SEPARATION PRINCIPLE FOR NONLINEAR SYSTEMS: A BILINEAR APPROACH

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In this paper we investigate the local stabilizability of single-input nonlinear affine systems by means of an estimated state feedback law given by a bilinear observer. The associated bilinear approximating system is assumed to be observable for any input and stabilizable by a homogeneous feedback law of degree zero. Furthermore, we discuss the case of planar systems which admit bad inputs (i.e. the ones that make bilinear systems unobservable). A separation principle for such systems is given.

Keywords: nonlinear systems, bilinear systems, observer, stabilization

1. Introduction

In this paper we study the local stabilization problem of single-input nonlinear systems of the form

$$\begin{cases} \dot{x} = f(x) + ug(x), \\ y = h(x), \end{cases} \tag{1}$$

where $x \in U$ is a neighborhood of the origin in $\mathbb{R}^n$, u stands for a scalar input, $y \in \mathbb{R}^p$ is the output, f and g denote smooth vector fields, and h is a real analytic function on $\mathbb{R}^n$, such that $f(0) = 0$ and $h(0) = 0$.

Several authors (Boothby and Marino, 1989; Dayawansa *et al.*, 1990) investigate the stabilizability problem when $g(0) \neq 0$. The common approach is to consider the corresponding linearized system. It is well-known that for nonlinear systems, in particular systems of the form (1), there exists a local exponential observer (i.e. a dynamic system of the form $\dot{\hat{x}} = \phi(\hat{x}, y, u)$) if and only if the linear approximation of the system at the origin is detectable. If, moreover, it is stabilizable by a state feedback, the problem of output feedback stabilization with state observer is solvable with a linear observer and a linear control law. This observer can be taken as $\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$, where the pair (A, C) is detectable and L signifies the gain matrix such that $(A - LC)$ is a Hurwitz matrix. Another approach to establish a local separation principle is to consider the system (1) in closed-loop with an observer-based output feedback control law $\dot{\hat{x}} = \phi(\hat{x}, y, u)$, $u = u(\hat{x})$. If the state feedback law $u(x)$

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stabilizes the system (1) locally and asymptotically at the origin, and if the system $\dot{\hat{x}} = \phi(\hat{x}, y, u)$ is a local observer, then the closed-loop system $\dot{x} = f(x) + u(\hat{x})g(x)$ is locally asymptotically stable. Suppose that there exists a local observer for system (1). It follows that this observer takes the form:

$$\dot{\hat{x}} = \phi(\hat{x}, y, u) = f(\hat{x}) + ug(\hat{x}) + k(\hat{x}, y, u),$$

where $k(\cdot, \cdot, \cdot)$ is a nonlinear smooth function that satisfies, on the whole, state space, and for all input signals u the equality $k(x, h(x), u) = 0$. This means that the observer and the plant have the same dynamics under the condition that the output function $\hat{y} = h(\hat{x})$ reproduces the output function $h(x)$. However, few results are known in the case where $g(0) = 0$ (Baccioti and Boieri, 1991; Chabour *et al.*, 1996; Chabour and Vivalda, 1991; Hammami and Jerbi, 1994). The principal difficulties arise from the fact that the linearized system is independent of the control and the vector field g is not locally rectifiable. To investigate the stabilization problem for such systems by means of a state-estimated feedback law given by an observer design, we consider a bilinear approximation system

$$\begin{cases} \dot{x} = Ax + uBx, \\ \tilde{y} = Cx, \end{cases} \tag{2}$$

where

$$A = \frac{\partial f}{\partial x}(0), \quad B = \frac{\partial g}{\partial x}(0) \quad \text{and} \quad C = \frac{\partial h}{\partial x}(0).$$

The aim of this paper is to study the stabilizability of the original system when the states of its bilinear approximating system are not available. The usual technique for asymptotically stabilizing this nonlinear system is to build an observer. The stabilization by means of a state-estimated feedback depends on both the existence of an observer and a stabilizing feedback law. In (Chabour and Hammouri, 1993; Gauthier and Kupka, 1992; Hammami, 1993) the authors solved the problem of stabilizing in observer design for some classes of nonlinear systems. Suppose that we have a stabilizable and observable bilinear system with states x. We use a state feedback law $u = u(x)$ to asymptotically stabilize the system (1).

