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AN ALGORITHM FOR CONSTRUCTION OF ε-VALUE FUNCTIONS FOR THE BOLZA CONTROL PROBLEM

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The problem considered is that of approximate numerical minimisation of the non-linear control problem of Bolza. Starting from the classical dynamic programming method of Bellman, an ε-value function is defined as an approximation for the value function being a solution to the Hamilton-Jacobi equation. The paper shows how an ε-value function which maintains suitable properties analogous to the original Hamilton-Jacobi value function can be constructed using a stable numerical algorithm. The paper shows the numerical closeness of the approximate minimum to the infimum of the Bolza functional.

Keywords: non-linear optimisation, Bolza problem, optimal control, Hamilton-Jacobi equation, dynamic programming, ε-value function, approximate minimum

1. Introduction

It is well-known (Bellman, 1957; Cesari, 1983; Fleming and Rishel, 1975) that in classical dynamic programming the whole family of problems with fixed initial points is considered. For one problem the initial point is fixed, but when a family of problems with different initial points is considered, the solutions to these problems are dependent on their initial points. This dependence is called the *value function.* The classical dynamic programming method describes the properties of this function, e.g. presents the necessary and sufficient conditions for the optimality of solutions. Starting from a description of the classical dynamic programming method for finding an approximate minimum (Nowakowski, 1990) of the Bolza functional (Cesari, 1983; Fleming and Rishel, 1975), the ε-value function is used to approximate the value function being a solution to the Hamilton-Jacobi equation (Cesari, 1983; Fleming and Rishel, 1975). It has been shown that the ε-value function has properties that are analogous to those of the value function itself.

The paper proposes a numerical algorithm and method for constructing an ε-value function which must satisfy the partial differential inequality of dynamic programming. According to this algorithm a formula for the ε-value function is obtained and an approximate minimum is calculated. In this study a formula is developed for

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the estimate of the difference between the calculated minimum and the infimum of the Bolza functional.

The value function approximation algorithm has been shown to be numerically stable by Jacewicz and Nowakowski (1995). An example in Section 6 illustrates the stability of the algorithm and the potential of this theory for the solution of non-linear optimal control problems based on the Hamilton-Jacobi equation. The new method of construction of the ε-value function is a major contribution in this research.

A numerical strategy for solving this problem was given by Polak (1997). In this approach, the partial differential equation of optimal control is discretised, i.e. the infinite-dimensional optimal control problem can be approximated by a sequence of finite-dimensional state-space problems. This paper provides some insight into the theory behind the derivation of the approximate minimum to the following Bolza functional and is not concerned with the calculation of the optimal control.

Problem Statement: Consider the Bolza functional

$$J(x,u) = \int_a^b L\big(t, x(t), u(t)\big)\,\mathrm{d}t + l\big(x(b)\big), \tag{1}$$

where the absolutely continuous trajectory $x: [a,b] \to \mathbb{R}^n$ and the Lebesgue measurable control function $u: [a,b] \to \mathbb{R}^m$ are subject to the non-linear controlled state-space system:

$$\dot{x}(t) = f\big(t, x(t), u(t)\big), \quad \text{a.e. in } [a,b], \tag{2}$$

$$u(t) \in U, \quad t \in [a,b], \tag{3}$$

$$x(a) = c, \tag{4}$$

$f: [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $L: [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $l: C \to \mathbb{R}$ are given functions, U is a compact subset of $\mathbb{R}^m$, C is a subset of $\mathbb{R}^n$, c is a point in the state space $\mathbb{R}^n$.

It is assumed that:

(Z) $(t,x,u) \to f(t,x,u)$ and $(t,x,u) \to L(t,x,u)$ are continuous and bounded functions in $[a,b] \times \mathbb{R}^n \times U$; they are Lipschitz functions with respect to t, x, u; $x \to l(x)$ is a Lipschitz function with respect to x.

Definition 1. A pair of functions $x(\cdot)$, $u(\cdot)$ is admissible if it satisfies (2), (3) and $t \to L(t, x(t), u(t))$ is summable; then the corresponding trajectory $t \to x(t)$ will be called *admissible*.

The value of the approximate minimum of the Bolza functional is sought for the admissible pair $x_\varepsilon(\cdot)$, $u_\varepsilon(\cdot)$, defined in $[a,b]$, $x_\varepsilon(a) = c$, and such that:

$$J(x_\varepsilon, u_\varepsilon) \le \inf\, J(x,u) + \varepsilon(b-a), \tag{5}$$

where the infimum is taken over all admissible pairs $x(\cdot)$, $u(\cdot)$ satisfying (4) and $\varepsilon > 0$ is any given number.

The value $J(x_\varepsilon, u_\varepsilon)$ is called an *approximate minimum* of the functional, and $(x_\varepsilon, u_\varepsilon)$ is an approximate solution to the problem under consideration (Nowakowski, 1990).

Remark 1. A pair $x_\varepsilon(\cdot)$, $u_\varepsilon(\cdot)$ satisfying (5) always exists, provided that $\inf J(x,u) > -\infty$.

The main problem considered in the literature is how to find an approximate solution to the Bolza problem, i.e. how to calculate a pair $x_\varepsilon(\cdot)$, $u_\varepsilon(\cdot)$ satisfying (5). The first answer for this problem belongs at least partially to Ekeland (1974, 1979), who formulated it in the form of a variational principle. This corresponds to the first variation in the ordinary extremum problem, i.e. for (1)–(4) it is simply the ε-maximum Pontryagin principle.

