is the correlation coefficient between A_j and A_k.

If all A_j and A_k are perfectly correlated such that $\rho_{jk} = +1$, then

$$V[\mathrm{PV}] = \left\{ \sum_{j=0}^{N} \sigma[A_j](1+i)^{-j} \right\}^2 . \tag{4}$$

In performing risk analyses involving correlated cash flows, Hillier (1963) suggests that the net cash flow in a year be seperated into those components of the cash flow one can reasonably expect to be independent from year to year and those that are correlated over time.

1.2. Fuzzy Set Theory and Fuzzy Numbers

To deal with the vagueness of human thought, Zadeh (1965) first introduced fuzzy set theory which was oriented to the rationality of uncertainty due to imprecision or vagueness. A major contribution of fuzzy set theory is its capability of representing vague knowledge. The theory also allows mathematical operators and programming to be applied to the fuzzy domain. A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function that assigns to each object a grade of membership ranging between zero and one.

Quite often in finance, future cash amounts and interest rates are estimated. One usually employs educated guesses, based on expected values or other statistical techniques, to obtain future cash flows and interest rates. A statement like 'approximately between 10% and 15%' must be translated into an exact amount such as '12.5%'. Appropriate fuzzy numbers can be used to capture the vagueness of 'approximately between 10% and 15%' (Buckley, 1987).

A fuzzy number is a normal and convex fuzzy set with a membership function $\mu_A(x)$ which satisfies the following conditions:

- normality:

$$\mu_A(x) = 1 \quad \text{for at least one} \quad x \in \mathbb{R}, \tag{5}$$

- convexity:

$$\mu_A(x') \geq \mu_A(x_1) \wedge \mu_A(x_2), \quad \forall\, x' \in [x_1, x_2], \tag{6}$$

where $\wedge$ stands for the min operator and $\mu_A(x) \in [0,1]$.

A tilde '$\sim$' will be placed above a symbol if the symbol represents a fuzzy set. Therefore, $\widetilde{P}$, $\widetilde{r}$ and $\widetilde{n}$ are all fuzzy sets. The membership functions for these fuzzy sets will be denoted by $\mu(x\,|\,\widetilde{P})$, $\mu(x\,|\,\widetilde{r})$ and $\mu(x\,|\,\widetilde{n})$, respectively. A triangular fuzzy number (TFN) is shown in Fig. 1. The membership function of a TFN is defined by

$$\mu(x\,|\,\widetilde{M}) = \Big(m_1, f_1(y\,|\,\widetilde{M})/m_2, m_2/f_2(y\,|\,\widetilde{M}), m_3\Big)\,, \tag{7}$$

where $m_1 \prec m_2 \prec m_3$, $f_1(y\,|\,\widetilde{M})$ is a continuous monotone increasing function of y for $0 \leq y \leq 1$ with $f_1(0\,|\,\widetilde{M}) = m_1$ and $f_1(1\,|\,\widetilde{M}) = m_2$, and $f_2(y\,|\,\widetilde{M})$ is a continuous monotone decreasing function of y for $0 \leq y \leq 1$ with $f_2(1\,|\,\widetilde{M}) = m_2$ and $f_2(0\,|\,\widetilde{M}) = m_3$. Here $\mu(x\,|\,\widetilde{M})$ is simply denoted by $(m_1/m_2, m_2/m_3)$ or (m_1, m_2, m_3). The parameters m_1, m_2 and m_3 respectively denote the smallest possible value, the most promising value, and the largest possible value that describes a fuzzy event.

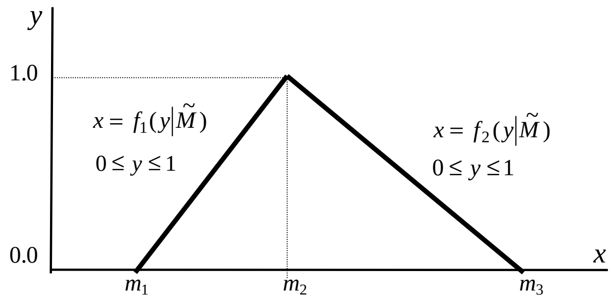

Fig. 1. A triangular fuzzy number, $\widetilde{M}$.