If the states are not available, we must construct a bilinear observer which is expected to produce an estimate $\hat{x}(t)$ of the state $x(t)$; then we apply the feedback $u = u(\hat{x})$. It turns out that for planar systems we can consider bilinear systems with bad inputs (i.e. the ones for which the system is not observable). There is at most only one input which is constant that makes the system unobservable. In this case, we show that the Kalman observer solves the problem. Moreover, for nonlinear systems of the form (1) we give a separation principle, including the case of bilinear systems.

2. Stabilization Using a State Estimation

Consider a single-input nonlinear system of the form (1). Since f, g and h are C^1, we can write

$$f(x) = Ax + f_1(x),$$
$$g(x) = Bx + g_1(x),$$

and

$$h(x) = Cx + h_1(x),$$

where f_1, g_1 and h_1 satisfy

$$\|f_1(x)\| \leq M_1\|x\|, \quad \|g_1(x)\| \leq M_2\|x\|, \quad \|h_1(x)\| \leq M_3\|x\|, \; \forall\, x \in U' \subset U \tag{3}$$

M_1, M_2 and M_3 being some positive constants. Throughout this paper we shall call (2) the approximating system for the system (1). Suppose now that the system (2) satisfies the following assumptions:

($\mathcal{A}_1$) The approximating system is observable for any input.

($\mathcal{A}_2$) There exists a homogeneous feedback law of degree zero $u(x)$, ($u(\lambda x) = u(x)$ for $\lambda \neq 0$), and of class C^1 (on $U \setminus \{0\}$), stabilizing the bilinear system (2).

Note that in (Chabour *et al.*, 1996) the authors gave a complete classification of planar homogeneous bilinear systems where for stabilizable bilinear systems, a given feedback u is smooth on $\mathbb{R}^2 \setminus \{0\}$ and homogeneous of degree zero.

Now, in order to investigate the stabilizability problem in observer design, we can consider a Kalman observer for the bilinear system (2) of the form

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + uB\hat{x} + S^{-1}{}^tC(\tilde{y} - C\hat{x}), \\ \dot{S} = -\theta S - {}^t(A + uB)S - S(A + uB) + {}^tCC, \end{cases} \tag{4}$$

where $\theta > 0$ and $S \in \mathcal{S}^+$, the cone of symmetric positive definite matrices on $\mathbb{R}^n$, which is invariant by the second equation of (4) (the symbol ${}^t(\cdot)$ denotes transposition). It is known from (Bornard *et al.*, 1989) that, under the assumption ($\mathcal{A}_1$), this observer converges for any bounded and small input u which is distant from bad inputs or just $u(t)$ is a regularly persistent input in the sense of (Hahn, 1967). Indeed, since the pair (A, C) is observable, $u = 0$ constitutes a universal input. Thus, for a small input u there exists $\varepsilon > 0$ such that for $|u| < \varepsilon$, u is universal too (i.e. *if it distinguishes the points, that is, for all initial conditions* $(x_0, \bar{x}_0)$ *there exists* $T > 0$ *such that* $C(x^u(T)) \neq C(\bar{x}^u(T))$, *where* $x^u(t)$ *is the solution of (2) such that* $x^u(0) = x_0$). Therefore, the Gramm observability matrix satisfies,

$$W_u(t) = \int_0^t \mathrm{e}^{-\theta(t-s)} \big(\dot{\Phi}_{u(s)}(t-s)\big)^{-1} C^T C \Phi^{-1}_{u(s)}(t-s)\, \mathrm{d}s \geq \alpha I, \quad \alpha > 0.$$

On the one hand, if we consider the error equation

$$\dot{e} = (A + uB)e - S^{-1}C^T Ce,$$

where $e = x - \hat{x}$, then, by using the same argument as in (Gauthier and Kupka, 1992), the error satisfies the estimate

$$\|e(t)\| \leq k e^{-\theta t/2}, \quad k > 0, \tag{5}$$

where the constant k depends only on the initial state of (2) and u. This implies that the system (4) is an exponential observer for (1).