However, we cannot infer that a pair satisfying this principle also satisfies (5). The situation is even worse: not every pair satisfying (5) also satisfies the ε-maximum Pontryagin principle. Nowakowski (1988, 1990) describes theories based on the generalisations of the field of extremals and Hilbert's independence integral, which allow us to state, under additional geometrical assumptions, that a pair $x_\varepsilon(\cdot)$, $u_\varepsilon(\cdot)$ satisfying the ε-maximum Pontryagin principle also satisfies (5), with an additional term on the right-hand side.

The aim of this work is to describe the classical dynamic programming method for an approximate minimum of the Bolza functional and to apply this method in order to approximate the value function that is a solution to the Hamilton-Jacobi equation. Using a new method of construction of the function approximating the value function, an effective formula for the ε-value function can be obtained. Clearly, when this formula is known, an approximate minimum of the Bolza functional can be calculated for the admissible pair $(x_\varepsilon, u_\varepsilon)$ satisfying (5) and the difference $J(x_\varepsilon, u_\varepsilon) - \inf J(x,u)$ can be estimated.

The remainder of the paper is organised as follows:
The definition of the value function is given in Section 2 and its most important properties described in terms of the classical dynamic programming are presented. It is then shown how the classical dynamic programming can be used to find an approximate minimum of the considered functional, i.e. the value function can be approximated by an ε-value function satisfying the partial differential inequality of dynamic programming (12).

Section 3 provides a description of the classical dynamic programming method for finding an approximate minimum of the Bolza functional. The ε-value function is defined and it is proved that it has properties analogous to those of the value function. The most important property is the so-called verification theorem that gives sufficient conditions for ε-optimality.

The purpose of Section 4 is to describe a method of constructing the function approximating the value function $(t,x) \to S(t,x)$, defined in a compact set $T \subset [a,b] \times \mathbb{R}^n$, satisfying the Lipschitz condition and being the solution to the Hamilton-Jacobi equation. An ε-value function is constructed that satisfies the partial differential inequality of dynamic programming. Hence, the value of the approximate minimum of the Bolza functional can be calculated and it is possible to estimate the difference:

$$J(x_\varepsilon, u_\varepsilon) - \inf J(x,u) \le \varepsilon(b-a),$$

where ε is a non-negative value, close to zero. This number is calculated while constructing consecutive functions in order to approximate the value function due to the numerical algorithm proposed in Section 5. Using this algorithm, Section 6 gives an example which serves to illustrate the power and effectiveness of the proposed method of constructing the ε-value function.

2. Definition and Properties of the Value Function

As stated in Section 1, in the classical dynamic programming (Bellman, 1957; Cesari, 1983; Fleming and Rishel, 1975) a whole family of problems with fixed initial points is considered. For one problem the initial point is fixed, but when a family of problems with different initial points is considered, the solutions to these problems are dependent on their initial points. This dependence is called the *value function* (see Definition 2). The classical dynamic programming method describes the properties of this function, e.g. it presents necessary and sufficient conditions for the optimality of solutions.

Let $T \subset [a,b] \times \mathbb{R}^n$ be a set with non-empty interior, covered by graphs of admissible trajectories, i.e. for every $(t_0, x_0) \in T$ there exists an admissible pair $x(\cdot)$, $u(\cdot)$, defined in $[t_0, b]$, such that $x(t_0) = x_0$ and $(s, x(s)) \in T$ for $s \in [t_0, b]$.

Definition 2. Function $(t,x) \to S(t,x)$ defined in T is called the *value function* if

$$S(t,x) = \inf \left\{ \int_t^b L\big(s, x(s), u(s)\big)\,\mathrm{d}s + l\big(x(b)\big) \right\},$$

where the infimum is taken over all admissible trajectories $s \to x(s)$, $s \in [t,b]$, which start from $(t,x) \in T$, $x(t) = x$, and their graphs are contained in T.

If only the value function $(t,x) \to S(t,x)$ is differentiable in the open set $T_0 \subset T$, then it satisfies the partial differential equation of dynamic programming known as the Hamilton-Jacobi equation (Cesari, 1983; Fleming and Rishel, 1975),

$$S_t(t,x) + H\big(t, x, S_x(t,x)\big) = 0, \quad (t,x) \in T_0 \tag{6}$$

with the boundary condition

$$S(b,x) = l(x), \quad (b,x) \in T_0, \tag{7}$$

where the Hamiltonian is given by

$$H(t,x,y) = yf\big(t,x,u(t,x)\big) + L\big(t,x,u(t,x)\big),$$

and $(t,x) \to u(t,x)$ is an optimal feedback control.

One can notice that for the considered Bolza problem (1)–(4) the above Hamilton-Jacobi equation can be re-written in the following way:

$$\frac{\partial}{\partial t}S(t,x) + \min_{u\in U}\left\{\frac{\partial}{\partial x}S(t,x)f(t,x,u) + L(t,x,u)\right\} = 0, \quad (t,x)\in T_0. \tag{8}$$

One of the most important properties of the value function is stated in Theorem 1:

Theorem 1. *If the functions* $(t,x,u)\to f(t,x,u)$, $(t,x,u)\to L(t,x,u)$ *and* $x\to l(x)$ *satisfy assumptions* **(Z)** *from the Bolza problem (1)–(4), then the value function* $(t,x)\to S(t,x)$ *satisfies a Lipschitz condition and is the solution to the Hamilton-Jacobi equation:*

$$\frac{\partial}{\partial t}S(t,x) + \min_{u\in U}\left\{\frac{\partial}{\partial x}S(t,x)f(t,x,u) + L(t,x,u)\right\} = 0 \text{ for a.e. } (t,x)\in T$$

with the boundary condition $S(b,x)=l(x)$, $(b,x)\in T$.