Each TFN has linear representations on its left- and right-hand sides, such that its membership function can be defined as

$$\mu(x\,|\,\widetilde{M}) = \begin{cases} 0, & x < m_1, \\ (x - m_1)/(m_2 - m_1), & m_1 \leq x \leq m_2, \\ (m_3 - x)/(m_3 - m_2), & m_2 \leq x \leq m_3, \\ 0, & x > m_3. \end{cases} \tag{8}$$

Inverse mappings from any given degree of membership to its corresponding x values can be defined, one on the left-hand side of the fuzzy number, another on the right-hand side of this number. Thus, a fuzzy number can always be given by its left and right representations of each degree of membership:

$$\begin{aligned}\widetilde{M} &= \left(M^{l(y)}, M^{r(y)}\right)\\ &= \big(m_1 + (m_2 - m_1)y, m_3 + (m_2 - m_3)y\big) \quad \forall\, y \in [0,1],\end{aligned} \tag{9}$$

where $l(y)$ and $r(y)$ denote the left and right-hand side representations of a fuzzy number, respectively. The algebraic operations with triangular fuzzy numbers are defined in the Appendix.

2. Benefit/Cost Ratio Analysis

The deterministic B/C ratio can be defined as the ratio of the equivalent value of benefits to the equivalent value of costs. The equivalent values can correspond to present, annual or future values. The B/C ratio (BCR) is formulated as

$$\mathrm{BCR} = B/C, \tag{10}$$

where B represents the equivalent value of the benefits associated with the project and C represents the project's net cost (Blank and Tarquin, 1989). A B/C ratio greater than or equal to 1.0 indicates that the project evaluated is economically advantageous.

In B/C analyses, costs are not preceded by a minus sign. The objective to be maximized behind the B/C ratio is to select an alternative with the largest net present value or with the largest net equivalent uniform annual value, because B/C ratios are obtained from the equations necessary to conduct an analysis on the incremental benefits and costs. Suppose that there are two mutually exclusive alternatives. In this case, for the incremental BCR analysis ignoring disbenefits, the following ratios must be used:

$$\frac{\Delta B_{2-1}}{\Delta C_{2-1}} = \frac{\Delta \mathrm{PVB}_{2-1}}{\Delta \mathrm{PVC}_{2-1}} \tag{11}$$

or

$$\frac{\Delta B_{2-1}}{\Delta C_{2-1}} = \frac{\Delta \mathrm{EUAB}_{2-1}}{\Delta \mathrm{EUAC}_{2-1}}, \tag{12}$$

where ΔB_{2-1} is the incremental benefit of Alternative 2 relative to Alternative 1, ΔC_{2-1} stands for the incremental cost of Alternative 2 relative to Alternative 1, $\Delta \mathrm{PVB}_{2-1}$ denotes the incremental present value of the benefits of Alternative 2 relative to Alternative 1, $\Delta \mathrm{PVC}_{2-1}$ signifies the incremental present value of costs of Alternative 2 relative to Alternative 1, $\Delta \mathrm{EUAB}_{2-1}$ means the incremental equivalent uniform annual benefit of Alternative 2 relative to Alternative 1, and $\Delta \mathrm{EUAC}_{2-1}$ is the incremental equivalent uniform annual cost of Alternative 2 relative to Alternative 2.

Thus, the concept of the B/C ratio includes the advantages of both NPV and NEUAV analyses. Because it does not require using a common multiple of the alternative lives (then the B/C ratio based on an equivalent uniform annual cash flow is used) and it is a more understandable technique relative to the rate of return analysis for many financial managers, the B/C analysis can be preferred to other techniques such as the present value analysis, the future value analysis, and the rate of return analysis.