On the other hand, let $\Phi_{u(t)}(t)$ be the matrix solution of $\dot{\Phi}_{u(t)} = (A + u(t)B)\Phi_{u(t)}$, for $|u(t)| \leq u_0$, where u_0 is a positive constant. Then

$$\begin{aligned} S(t) \;&=\; \mathrm{e}^{-\theta t}(\dot{\Phi}_{u(t)})^{-1} S_0 \Phi^{-1}_{u(t)} \\ &\quad + \int_0^t \mathrm{e}^{-\theta(t-s)}\big(\dot{\Phi}_{u(s)}(t-s)\big)^{-1} C^T C \Phi^{-1}_{u(s)}(t-s)\,\mathrm{d}s \end{aligned}$$

with $S_0 \in \mathcal{S}^+$ being the initial condition for the solution $S(t)$ (i.e. $S(0) = S_0$). Hence

$$\begin{aligned} \|S(t)\| \;&\leq\; \mathrm{e}^{-\theta t}\|(\dot{\Phi}_{u(t)})^{-1}\|\,\|S_0\|\,\|\Phi^{-1}_{u(t)}\| \\ &\quad + \int_0^t \mathrm{e}^{-\theta(t-s)}\|(\dot{\Phi}_{u(s)}(t-s))^{-1}\|\,\|C^T C\|\,\|\Phi^{-1}_{u(s)}(t-s)\|\,\mathrm{d}s. \end{aligned}$$

Let

$$\lambda = \sup_{|u(t)| \leq u_0} \|A + uB\|.$$

Then $\|\Phi_u^{-1}(t)\|$ and $\|(\dot{\Phi}_{u(t)})^{-1}\|$ are bounded by $\mathrm{e}^{\lambda t}, \lambda > 0$. Indeed, we have

$$\dot{\Phi}_u^{-1} = -\Phi_u^{-1}(A + uB) \;\text{ and }\; \dot{\Phi}_u^{-1} = I - \int_0^t \Phi_u^{-1}(s)(A + sB)\,\mathrm{d}s.$$

Then

$$\|\dot{\Phi}_u^{-1}(t)\| \leq 1 + \lambda \int_0^t \|\Phi_u^{-1}(s)\|\,\mathrm{d}s.$$

Using Gronwall's inequality, we obtain

$$\|\Phi_u^{-1}(t)\| \leq \mathrm{e}^{\lambda t}.$$

Therefore

$$\|S(t)\| \leq \mathrm{e}^{-\theta t}\|S(0)\|\mathrm{e}^{2\lambda t} + \int_0^t \mathrm{e}^{-\theta(t-s)}\|C^T C\|\mathrm{e}^{2\lambda(t-s)}\,\mathrm{d}s.$$

Hence

$$\|S(t)\| \leq \|S(0)\| + \frac{\|C^T C\|}{\theta - 2\lambda} \quad \text{for} \quad \theta > 2\lambda. \tag{6}$$

This implies that for a bounded control $u(t)$ the matrix $S(t)$ is bounded. Since $u = u(x)$ is bounded, where the feedback $u(x)$ is given in $(\mathcal{A}_2)$, from (6) we see that $S(t)$ is bounded with respect to $u = u(x) \leq u_0$. Therefore, since the bilinear system is observable for any input and the matrix $S(t)$ is bounded, by using some techniques regarding the Ricatti equations (Gauthier and Kupka, 1992) we can show that the matrix $S^{-1}(t)$ is also bounded. So, there exists a positive constant η such that for all $t \geq 0$,

$$\|S^{-1}(t)\| \leq \eta. \tag{7}$$

Now, let us consider the equation

$$\dot{\hat{x}} = A\hat{x} + uB\hat{x} - S^{-1}(t)^t C(C\hat{x} - y), \tag{8}$$

where we take $y = h(x)$ as the output of the original system (1). Letting $\varepsilon = x - \hat{x}$, where x is the state of (1) and $\hat{x}$ satisfies (8). The derivative of the error ε is given by

$$\begin{aligned}
\dot{\varepsilon} &= f(x) + ug(x) - A\hat{x} - uB\hat{x} + S^{-1t}C(C\hat{x} - y) \\
&= Ax + f_1(x) + uBx + ug_1(x) - A\hat{x} - uB\hat{x} + S^{-1t}C\big(C\hat{x} - Cx - h_1(x)\big) \\
&= (A + uB)\varepsilon + f_1(x) + ug_1(x) - S^{-1}(t)^t CC\varepsilon - S^{-1t}Ch_1(x) \\
&= (A + uB - S^{-1t}CC)\varepsilon + f_1(x) + ug_1(x) - S^{-1}(t)^t Ch_1(x).
\end{aligned}$$