Proof. See (Fleming and Rishel, 1975, Ch. IV, Th. 4.2). ■

A simple procedure for finding an approximate minimum of the Bolza functional from the problem (1)–(4) will now be described. The classical dynamic programming method for the approximate minimum is very useful for this purpose.

According to Fleming and Rishel (1975) and Cesari (1983), in the classical dynamic programming the sufficient condition for optimality of the solution to the considered problem is expressed as the solution to the Hamilton-Jacobi equation so that following Theorem 2 holds.

Theorem 2. *Let* $(t,x)\to G(t,x)$ *be a solution of the class* $C^1(T)$ *to the Hamilton-Jacobi equation*

$$G_t(t,x) + H\big(t,x,G_x(t,x)\big) = 0, \quad (t,x)\in T_0$$

with the boundary condition

$$G(b,x) = l(x), \quad (b,x)\in T_0,$$

where $T_0\subset T$ *is an open set, the Hamiltonian is given by the formula* $H(t,x,y) = yf(t,x,u(t,x)) + L(t,x,u(t,x))$, *and* $(t,x)\to u(t,x)$ *is an optimal feedback control. If* $x=x(t)$ *and a pair* $x(\cdot)$, $u(\cdot)$, *defined in* $[a,b]$, $x(a)=c$, *is admissible and such that*

$$\frac{\partial}{\partial t}G\big(t,x(t)\big) + \frac{\partial}{\partial x}G\big(t,x(t)\big)f\big(t,x(t),u(t)\big) + L\big(t,x(t),u(t)\big) = 0,$$

then the pair $x(\cdot)$, $u(\cdot)$ *is optimal, and also* $G(t,x)=S(t,x)$, $(t,x)\in T_0$, *where* $S(\cdot,\cdot)$ *is the value function.*

Proof. see (Fleming and Rishel, 1975, Ch. IV, Th. 4.4). ■

It can be seen that some regularity of the function $(t,x)\to G(t,x)$, being the solution to the Hamilton-Jacobi equation, is required, i.e. it must be at least a Lipschitz function (see Th. 1).

There can be given some examples of control problems (Cesari, 1983) whose solutions cannot be characterised in that way. The main reason behind this is an 'insufficient regularity' of the function $(t, x, y) \to H(t, x, y)$.

For that kind of problems the following equation is considered:

$$G_t(t, x) + H\big(t, x, G_x(t, x), \varepsilon\big) = 0, \quad (t, x) \in T_0 \tag{9}$$

with the boundary condition

$$G(b, x) = l(x, \varepsilon), \quad (b, x) \in T_0, \tag{10}$$

where $(t, x, y, \varepsilon) \to H(t, x, y, \varepsilon)$ and $(x, \varepsilon) \to l(x, \varepsilon)$ are regular (smooth) enough, and the problem (9)–(10) can be solved analytically or numerically.

Moreover, it is assumed that $H(t, x, y, \varepsilon) \to H(t, x, y)$ uniformly with respect to t, x, y, as $\varepsilon \to 0$ and that $l(x, \varepsilon) \to l(x)$ uniformly with respect to x, as $\varepsilon \to 0$, i.e. it is assumed that

$$\begin{aligned} \big|H(t, x, y, \varepsilon) - H(t, x, y)\big| \le \varepsilon \ \text{ for } \ (t, x) \in T_0, \quad y \in \mathbb{R}^n, \\ \big|l(x, \varepsilon) - l(x)\big| \le \varepsilon \ \text{ for } \ (b, x) \in T_0. \end{aligned} \tag{11}$$

Hence, one can infer that if the function $(t, x) \to G(t, x)$ is a solution to (9)–(10), it satisfies the inequality

$$-\varepsilon \le G_t(t, x) + H\big(t, x, G_x(t, x)\big) \le \varepsilon, \quad (t, x) \in T_0.$$

A new function, shifted to the left-hand side, is defined by the formula

$$G_\varepsilon(t, x) := G(t, x) + \varepsilon(b - t), \quad (t, x) \in T_0.$$

This satisfies the inequality

$$-2\varepsilon \le G_{\varepsilon t}(t, x) + H\big(t, x, G_{\varepsilon x}(t, x)\big) \le 0, \quad (t, x) \in T_0. \tag{12}$$

The function $(t, x) \to G_\varepsilon(t, x)$ satisfies the properties of an ε-value function in T_0, i.e. it satisfies

$$S(t, x) \le G_\varepsilon(t, x) \le S(t, x) + 3\varepsilon, \quad (t, x) \in T_0,$$

$$l(x) - \varepsilon \le G_\varepsilon(b, x) \le l(x) + \varepsilon, \quad (b, x) \in T_0,$$

where $(t, x) \to S(t, x)$ is the value function, and $\varepsilon \ge 0$ is arbitrary and fixed for further considerations.

If an admissible pair $x_\varepsilon(\cdot)$, $u_\varepsilon(\cdot)$ can be found as defined in $[a, b]$, $x_\varepsilon(a) = c$, and satisfying the inequality:

$$G_\varepsilon(a, c) \ge \int_a^b L\big(t, x_\varepsilon(t), u_\varepsilon(t)\big)\, \mathrm{d}t + l\big(x_\varepsilon(b)\big),$$

then $t \to x_\varepsilon(t)$ is called an *ε-optimal trajectory associated with* $(t, x) \to G_\varepsilon(t, x)$.