2.1. Probabilistic B/C Ratio Analysis

If B and C are normally distributed random variables with means μ_j, variances σ_j^2 ($j = 1$ for B and $j = 2$ for C), and correlation coefficient ρ, the probability distribution of R_A is defined as follows (Park and Sharp-Bette, 1990): Let $R_A = B/C$. Consequently, the probability that R_A is less than or equal to a is

$$\begin{aligned} P(R_A \leq a) &= P\left(B/C \leq a\right) \\ &= P(B - aC \leq 0 \,|\, C \succ 0)P(C \succ 0) \\ &\quad + P(B - aC \succ 0 \,|\, C \prec 0)P(C \prec 0) \\ &= L(\eta, -\kappa; \lambda) + L(-\eta, \kappa; \lambda), \end{aligned} \tag{13}$$

where

$$\eta = \frac{\mu_1 - \mu_2 a}{G(a)}, \tag{14}$$

$$\kappa = \frac{\mu_2}{\sigma_2}, \tag{15}$$

$$\lambda = \frac{\sigma_2 a - \rho\sigma_1}{G(a)}, \tag{16}$$

$$G(a) = \sigma_1\sigma_2 \left(\frac{a^2}{\sigma_1^2} - \frac{2\rho a}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right)^{1/2}, \tag{17}$$

and $L(\eta, \kappa; \lambda)$ is the standard bivariate normal integral tabulated by the National Bureau of Standards. Note that if $\kappa \to \infty$, i.e. as $P(C \succ 0) \to 1$, eqn. (13) becomes

$$P(R_A \leq a) \to \Phi(-\eta) \tag{18}$$

or, with $P(C \succ 0) = 1$,

$$P(R_A \leq a) = \Phi(-\eta), \tag{19}$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2}\, \mathrm{d}u. \tag{20}$$

For the incremental probabilistic B/C ratio analysis, we have

$$\begin{aligned}&P(\text{Project 2 is preferred over Project 1})\\&\quad = P(\Delta B-\Delta C \le 0 \mid \Delta C \succ 0)P(\Delta C \succ 0)\\&\qquad + P(\Delta B-\Delta C \prec 0 \mid \Delta C \prec 0)P(\Delta C \prec 0). \qquad (21)\end{aligned}$$

Since we assume normal distributions for both ΔB and ΔC with means μ_j, variances σ_j^2 ($j=1$ for ΔB and $j=2$ for ΔC), $a=1$, and correlation coefficient ρ, if any, eqn. (21) becomes

$$\begin{aligned}P(\text{Project 1} \prec \text{Project 2}) &= L(\eta,-\kappa;\lambda) + \big[P(\Delta C \prec 0) - L(-\eta,\kappa;\lambda)\big]\\&= L(\eta,-\kappa;\lambda) + \big[\Phi(-\kappa) - L(-\eta,\kappa;\lambda)\big]. \qquad (22)\end{aligned}$$

2.2. Fuzzy B/C Ratio Analysis

In the case of fuzziness, the steps of the fuzzy B/C analysis are given in the following:

Step 1. Calculate the overall fuzzy measure of the benefit to the cost ratio and eliminate the alternatives for which we have

$$\widetilde{B}/\widetilde{C} = \left(\frac{\sum_{t=0}^{n} B_t^{l(y)}\left(1+r^{r(y)}\right)^{-t}}{\sum_{t=0}^{n} C_t^{r(y)}\left(1+r^{l(y)}\right)^{-t}}, \frac{\sum_{t=0}^{n} B_t^{r(y)}\left(1+r^{l(y)}\right)^{-t}}{\sum_{t=0}^{n} C_t^{l(y)}\left(1+r^{r(y)}\right)^{-t}} \right) \prec \widetilde{1}, \qquad (23)$$

where $\widetilde{r}$ is the fuzzy interest rate, $\widetilde{1}$ is $(1,1,1)$, and n denotes the crisp useful life.

Step 2. Assign the alternative that has the lowest initial investment cost as the defender and the next-to-lowest acceptable alternative as the challenger.

Step 3. Determine the incremental benefits and the incremental costs between the challenger and the defender.

Step 4. Calculate the $\Delta\widetilde{B}/\Delta\widetilde{C}$ ratio, assuming that the largest possible value for the cash in year t of the alternative with the lowest initial investment cost is less than the least possible value for the cash in year t of the alternative with the next-to-lowest initial investment cost.