Then the last expression taken in conjunction with the system (1) in the closed loop with the estimated feedback law

$$u = u(x - \varepsilon) \tag{9}$$

yields

$$\begin{pmatrix} \dot{x} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} (A + u(x-\varepsilon)B)x \\ (A + u(x-\varepsilon)B - S^{-1}(t)^t CC)\varepsilon \end{pmatrix} + \begin{pmatrix} f_1(x) + u(x-\varepsilon)g_1(x) \\ f_1(x) + u(x-\varepsilon)g_1(x) - S^{-1}(t)^t Ch_1(x) \end{pmatrix}. \tag{10}$$

Set

$$\phi(x,\varepsilon) = \begin{pmatrix} (A + u(x-\varepsilon)B)x \\ (A + u(x-\varepsilon)B - S^{-1t}CC)\varepsilon \end{pmatrix}$$

and

$$\psi(x,\varepsilon) = \begin{pmatrix} f_1(x) + u(x-\varepsilon)g_1(x) \\ f_1(x) + u(x-\varepsilon)g_1(x) - S^{-1t}Ch_1(x) \end{pmatrix}.$$

Since u is of class C^1 on $U' \setminus \{0\} \subset U$, it can be seen that ϕ and ψ are locally Lipschitz. Moreover, since u is homogeneous of degree zero, we have

$$u(z) = u\left(\frac{z}{\|z\|}\right) \quad \text{for every } z \neq 0.$$

Hence there exists M_0 such that

$$|u(z)| \leq M_0 \quad \text{for every } z \neq 0.$$

M_0 can be taken as the maximum of $u(z)$ on the unit sphere

$$S^1 = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

which is a compact set. On the one hand, it can be seen that ϕ is homogeneous of degree one. By using (3) and (7), one can verify that ψ satisfies

$$\|\psi(x,\varepsilon)\| \leq M\|(x,\varepsilon)\|, \ \ \forall (x,\varepsilon) \in U' \times U',$$

where M is a positive constant depending on M_0, M_1, M_2, M_3, η and $\|C\|$.

On the other hand, the system

$${}^t(\dot{x},\dot{\varepsilon}) = \phi(x,\varepsilon)$$

is globally asymptotically stable (the proof is given as a special case for bilinear systems from Theorem 2, see Remark 2). From a theorem of (Massera, 1956), it follows that the solution $(x,\varepsilon) = (0,0)$ of the differential equation

$${}^t(\dot{x},\dot{\varepsilon}) = \phi(x,\varepsilon) + \psi(x,\varepsilon)$$

is asymptotically stable.

Therefore, using this fact, we can formulate the following result.

Theorem 1. *If the approximating system (2) is observable for any input and stabilizable by means of a homogeneous feedback $u(x)$ of degree zero and of class C^1 on $U \setminus \{0\}$, then it is stabilizable by means of a state estimate feedback given by the bilinear observer (4), and the feedback law $u = u(x-\varepsilon)$ given in (9) makes the origin of the original system (1) a locally asymptotically stable equilibrium point.*

Example 1. Consider the following planar system:

$$\begin{cases} \dot{x}_1 = 3\sin x_1 + 3x_2 + u\sin x_1, \\ \dot{x}_2 = -2x_1 + 3\sin x_2 - u\sin x_2. \end{cases} \tag{11}$$

Then the approximating system for (11) is given by

$$\begin{cases} \dot{x}_1 = 3x_1 + 3x_2 + ux_1, \\ x_2 = -2x_1 + 3x_2 - ux_2. \end{cases} \tag{12}$$