The above approximation of the value function by the ε-value function is numerically stable in the following sense (Jacewicz and Nowakowski, 1995): for every $\varepsilon > 0$ there exists an $M_s > 0$ such that for all $H(t, x, y, \varepsilon_1)$ and $H(t, x, y, \varepsilon_2)$ satisfying (11) with ε on the right-hand side of these inequalities and corresponding to them ε_1-optimal and ε_2-optimal trajectories $x_{\varepsilon_1}(\cdot)$, $x_{\varepsilon_2}(\cdot)$ and controls $u_{\varepsilon_1}(\cdot)$, $u_{\varepsilon_2}(\cdot)$, the following inequality is satisfied:

$$\left|J(x_{\varepsilon_1}, u_{\varepsilon_1}) - J(x_{\varepsilon_2}, u_{\varepsilon_2}\right| \le M_s\varepsilon.$$

This means that the difference between the values of the Bolza functional calculated for two different approximate solutions $(x_{\varepsilon_1}, u_{\varepsilon_1})$ and $(x_{\varepsilon_2}, u_{\varepsilon_2})$ is limited by the positive number $M_s\varepsilon$.

3. Classical Dynamic Programming Approach

In this section the classical dynamic programming method for finding approximate minimum for the Bolza problem (1)–(4) is described. The ε-value function is defined and its most important properties, which are necessary and sufficient conditions for ε-optimality, are proved.

Let $T \subset [a, b] \times \mathbb{R}^n$ be a set with non-empty interior covered by graphs of admissible trajectories (cf. Section 2).

Definition 3. The function $(t, x) \to S_\varepsilon(t, x)$, defined in the set T, is called the *ε-value function* if

$$\begin{aligned} &S(t, x) \le S_\varepsilon(t, x) \le S(t, x) + \varepsilon(b - a), \quad (t, x) \in T, \\ &S_\varepsilon(b, x) = l(x), \quad (b, x) \in T, \end{aligned} \tag{13}$$

where $(t, x) \to S(t, x)$ is the value function, $x \to l(x)$ is the function described in the Bolza problem (1)–(4) and satisfying the assumptions (Z), $\varepsilon > 0$ is arbitrary and fixed for further consideration.

It is clear that the ε-value functions for fixed $\varepsilon > 0$ are non-unique. However, for $\varepsilon = 0$, the formulae (13) define a unique value function.

It is assumed here that the function $(t, x) \to S_\varepsilon(t, x)$ is finite in T, i.e. the points from T for which the function has values $\pm\infty$ should be excluded from this set.

Definition 4. An admissible trajectory $s \to x_\varepsilon(s)$, $s \in [t, b]$, $x_\varepsilon(t) = x$ is called *ε-optimal* if

$$S_\varepsilon(t, x) \ge \int_t^b L\big(s, x_\varepsilon(s), u_\varepsilon(s)\big)\, \mathrm{d}s + l\big(x_\varepsilon(b)\big)$$

for a given, fixed $S_\varepsilon(t, x)$, $(t, x) \in T$.

Proposition 1 shows that the ε-value function has properties analogous to those of the value function.

Proposition 1. *(i) Let* $x(\cdot)$*,* $u(\cdot)$ *be an admissible pair defined in* $[t_0, b]$*, for which the trajectory* $t \to x(t)$ *lies in* T *and starts at the point* $(t_0, x_0) \in T$*. Then the function* $(t, x) \to R(t, x)$ *defined by*

$$R(t,x) = S_\varepsilon(t,x) - \int_t^b L\big(s, x(s), u(s)\big)\,\mathrm{d}s,$$

evaluated along an admissible trajectory $t \to x(t)$*, satisfies the inequalities*

$$R\big(t_1, x(t_1)\big) \le R\big(t_2, x(t_2)\big) + \varepsilon(b-a), \quad t_0 \le t_1 \le t_2 \le b.$$

(ii) For the ε*-optimal trajectory* $t \to x_\varepsilon^0(t)$*,* $t_0 \le t \le b$*, the following inequalities are valid:*

$$R(t_0, x_0) \ge l\big(x_\varepsilon^0(b)\big)$$

and $S_\varepsilon(t, x_\varepsilon^0(t)) \le \int_t^b L(s, x_\varepsilon^0(s), u_\varepsilon^0(s))\,\mathrm{d}s + l(x_\varepsilon^0(b)) + \varepsilon(b-a)$*,* $t_0 < t \le b$*.*

Proof. Since part (ii) is a direct consequence of part (i) and the definition of the ε-optimal trajectory, only part (i) must be proved.

Let $x(\cdot)$, $u(\cdot)$ be an admissible pair defined in $[t_0, b]$, $x(t_0) = x_0$. Then the pair $x(\cdot)$, $u(\cdot)$ restricted to $[t_1, b]$ is also admissible. Let $x_2(\cdot)$, $u_2(\cdot)$ be an admissible pair defined in $[t_2, b]$, $x_2(t_2) = x(t_2)$.

Let the function $t \to u_1(t)$ be defined in $[t_1, b]$ as follows:

$$u_1(t) = \begin{cases} u(t) & \text{for} \quad t_1 \le t \le t_2, \\ u_2(t) & \text{for} \quad t_2 \le t \le b. \end{cases}$$

Then the pair $x_1(\cdot)$, $u_1(\cdot)$ defined in $[t_1, b]$ with $x_1(\cdot)$ corresponding to $u_1(\cdot)$ due to (2) is admissible.

Hence

$$S_\varepsilon\big(t_1, x(t_1)\big) \le \int_{t_1}^b L\big(s, x_1(s), u_1(s)\big)\,\mathrm{d}s + l\big(x_1(b)\big) + \varepsilon(b-a).$$

As $x_2(\cdot)$ defined in $[t_2, b]$ is an arbitrary admissible trajectory on $[t_2, b]$, the following inequality holds:

$$S_\varepsilon\big(t_1, x(t_1)\big) - \int_{t_1}^b L\big(s, x(s), u(s)\big)\,\mathrm{d}s$$

$$\le \inf\left\{\int_{t_2}^b L\big(s, \bar{x}(s), \bar{u}(s)\big)\,\mathrm{d}s + l\big(\bar{x}(b)\big)\right\} - \int_{t_2}^b L\big(s, x(s), u(s)\big)\,\mathrm{d}s + \varepsilon(b-a),$$

where the infimum is taken over all admissible trajectories $\bar{x}(\cdot)$ defined in $[t_2, b]$, which start at $(t_2, x(t_2))$. The result is that (i) above is satisfied. ■

It is easy to see that these properties are necessary conditions for ε-optimality. Proposition 2 asserts that these properties also satisfy sufficiency.