The fuzzy incremental BCR is

$$\Delta\widetilde{B}/\Delta\widetilde{C} = \left(\frac{\sum_{t=0}^{n}\left(B_{2t}^{l(y)}-B_{1t}^{r(y)}\right)\left(1+r^{r(y)}\right)^{-t}}{\sum_{t=0}^{n}\left(C_{2t}^{r(y)}-C_{1t}^{l(y)}\right)\left(1+r^{l(y)}\right)^{-t}}, \frac{\sum_{t=0}^{n}\left(B_{2t}^{r(y)}-B_{1t}^{l(y)}\right)\left(1+r^{l(y)}\right)^{-t}}{\sum_{t=0}^{n}\left(C_{2t}^{l(y)}-C_{1t}^{r(y)}\right)\left(1+r^{r(y)}\right)^{-t}} \right). \qquad (24)$$

If $\Delta\widetilde{B}/\Delta\widetilde{C}$ is greater than or equal to $(1,1,1)$, Alternative 2 is preferred.

In the case of a regular annuity, the fuzzy $\widetilde{B}/\widetilde{C}$ ratio of a single investment alternative is

$$\widetilde{B}/\widetilde{C} = \left(\frac{A^{l(y)}\gamma\left(n, r^{r(y)}\right)}{C^{r(y)}}, \frac{A^{r(y)}\gamma\left(n, r^{l(y)}\right)}{C^{l(y)}} \right), \tag{25}$$

where $\widetilde{C}$ is the first cost, $\widetilde{A}$ is the net annual benefit, and $\gamma(n,r) = ((1+r)^n - 1)/(1+r)^n r)$.

The $\Delta\widetilde{B}/\Delta\widetilde{C}$ ratio in the case of a regular annuity is

$$\Delta\widetilde{B}/\Delta\widetilde{C} = \left(\frac{\left(A_2^{l(y)} - A_1^{r(y)}\right)\gamma\left(n, r^{r(y)}\right)}{C_2^{r(y)} - C_1^{l(y)}}, \frac{\left(A_2^{r(y)} - A_1^{l(y)}\right)\gamma\left(n, r^{l(y)}\right)}{C_2^{l(y)} - C_1^{r(y)}} \right). \tag{26}$$

Step 5. Repeat Steps 3 and 4 until only one alternative is left. Thus the optimal alternative is obtained.

The cash-flow set $\{A_t = A : \; t = 1, 2, \ldots, n\}$, consisting of n cash flows, each of the same amount A, at times $1, 2, \ldots, n$, with no cash flow at time zero, is called the equal-payment series. An older name for it is the uniform series, and it has been called an annuity, since one of the meanings of 'annuity' is a set of fixed payments for a specified number of years. To find the fuzzy present value of a regular annuity $\{\widetilde{A}_t = \widetilde{A} : \; t = n\}$, we will use (13). The membership function $\mu(x \,|\, \widetilde{P}_n)$ for $\widetilde{P}_n$ is determined by

$$f_{ni}(y \,|\, \widetilde{P}_n) = f_i(y \,|\, \widetilde{A})\gamma\big(n, f_{3-i}(y \,|\, \widetilde{r})\big) \tag{27}$$

for $i = 1, 2$ and $\gamma(n, r) = (1 - (1+r)^{-n})/r$. Both $\widetilde{A}$ and $\widetilde{r}$ are positive fuzzy numbers. Here $f_1(\cdot)$ and $f_2(\cdot)$ stand for the left and right representations of the fuzzy numbers, respectively.