Let $y = x_1$ be the output of (11). Then the system (12) is observable for any input. Besides, it is shown in (Chabour and Vivalda, 1991) that the homogeneous zeroth-degree feedback law $u(x)$ given by

$$u(x_1, x_2) = \frac{-14x_1^2 + 14x_1x_2 + 21x_2^2}{2x_1^2 + 3x_2^2} \quad \text{if } (x_1, x_2) \neq (0, 0)$$

and $u(x) = 0$ if $x = 0$, stabilizes the bilinear subsystem (12). Indeed, let $V(x)$ be the Lyapunov function given by

$$V(x) = 4x_1^2x_2^2 + 2x_1x_2(-2x_1^2 + 6x_1x_2 + 3x_2^2) + (-2x_1^2 + 6x_1x_2 + 3x_2^2)^2,$$

where $x = (x_1, x_2)$. A simple computation shows that the derivative of $V(x)$ with respect to the resulting closed loop bilinear system is $\dot{V}(x) = -2V(x)$ and so $x = 0$ is a globally asymptotically stable equilibrium point of the system (12) in closed-loop. Since V is a homogeneous Lyapunov function, it follows that there exists a positive constant α such that $\|\nabla V(x)\| \leq \alpha(1 + V(x))$. Then one can deduce that the system (12) is stabilizable by the feedback $u(x-e)$ given by the observer (4) and, by applying Theorem 1, one can stabilize the system (11) by the estimated feedback law $u(x - \varepsilon)$, where $\varepsilon = x - \hat{x}$ and $\hat{x}$ satisfies(8). ♦

Remark 1. If the states of the bilinear system are available, we can formulate the stabilization problem of the system (1) as follows: Consider the system (1) defined in a neighbourhood of the origin of $\mathbb{R}^n$, where we suppose that $f(0) = g(0) = 0$. *A function φ is said to be positively homogeneous of degree $m \geq 0$, if for any vector x and any real positive λ, we have $\varphi(\lambda x) = \lambda^m \varphi(x)$.* Therefore, if the bilinear approximation system (2) is stabilizable by means of a positively homogeneous feedback of degree zero and of class C^1 on $\mathbb{R}^n \setminus \{0\}$, then the system (1) is locally stabilizable.

Now, consider the following class of bilinear systems:

$$\begin{cases} \dot{x} = Ax + uBx, & u \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ y = Cx, \end{cases} \tag{13}$$

A, B being $(n \times n)$ constant matrices, where the drift of the system (13) is dissipative (i.e. $x^T Ax \leq 0$, $\forall\, x \in \mathbb{R}^n$, including the case where A is skew symmetric).

Consider

$$V(x) = \|x\|^2 = \langle x \mid x \rangle$$

as a Lyapunov function candidate and assume that (13) satisfies the ad-condition, i.e.

$$\left\{x \in \mathbb{R}^n \mid A^{k+1}V(x) = A^kBV(x) = 0,\ \forall k \in \mathbb{N}\right\} = \{0\}.$$

Using the same techniques as in (Gauthier and Kupka, 1992), we can prove that there exists a small feedback which makes the origin of (13) a globally asymptotically stable equilibrium point. This feedback can be taken as

$$u(x) = -\frac{\langle Bx \mid x \rangle}{M\|x\|^2}, \quad M > 0 \text{ if } x \neq 0 \text{ and } u(0) = 0,$$

which is bounded and of class C^∞ in $\mathbb{R}^n \setminus \{0\}$, and can be chosen small enough.

Furthermore, if the pair (A, C) is observable, then the input $u = \tilde{0}$ is universal. In this case the bilinear system becomes a linear one. Besides, under this assumption, there exists $\eta > 0$ such that each input u which verifies $\|u\| < \eta$, distinguishes the points of the bilinear system (13). Therefore the observer design problem is solved by the classical Kalman observer as the dynamical system given in (4). Hence, for $M > 0$ large enough, from Theorem 1 it follows that the feedback

$$u(\hat{x}) = -\frac{\langle B\hat{x} \mid \hat{x}\rangle}{M\|\hat{x}\|^2}, \quad u(0) = 0,$$

makes the origin of the original system (1) a locally asymptotically stable equilibrium point, provided that (13) is considered as the approximating system for (1).