Proposition 2. *Let $(t,x) \to G(t,x)$ be any real-valued function defined in T and such that $G(b,x) = l(x)$. Let $(t_0, x_0) \in T$ be a given initial condition, and suppose that for each admissible trajectory $x(\cdot)$ defined in $[t_0, b]$, $x(t_0) = x_0$, $G(\cdot, x(\cdot))$ is finite in $[t_0, b]$ and*

$$G\big(t_1, x(t_1)\big) - \int_{t_1}^{b} L\big(t, x(t), u(t)\big)\,\mathrm{d}t \le G\big(t_2, x(t_2)\big) - \int_{t_2}^{b} L\big(t, x(t), u(t)\big)\,\mathrm{d}t + \varepsilon(b-a) \quad (14)$$

for all $t_0 \le t_1 \le t_2 \le b$. If an admissible trajectory $x_\varepsilon^0(\cdot)$ defined in $[t_0, b]$, $x_\varepsilon^0(t_0) = x_0$, is such that

$$G(t_0, x_0) \ge \int_{t_0}^{b} L\big(t, x_\varepsilon^0(t), u_\varepsilon^0(t)\big)\,\mathrm{d}t + l\big(x_\varepsilon^0(b)\big) \quad (15)$$

and

$$G\big(t_0, x_\varepsilon^0(t)\big) \le \int_{t}^{b} L\big(s, x_\varepsilon^0(s), u_\varepsilon^0(s)\big)\,\mathrm{d}s + l\big(x_\varepsilon^0(b)\big) + \varepsilon(b-a), \quad (16)$$

for $t_0 < t \le b$, then $x_\varepsilon^0(\cdot)$ is the ε-optimal trajectory for $S_\varepsilon(t_0, x_0) = G(t_0, x_0)$.

Proof. Let $x(\cdot)$ be an admissible trajectory defined in $[t_0, b]$, $x(t_0) = x_0$. Then

$$G(t_0, x_0) - \int_{t_0}^{b} L\big(t, x(t), u(t)\big)\,\mathrm{d}t \le l\big(x(b)\big) + \varepsilon(b-a).$$

Hence

$$G(t_0, x_0) \le \inf\left\{\int_{t_0}^{b} L\big(t, \bar{x}(t), \bar{u}(t)\big)\,\mathrm{d}t + l\big(\bar{x}(b)\big)\right\} + \varepsilon(b-a),$$

where the infimum is taken over all admissible trajectories $\bar{x}(\cdot)$ defined in $[t_0, b]$, $\bar{x}(t_0) = x_0$. For $x_\varepsilon^0(\cdot)$ the following inequality holds:

$$G(t_0, x_0) \ge \int_{t_0}^{b} L\big(t, x_\varepsilon^0(t), u_\varepsilon^0(t)\big)\,\mathrm{d}t + l\big(x_\varepsilon^0(b)\big).$$

Consequently, $x_\varepsilon^0(\cdot)$ is the ε-optimal trajectory for $S_\varepsilon(t_0, x_0) = G(t_0, x_0)$. ■

A more important property of the ε-value function, from a practical standpoint, is the so-called *verification theorem*.

Proposition 3. *Let $T \subset [a,b] \times \mathbb{R}^n$ be an open set and the function $(t,x) \to G(t,x)$, defined in T, be a $C^1(T)$ solution to the following inequality:*

$$-\varepsilon \le G_t(t,x) + \inf\big\{G_x(t,x) f(t,x,u) + L(t,x,u) \colon\ u \in U\big\} \le 0, \quad (17)$$

which satisfies the boundary condition $G(b,x)=l(x),\ (b,x)\in T$.

If $x_\varepsilon(\cdot),\ u_\varepsilon(\cdot)$ *is an admissible pair defined in* $[a,b],\ x_\varepsilon(a)=c$, *such that*

$$-\varepsilon \le G_t\big(t,x_\varepsilon(t)\big)+G_x\big(t,x_\varepsilon(t)\big)f\big(t,x_\varepsilon(t),u_\varepsilon(t)\big)+L\big(t,x_\varepsilon(t),u_\varepsilon(t)\big)\le 0, \quad (18)$$

then $x^0_\varepsilon(\cdot)$ *is the* ε*-optimal trajectory for the* ε*-value function* $S_\varepsilon(t,x)=G(t,x)$, $(t,x)\in T$.

Proof. The dynamic programming partial differential inequality (17) for an admissible pair $x(\cdot),\ u(\cdot)$ implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}G\big(t,x(t)\big) &= G_t\big(t,x(t)\big)+G_x\big(t,x(t)\big)f\big(t,x(t),u(t)\big)\\ &\ge -\varepsilon - L\big(t,x(t),u(t)\big).\end{aligned}$$

Hence, we infer that $(t,x)\to G(t,x)$ satisfies (14).