In the case of a regular annuity, the fuzzy $\widetilde{B}/\widetilde{C}$ ratio can be calculated as follows: the fuzzy $\widetilde{B}/\widetilde{C}$ ratio of a single investment alternative is

$$\widetilde{B}/\widetilde{C} = \left(\frac{A^{l(y)}\gamma\left(n, r^{r(y)}\right)}{FC^{r(y)}}, \frac{A^{r(y)}\gamma\left(n, r^{l(y)}\right)}{FC^{l(y)}} \right), \tag{28}$$

where $F\widetilde{C}$ is the first cost and $\widetilde{A}$ is the net annual benefit. The $\Delta\widetilde{B}/\Delta\widetilde{C}$ ratio in the case of a regular annuity is

$$\Delta\widetilde{B}/\Delta\widetilde{C} = \left(\frac{\left(A_2^{l(y)} - A_1^{r(y)}\right)\gamma\left(n, r^{r(y)}\right)}{FC_2^{r(y)} - FC_1^{l(y)}}, \frac{\left(A_2^{r(y)} - A_1^{l(y)}\right)\gamma\left(n, r^{l(y)}\right)}{FC_2^{l(y)} - FC_1^{r(y)}} \right). \tag{29}$$

Up to this point, we assumed that the alternatives had equal lives. When the alternatives have useful lives different from the analysis period, a common multiple of the alternative lives (CMALs) is calculated for the analysis period. Frequently, a CMALs for the analysis period hardly seems realistic (e.g. CMALs is $(7, 13) = 91$

years). Instead of conducting an analysis based on the present value method, it is appropriate to compare the annual cash flows computed for the alternatives based on their own service lives. In the case of unequal lives, the following fuzzy $\widetilde{B}/\widetilde{C}$ and $\Delta\widetilde{B}/\Delta\widetilde{C}$ ratios will be used:

$$\widetilde{B}/\widetilde{C} = \left(\frac{\mathrm{PVB}^{l(y)}\beta\left(n, r^{l(y)}\right)}{\mathrm{PVC}^{r(y)}\beta\left(n, r^{r(y)}\right)}, \frac{\mathrm{PVB}^{r(y)}\beta\left(n, r^{r(y)}\right)}{\mathrm{PVC}^{l(y)}\beta\left(n, r^{l(y)}\right)} \right), \tag{30}$$

$$\Delta\widetilde{B}/\Delta\widetilde{C} = \left(\frac{\mathrm{PVB}_2^{l(y)}\beta\left(n, r^{l(y)}\right) - \mathrm{PVB}_1^{r(y)}\beta\left(n, r^{r(y)}\right)}{\mathrm{PVC}_2^{r(y)}\beta\left(n, r^{r(y)}\right) - \mathrm{PVC}_1^{l(y)}\beta\left(n, r^{l(y)}\right)}, \right.$$
$$\left. \frac{\mathrm{PVB}_2^{r(y)}\beta\left(n, r^{r(y)}\right) - \mathrm{PVB}_1^{l(y)}\beta\left(n, r^{l(y)}\right)}{\mathrm{PVC}_2^{l(y)}\beta\left(n, r^{l(y)}\right) - \mathrm{PVC}_1^{r(y)}\beta\left(n, r^{r(y)}\right)} \right), \tag{31}$$

where PVB stands for the present value of benefits, PVC is the present value of costs and $\beta(n, r) = (1+n)^n i/((1+r)^n - 1)$.

A numeric application of the fuzzy $\widetilde{B}/\widetilde{C}$ ratio analysis to manufacturing technologies can be found in (Kahraman *et al.*, 2000).

3. Ranking Fuzzy Numbers

The fuzzy B/C ratio method requires ranking fuzzy outcomes. There are a number of methods that are devised to rank mutually exclusive projects such as Chang's method (1981), Jain's method (1976), Dubois and Prade's method (1983), Yager's method (1980), or Baas and Kwakernaak's method (1977). However, certain shortcomings of some of the methods were reported in (Bortolan and Degani, 1985; Chen, 1985; Kim and Park, 1990). Because the ranking methods might give different ranking results, they must be used together to obtain the true rank. Chiu and Park (1994) compare some ranking methods by using a numerical example, and determine which methods give the same or very close results to one another. In their paper, several dominance methods are selected and discussed. Most methods are tedious in graphic manipulation and require complex mathematical calculations. Chiu and Park's weighting method as well as Kaufmann and Gupta's (1988) method are the methods giving the same rank for the considered alternatives, are easy to calculate and require no graphical representation. Therefore these two methods will be outlined in the following and used in the application section.