In the two-dimensional case, there is at most one input u_b which is constant and makes the bilinear system unobservable. It is given by the linear equation

$$\det\begin{pmatrix} C \\ C(A + uB) \end{pmatrix} = 0.$$

From (Jerbi, 1994), for any bounded and analytic (on $\mathbb{R}^2 \setminus \{0\}$) stabilizing feedback law $u(x)$, there exists $\delta > 0$ such that

$$u(x) > u_b - \delta \text{ and } u(x) < u_b + \delta, \ \forall x \in \mathbb{R}^2,$$

where u_b is a bad input. Consider now the system (2) with $x \in \mathbb{R}^2$ and assume that there exists a bounded and analytic (on $\mathbb{R}^2 \setminus \{0\}$) stabilizing feedback law $u(x)$ such that for any bad input u_b, there exists $\epsilon > 0$ for which

$$u(x) \notin (u_b - \epsilon, u_b + \epsilon), \ \forall x \in \mathbb{R}^2$$

and $u(x)$ is homogeneous of degree zero. Then the system (7) is globally asymptotically stable. Indeed, since the feedback law $u(x_1, x_2)$ is bounded and analytic (on $\mathbb{R}^2 \setminus \{0\}$), there exists $\delta > 0$ such that

$$-\delta < u(x_1, x_2) < u_b - \epsilon, \ \forall (x_1, x_2) \in \mathbb{R}^2,$$

or

$$u_b + \epsilon < u(x_1, x_2) < \delta, \ \forall (x_1, x_2) \in \mathbb{R}^2,$$

where $x = (x_1, x_2)$. Suppose that

$$u_b + \epsilon < u(x_1, x_2) < \delta, \ \forall (x_1, x_2) \in \mathbb{R}^2,$$

(for the other case, the proof is the same). Then, under a change in the input space of the form $u \to u + \delta$, the system (2) becomes

$$\begin{cases} \dot{x} = \tilde{A}x + uBx, \\ \tilde{y} = Cx, \end{cases}$$

where $\tilde{A} = A + \delta B$.

Denoting by $\tilde{u}_b$ the only bad input of the above system and by $\tilde{u}(x_1, x_2)$ the stabilizing feedback law, we get

$$\tilde{u}_b = u_b - \delta \quad \text{and} \quad \tilde{u}(x_1, x_2) = u(x_1, x_2) - \delta.$$

However, $\tilde{u}(x_1, x_2)$ is a homogeneous feedback of degree zero, which satisfies

$$|\tilde{u}(x_1, x_2)| < |\tilde{u}_b| - \epsilon < |\tilde{u}_b|.$$

Besides, the origin of the error equation associated with the observer given in (4) is achieved with an exponential rate of convergence, and we have an estimate as the one given in (5). Hence, by using Theorem 1, we can stabilize the system (1) by the estimated feedback law.

Example 2. (Jerbi, 1994) We consider the following planar bilinear system:

$$\begin{cases} \dot{x} = Ax + uBx, \\ y = Cx, \end{cases}$$

$$x = (x_1, x_2), \quad A = \begin{pmatrix} -1 & 6 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = (1, 3).$$

This system is not stabilizable by any feedback law continuous at the origin. Moreover, since (A,C) is not observable, the null input is a bad one. Thus the feedback

$$u(\hat{x}_1, \hat{x}_2) = \frac{-3\left(14\hat{x}_1^2 + 50\hat{x}_1\hat{x}_2 + 207\hat{x}_2^2\right)}{2\hat{x}_1^2 - 2\hat{x}_1\hat{x}_2 + 15\hat{x}_2^2} \quad \text{and} \quad u(0,0) = 0$$

stabilizes the system with the estimate $\hat{x}$ given by the observer (4). ♦

3. Separation Principle: Application to Bilinear Systems

In this section we examine the class of nonlinear systems that can be modelled by the equation

$$\begin{cases} \dot{x} = f(x) + ug(x), \\ y = Cx, \end{cases} \tag{14}$$

where the output is linear. We suppose that the system (14) is observable for any inputs and the following assumptions required for stabilization by an estimated feedback:

($\mathcal{A}_3$) There exists an exponential observer for the system (14) of the form

$$\dot{\hat{x}} = f(\hat{x}) + ug(\hat{x}) - L(C\hat{x} - y),$$

where L is the gain matrix.