Similarly, from (18), the inequalities (15) and (16) can be obtained. Thus, by Proposition 2, $x_\varepsilon(\cdot)$ is the ε-optimal trajectory for the ε-value function $S_\varepsilon(t,x)=G(t,x),\ (t,x)\in T$. ■

Note that the inequality (17) of dynamic programming has an important practical meaning as the function $(t,x)\to G(t,x),\ (t,x)\in T$ satisfying this inequality is the ε-value function $S_\varepsilon(t,x)=G(t,x)$ only if it is regular enough, i.e. it is at least a function of class $C^1(T)$. This property is then used in Section 4 while constructing a function approximating the value function.

4. Approximation of the Value Function

This section describes a method of constructing an ε-value function $(t,x)\to S_\varepsilon(t,x)$ which approximates the value function $(t,x)\to S(t,x)$ defined in the compact set $T\subset[a,b]\times\mathbb{R}^n$, satisfying the Lipschitz condition and being the solution to the Hamilton-Jacobi equation for the considered Bolza problem (1)–(4) with the assumptions (Z). The proposed way of constructing the ε-value function of class $C^1(T)$ ensures that the dynamic programming partial differential inequality (17) from Proposition 3 is satisfied. Thus the value of the approximate minimum for this problem can be calculated and the difference from the exact solution, i.e. from the infimum of the Bolza functional, can be estimated. An arbitrary function $(t,x)\to w(t,x)$ of class $C^1(T)$ can be chosen and in a few steps of the construction it is modified until the resulting function $(t,x)\to w^{\beta,i}_{3,j}(t,x)$ satisfies the dynamic programming inequality (17), i.e. it is the ε-value function.

Gonzalez proved (1976) that there exists a maximum solution to the Hamilton-Jacobi equation which satisfies the Lipschitz condition and which is also the value function. The results presented here are: the construction of the ε-value function approximating the value function and giving an answer to the question of how much the approximate minimum differs from the infimum of the Bolza functional. To get a better result, i.e. to reduce the error in the estimate of the minimum, this procedure should be repeated using the numerical algorithm proposed in Section 5.

Let $T \subset [a,b] \times \mathbb{R}^n$ be a compact set with a non-empty interior covered by the graphs of admissible trajectories.

The construction of the ε-value function starts with the choice of some arbitrary function $(t,x) \to w(t,x)$ of class $C^1(T)$. The lack of any other limitations (restrictions) connected with the choice of the start function is one of the advantages of the method described in this paper.

The dynamic programming inequality (17) is an estimation from both sides of the values of the left-hand side of the Hamilton-Jacobi equation. Hence, to make the notation shorter and simpler, a function $(t,x) \to F(t,x)$ is defined in the set T by the following formula:

$$F(t,x) := \frac{\partial}{\partial t} w(t,x) + \min_{u \in U} \left\{ \frac{\partial w}{\partial x}(t,x) f(t,x,u) + L(t,x,u) \right\}, \tag{19}$$

where the functions $(t,x,u) \to f(t,x,u)$ and $(t,x,u) \to L(t,x,u)$ satisfy the assumptions (Z) from Section 1, $(t,x) \to w(t,x)$ is an arbitrarily chosen function of class $C^1(T)$, and the infimum is replaced with the minimum taken over all controls u from the compact set U.

The function $(t,x) \to F(t,x)$ defined so is considered in a compact set T. As the function is continuous, due to the Weierstrass theorem, it reaches its infimum and supremum (they are finite) in T. Denoting these extrema by κ_d and κ_g, respectively, the values of the function $F(\cdot,\cdot)$ can be estimated as follows:

$$\kappa_d \le F(t,x) \le \kappa_g \quad \text{for all } (t,x) \in T.$$

The function $(t,x) \to F(t,x)$, defined by (19) using $(t,x) \to w(t,x)$, can have values of different signs, although it should satisfy the dynamic programming inequality (17), i.e. it should have non-positive values, close to zero. Accordingly, a new function $(t,x) \to w_{1,j}(t,x)$ must be constructed, and additionally, a new function $(t,x) \to F_{1,j}(t,x)$ must be defined in a way analogous to the definition of the function $(t,x) \to F(t,x)$. The function $(t,x) \to F_{1,j}(t,x)$ should also satisfy (17).

Now a new function $(t,x) \to w_{1,j}(t,x)$ must be defined in non-intersecting subsets P_j of the compact set T, which cover T completely. First, the domain of this function must be constructed.

This is the reason why the interval $[\kappa_d, \kappa_g] \subset \mathbb{R}$, as the image of the function $(t,x) \to F(t,x)$, needs to be divided into $r+k$ subintervals using the following points:

(a) if infimum κ_d and supremum κ_g are of different signs, then

$$\kappa_d = y_{-r} < y_{-r+1} < \dots < y_{-1} < y_0 < y_1 < \dots < y_k = \kappa_g,$$

where $y_0 = 0$ and $r,\ k \in \mathbb{N}$;

(b) if infimum κ_d is non-negative, then

$$\kappa_d = y_0 < y_1 < \dots < y_k = \kappa_g,$$

where $k \in \mathbb{N}$;

(c) if supremum κ_g is non-positive, then

$$\kappa_d = y_{-r} < y_{-r+1} < \cdots < y_{-1} < y_0 = \kappa_g,$$

where $r \in \mathbb{N}$.

Later in the paper we consider the case when the infimum and supremum (κ_d and κ_g) of the function $(t,x) \to F(t,x)$ are of different signs. The remaining two cases are simpler.