Chiu and Park's (1994) weighted method for ranking TFNs with parameters (a, b, c) is formulated as

$$\frac{1}{3}(a+b+c) + wb,$$

where w is a value determined by the nature and the magnitude of the most promising value.

Kaufmann and Gupta (1988) suggest three criteria for ranking TFNs with parameters (a, b, c). The dominance sequence is determined according to the priority of:

1. comparing the ordinary number $(a + 2b + c)/4$,
2. comparing the mode (the corresponding most promise value), b, of each TFN,
3. comparing the range, c–a, of each TFN.

The preference of projects is determined by the amount of their ordinary numbers. The project with a larger ordinary number is preferred. If the ordinary numbers are equal, the project with a larger most promising value is preferred. If projects have the same ordinary numbers and most promising values, the project with a larger range is preferred.

4. Conclusions

An effective probabilistic risk analysis requires substantial amounts of both internal and external data. These data, such as estimates of the cash flow, the cost of capital, and economic life, are obtained by updating previous forecasts or from internal historical record files. External future information in many cases has to be collected through other, unstructured, inconsistent and less reliable channels. Without a supportive information system, it is very difficult to carry out any sophisticated or meaningful risk analysis. As an alternative to the probabilistic B/C ratio, a fuzzy benefit/cost ratio analysis was developed. Fuzzy sets present an alternative to having to use exact amounts for the parameters used in the justification process. This prevents us from making some erroneous estimates of the parameters.

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Appendix

One of the most basic concepts of fuzzy set theory which can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. Let X be the Cartesian product of universes, $X = X_1 \times \cdots \times X_r$, and $\widetilde{A}_1, \ldots, \widetilde{A}_r$ be r fuzzy sets in $X_1, \ldots, X_r$, respectivelty. Let f be a mapping from X to a universe Y, $y = f(x_1, \ldots, x_r)$. Then the extension principle allows us to define a fuzzy set $\widetilde{B}$ in Y by

$$\widetilde{B} = \big\{(y, \mu_{\widetilde{B}}(y)) \,|\, y = f(x_1, \ldots, x_r), (x_1, \ldots, x_r) \in X\big\}, \tag{A1}$$

where

$$\mu_{\widetilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_r)\in f^{-1}(y)} \min\big\{\mu_{\widetilde{A}_1}(x_1), \ldots, \mu_{\widetilde{A}_r}(x_r)\big\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{A2}$$

f^{-1} being the inverse of f.

Assume that $\widetilde{P} = (a, b, c)$ and $\widetilde{Q} = (d, e, f)$, where a, b, c, d, e, f are all positive numbers. With this notation and by the extension principle, some of the extended algebraic operations of triangular fuzzy numbers are expressed as follows:

Changing Sign:

$$-(a, b, c) = (-c, -b, -a) \tag{A3}$$

or

$$-(d, e, f) = (-f, -e, -d). \tag{A4}$$

Addition:

$$\widetilde{P} \oplus \widetilde{Q} = (a+d, b+e, c+f) \tag{A5}$$

and

$$k \oplus (a, b, c) = (k+a, k+b, k+c) \tag{A6}$$

or

$$k \oplus (d, e, f) = (k+d, k+e, k+f) \tag{A7}$$

if k is an ordinary number (a constant).

Subtraction:

$$\widetilde{P} - \widetilde{Q} = (a-f, b-e, c-d) \tag{A8}$$

and

$$(a, b, c) - k = (a-k, b-k, c-k) \tag{A9}$$

or

$$(d, e, f) - k = (d-k, e-k, f-k) \tag{A10}$$

if k is an ordinary number.

Multiplication:

$$\widetilde{P} \otimes \widetilde{Q} = (ad, be, cf) \tag{A11}$$

and

$$k \otimes (a, b, c) = (ka, kb, kc) \tag{A12}$$

or

$$k \otimes (d, e, f) = (kd, ke, kf) \tag{A13}$$

if k is an ordinary number.

Division:

$$\widetilde{P} \oslash \widetilde{Q} = (a/f, b/e, c/d). \tag{A14}$$