($\mathcal{A}_4$) There exists a feedback law making the equilibrium point of the system (14) globally asymptotically stable and such that $f(x) + u(x)g(x)$ is homogeneous.

We consider the system (14) controlled by the feedback $u(x)$ given in ($\mathcal{A}_4$) and estimated with the observer ($\mathcal{A}_3$).

Theorem 2. *Under assumptions* ($\mathcal{A}_3$) *and* ($\mathcal{A}_4$), *the system*

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}) + u(\hat{x})g(\hat{x}) - LCe, \\ \dot{e} = f(\hat{x}) - f(\hat{x} - e) + u(\hat{x})(g(\hat{x}) - g(\hat{x} - e)) - LCe, \end{cases} \tag{15}$$

where $e = \hat{x} - x$, *is globally asymptotically stable.*

Proof. By ($\mathcal{A}_3$), there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\|e(t)\| \leq \lambda_1 \|e(0)\| e^{-\lambda_2 t}, \tag{16}$$

which implies the global exponential stability of the error equation. Moreover, the system

$$\dot{x} = \tilde{F}(x) = f(x) + u(x)g(x)$$

is globally asymptotically stable. Consequently, the system (15) is locally asymptotically stable (Vidyasagar, 1980). In order to show the global asymptotic stability, by using the argument of Seibert and Suarez (1990), it suffices to prove the boundedness of any trajectories $(e(t), \hat{x}(t))$, $t \geq 0$ of the system (15). Since $e(t)$ given in (16) is bounded, it suffices to show the boundedness of the component $\hat{x}(t)$. From ($\mathcal{A}_4$) and the fact that $\tilde{F}$ is a homogeneous vector field, there exists a C^1-homogeneous Lyapunov function V such that $V(x) > 0$, $\forall\, x \neq 0$, $V(0) = 0$,

$$\dot{V}(x) = \nabla V(x)\big(\tilde{F}(x)\big) < 0, \quad \forall\, x \neq 0,$$

for which $\|\nabla V(x)\| \leq \alpha(1 + V(x))$, $\alpha > 0$, $\forall\, x \in \mathbb{R}^n$. These properties can be found in (Hahn, 1967). Therefore, the derivative of V along the trajectories of the time-varying differential equation

$$\dot{\hat{x}} = \tilde{F}(\hat{x}) - LCe(t)$$

satisfies

$$\dot{V}(\hat{x}) = \nabla V(\hat{x})\big(\tilde{F}(\hat{x})\big) - \nabla V\big(\hat{x})(LCe(t)\big).$$

Since $\nabla V(\hat{x})(\tilde{F}(\hat{x})) < 0$, we have

$$\dot{V}(\hat{x}) \leq -\nabla V(\hat{x})\big(LCe(t)\big) \leq \big\|\nabla V(\hat{x})\big(LCe(t)\big)\big\|.$$

Thus

$$\dot{V}(\hat{x}) \leq \|\nabla V(\hat{x})\|\, \|LC\|\, \|e(t)\|.$$

Therefore, from (16) and the fact that $\|\nabla V(\hat{x})\|$ is bounded by $\alpha(1 + V(x))$ as pointed out above, we obtain $\dot{V}(\hat{x}) \leq \mu e^{-\lambda_2 t}(1 + V(\hat{x}))$, where $\mu = \alpha\lambda_1 \|LC\|\, \|e(0)\|$.