Applying the approach analogous to that used for defining the Lebesgue integral, some subsets of the set T can be defined. In these subsets the function $(t,x) \to F(t,x)$ takes the values from the subintervals determined by each pair of adjacent points from the division of the interval $[\kappa_d, \kappa_g] \subset \mathbb{R}$ given above:

$$P_j := \big\{(t,x) \in T : \ y_j \le F(t,x) < y_{j+1}\big\}, \quad j \in \{-r, \ldots, -1\},$$

$$P_j := \big\{(t,x) \in T : \ y_{j-1} \le F(t,x) \le y_j\big\}, \quad j = 1,$$

$$P_j := \big\{(t,x) \in T : \ y_{j-1} < F(t,x) \le y_j\big\}, \quad j \in \{2, \ldots, k\}.$$

Clearly, the pairs of defined subsets $P_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ of the set T are non-intersecting, i.e. for all $i, j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\},\ i \ne j,\ P_i \cap P_j = \emptyset$, and all these subsets cover the whole set T, i.e. $\bigcup_{j=-r}^{k} P_j = T$.

Let $\bar{P}_j$ denote the *closure* of the set $P_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$.

In such subsets $P_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ of T defined above a new function $(t,x) \to w_{1,j}(t,x)$ will now be defined.

The function $(t,x) \to F(t,x)$ can have values of different signs on T, so the following two cases must be considered:

Case 1. Let $F(t,x) \ge 0,\ (t,x) \in P_j,\ j \in \{1, \ldots, k\}$, i.e.

$$y_{j-1} \le F(t,x) \le y_j, \quad (t,x) \in P_j, \quad j \in \{1, \ldots, k\}. \tag{20}$$

A new (transformed) function in subsets $P_j,\ j \in \{1, \ldots, k\}$ can be defined as

$$w_{1,j}(t,x) := w(t,x) + \gamma_j y_j (b - t), \tag{21}$$

where the function $(t,x) \to w(t,x)$ was chosen earlier and used in the definition of the function $(t,x) \to F(t,x)$ satisfying (20). The positive numbers $y_j,\ j \in \{1, \ldots, k\}$ are the points from the division of the interval $[\kappa_d, \kappa_g] \subset \mathbb{R}$, and the numbers $1 < \gamma_j < 2$, $j \in \{1, \ldots, k\}$ were chosen to estimate the non-negative values, close to zero, of a new function $(t,x) \to F_{1,j}(t,x)$ defined in the subsets $P_j,\ j \in \{1, \ldots, k\}$ by

$$F_{1,j}(t,x) := \frac{\partial}{\partial t} w_{1,j}(t,x) + \min_{u \in U} \left\{ \frac{\partial w_{1,j}}{\partial x}(t,x) f(t,x,u) + L(t,x,u) \right\}. \tag{22}$$

Using the definition of $w_{1,j}(\cdot,\cdot)$ and substituting $w_{1,j}(\cdot,\cdot)$ in the formula (22) for $F_{1,j}(\cdot,\cdot)$, we obtain a relation between functions $F(\cdot,\cdot)$ and $F_{1,j}(\cdot,\cdot)$, where both the functions are defined in the subsets P_j, $j \in \{1,\ldots,k\}$, i.e.

$$\begin{aligned} F_{1,j}(t,x) &= \frac{\partial}{\partial t} w(t,x) - \gamma_j y_j + \min_{u\in U}\left\{\frac{\partial w}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\} \\ &= F(t,x) - \gamma_j y_j. \end{aligned}$$

From this result and using (20), the following inequality can be derived:

$$-\mu_j \le F_{1,j}(t,x) \le -\eta_j, \quad (t,x)\in P_j, \quad j\in\{1,\ldots,k\},$$

where $-\mu_j = y_{j-1} - \gamma_j y_j$, $-\eta_j = y_j - \gamma_j y_j$, $1 < \gamma_j < 2$. This estimation improves as γ_j tends to 1.

It can easily be seen that the function $w_{1,j}(\cdot,\cdot)$, defined and continuous in P_j, $j \in \{1,\ldots,k\}$, can be extended to the closure $\bar{P}_j \subset T$, $j \in \{1,\ldots,k\}$, using

$$w_{1,j}(t,x) := w(t,x) + \gamma_j y_j (b-t),$$

for all $(t,x) \in \bar{P}_j \backslash P_j$, $j \in \{1,\ldots,k\}$.

It is also clear that the extended function $w_{1,j}(\cdot,\cdot)$ and consequently the function $F_{1,j}(\cdot,\cdot)$ will also be continuous in $\bar{P}_j \subset T$, $j \in \{1,\ldots,k\}$ as the function $w(\cdot,\cdot)$ is continuous in the set T, and the functions $f(\cdot,\cdot,\cdot)$ and $L(\cdot,\cdot,\cdot)$ are continuous in the set $T \times U$.

Case 2. Let $F(t,x) < 0$, $(t,x) \in P_j$, $j \in \{-r,\ldots,-1\}$, which implies

$$y_j \le F(t,x) \le y_{j+1}, \quad (t,x)\in P_j, \quad j\in\{-r,\ldots,-1\}. \tag{23}$$

In much the same way as in the previous case (21), a new function in the subsets P_j, $j \in \{-r,\ldots,-1\}$ can be defined as follows:

$$w_{1,j}(t,x) := w(t,x) + \delta_j y_{j+1}(b-t), \tag{24}$$

where the function $(t,x) \to w(t,x)$ was chosen earlier and can be seen in the definition of the function $(t,x) \to F(t,x)$ satisfying (23). Negative numbers y_j, $j \in \{-r,\ldots,-1\}$ are the points from the division of the interval $[\kappa_d,\kappa_g] \subset \mathbb{R}$. The numbers $0 < \delta_j < 1$ were chosen to estimate the non-negative values, close to zero, of a new function $(t,x) \to F_{1,j}(t,x)$. This new function is defined in subsets P_j, $j \in \{-r,\ldots,-1\}$ as follows:

$$F_{1,j}(t,x) := \frac{\partial}{\partial t} w_{1,j}(t,x) + \min_{u\in U}\left\{\frac{\partial w_{1,j}}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\}. \tag{25}$$