Therefore $\text{Log}(1+V(\hat{x}))$ is bounded by a positive constant and hence $\hat{x}(t)$ is bounded. It follows that (14) is globally asymptotically stabilizable by $u(\hat{x})$. ■

Remark 2. Note that this class of systems includes the bilinear case. Since the bilinear system is observable for any input, the assumption $(\mathcal{A}_1)$ implies $(\mathcal{A}_3)$ (see Bornard *et al.*, 1989) and one can consider the Kalman observer as the dynamical system given in (4) with an exponential rate of convergence (5). Also, it is clear that $(\mathcal{A}_2)$ implies $(\mathcal{A}_4)$. By assumption $(\mathcal{A}_2)$, since the feedback $u(x)$ is homogeneous of degree zero, the closed-loop system $\dot{x} = Ax + u(x)Bx$ is a continuous homogeneous vector field of degree one. Therefore, according to (Hahn, 1967), there exists a homogeneous Lyapunov function V for the above differential equation. Since its partial derivatives are also homogeneous, it follows that there exists a positive constant α such that $\|\nabla V(x)\| \leq \alpha(1 + V(x))$. Thus, under assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, the system

$$\begin{cases} \dot{x} = Ax + u(x-e)Bx, \\ \dot{e} = (A + u(x-e)B - S^{-1t}CC)e \end{cases}$$

is globally asymptotically stable.

4. Conclusion

In this paper we have studied the problem of stabilizing a class of control affine systems by an estimated state feedback law. We considered nonlinear systems whose linear approximation is not stabilizable. In this case we introduced a bilinear approximation system. We showed that the system can be locally asymptotically stabilizable by a state estimate feedback law given by considering a Kalman observer associated with the bilinear system. As an illustration, we gave an example in two dimensions. Furthermore, we considered the case of bilinear systems with inputs making the system unobservable. Moreover, a separation principle with an application to bilinear systems was given.

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References

Baccioti A. and Boieri P. (1991): *A characterization of single input planar bilinear systems which admit a smooth stabilizer.* — Syst. Contr. Lett., Vol.16, pp.139–143.

Boothby W. and Marino R. (1989): *Feedback stabilization of planar nonlinear systems.* — Syst. Contr. Lett., Vol.12, pp.87–92.

Bornard G., Couenne N. and Celle F. (1989): *Regularly persistent observers for bilinear systems.* — Berlin: Springer, Contr. Inform. Sci., pp.130–140.

Chabour R. and Hammouri H. (1993): *Stabilization of planar bilinear systems using an observer configuration.* — Appl. Math. Lett., Vol.6, pp.7–10.

Chabour R., Sallet G. and Vivalda J.C. (1996): *Stabilization of nonlinear two dimensional systems: A bilinear approach.* — Math. Contr. Signal Syst., pp.224–246.

Chabour R. and Vivalda J.C. (1991): *Stabilisation des systèmes bilinéaires dans le plan par une commande non réguliere.* — Proc. European Control Conference, ECC'91, Grenoble, France, pp.485–487.

Dayawansa W.P., Martin C.F. and Knowles G. (1990): *Asymptotic stabilization of a class smooth two-dimensional systems.* — SIAM J. Contr. Optim., Vol.28, pp.1321–1349.

Gauthier J.P. and Kupka I. (1992): *A separation principle for bilinear systems with dissipative drift.* — IEEE Trans. Automat. Contr., Vol.AC–37, No.12, pp.1970–1974.

Hahn W. (1967) *Stability of Motion.* — Berlin: Springer.

Hammami M.A. (1993): *Stabilization of a class of nonlinear systems using an observer design.* — Proc. 32nd IEEE Conf. Decision and Control, San Antonio, Texas, Vol.3, pp.1954–1959.

Hammami M.A. and Jerbi H. (1994): *On the stabilization of homogeneous cubic vector fields in the plane.* — Appl. Math. Lett., Vol.7, No.4, pp.95–99.

Jerbi H. (1994): *Quelques résultats sur la stabilization des systèmes non linéaires par estimation et retour d'état.* — Ph.D. Thesis, University of Metz, France.

Massera J.L. (1956): *Contribution to stability theory.* — Annals of Mathematics, Vol.64, pp.182–206.

Seibert P. and Suarez R. (1990): *Global stabilization of nonlinear cascade systems.* — Syst. Contr. Lett., Vol.14, pp.347–352.

Vidyasagar M. (1980): *On the stabilization of nonlinear systems using state detection.* — IEEE Trans. Automat. Contr., Vol.AC–25, pp.504–509.

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