Using the definition of $w_{1,j}(\cdot,\cdot)$ and substituting $w_{1,j}(\cdot,\cdot)$ in the formula (25) for $F_{1,j}(\cdot,\cdot)$, we obtain a relation between the functions $F(\cdot,\cdot)$ and $F_{1,j}(\cdot,\cdot)$ defined in the subsets P_j, $j \in \{-r,\ldots,-1\}$, i.e.

$$\begin{aligned} F_{1,j}(t,x) &= \frac{\partial}{\partial t} w(t,x) - \delta_j y_{j+1} + \min_{u\in U}\left\{\frac{\partial w}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\} \\ &= F(t,x) - \delta_j y_{j+1}. \end{aligned}$$

From this result and using (23), the following inequality can be derived:

$$-\mu_j \le F_{1,j}(t,x) \le -\eta_j, \quad (t,x) \in P_j, \quad j \in \{-r,\dots,-1\},$$

where $-\mu_j = y_j - \delta_j y_{j+1}$, $-\eta_j = y_{j+1} - \delta_j y_{j+1}$, $0 < \delta_j < 1$. This estimation improves as δ_j tends to 1.

It can easily be seen that the function $w_{1,j}(\cdot,\cdot)$, defined and continuous in P_j, $j \in \{-r,\dots,-1\}$ can be extended to the closure $\bar{P}_j \subset T$, $j \in \{-r,\dots,-1\}$ by means of the relationship

$$w_{1,j}(t,x) := w(t,x) + \delta_j y_{j+1}(b-t),$$

for all $(t,x) \in \bar{P}_j \backslash P_j$, $j \in \{-r,\dots,-1\}$.

It is clear that the extended function $w_{1,j}(\cdot,\cdot)$ and consequently the function $F_{1,j}(\cdot,\cdot)$ will also be continuous in $\bar{P}_j \subset T$, $j \in \{-r,\dots,-1\}$ as the function $w(\cdot,\cdot)$ is continuous in the set T, and the functions $f(\cdot,\cdot,\cdot)$ and $L(\cdot,\cdot,\cdot)$ are continuous in the set $T \times U$.

The result of the first step of the construction method for the function approximating the value function is the construction of the function $w_{1,j}(\cdot,\cdot)$ in all subsets P_j, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, and then defining the function $F_{1,j}(\cdot,\cdot)$ in the same domain, which has only non-positive values, close to zero.

This objective is achieved as the values of the function $F_{1,j}(\cdot,\cdot)$ can be estimated as follows:

$$-\mu_j \le F_{1,j}(t,x) \le -\eta_j, \quad (t,x) \in P_j, \tag{26}$$

where

$$\mu_j = \begin{cases} -y_{j-1} + \gamma_j y_j & \text{for} \quad j \in \{1,\dots,k\}, \\ -y_j + \delta_j y_{j+1} & \text{for} \quad j \in \{-r,\dots,-1\}, \end{cases}$$

and

$$\eta_j = \begin{cases} -y_j + \gamma_j y_j & \text{for} \quad j \in \{1,\dots,k\}, \\ -y_{j+1} + \delta_j y_{j+1} & \text{for} \quad j \in \{-r,\dots,-1\}, \end{cases}$$

while $1 < \gamma_j < 2$ for $j \in \{1,\dots,k\}$, $0 < \delta_j < 1$ for $j \in \{-r,\dots,-1\}$.

If all the numbers γ_j and δ_j are close to 1, and the number of points y_j, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ of the division of the interval $[\kappa_d, \kappa_g] \subset \mathbb{R}$ is very large, i.e. r and k are very large natural numbers, then the numbers $-\mu_j$ and $-\eta_j$ are non-positive and very close to zero.

As the estimation of the values of the function $F_{1,j}(\cdot,\cdot)$ given by (26) is valid, the function $w_{1,j}(\cdot,\cdot)$ defined by (21) and (24) and being used in formulae (22) and (25) for the function $F_{1,j}(\cdot,\cdot)$ would satisfy the dynamic programming inequality (17), i.e. it would approximate the value function for the Bolza problem (1)–(4), if only it had been regular (smooth) enough, i.e. at least of class $C^1(T)$. This would have been the last step of this construction.

However, the function $w_{1,j}(\cdot,\cdot)$ has just been defined in the subsets $P_j \subset T$, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$, and although it is continuous in these subsets and even in their closures, it is only piecewise continuous in the set T and may not be sufficiently regular. To ensure sufficient regularity, the function $w_{1,j}(\cdot,\cdot)$ must be convolved with a function of class $C_0^\infty(\mathbb{R}^{n+1})$ having a compact support.

Thus a new function $(t,x) \to w_{2,j}^{\beta,i}(t,x)$ must be constructed for an arbitrary and fixed $\beta > 0$, $i \in \mathbb{N}\backslash\{0,1,2,3\}$, defined in the subsets P_j, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$ of the set T, by using the convolution of the function $w_{1,j}(\cdot,\cdot)$ with an infinitely smooth function, having a compact support and shifting this convolution to the left as follows:

$$w_{2,j}^{\beta,i}(t,x) := (w_{1,j} * \rho_\beta)(t,x) - \frac{i-2}{i}\eta_j(b-t). \tag{27}$$

The function $(t,x) \to w_{1,j}(t,x)$, constructed earlier (see (21) and (24)), was used in the definition of the function $(t,x) \to F_{1,j}(t,x)$, given in (22) and (25), having non-positive values, close to zero (see (26)). The function $F_{1,j}(\cdot,\cdot)$ is bounded from above by the numbers η_j, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$ in the subsets P_j, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$. In (27) we have the number $i \in \mathbb{N}\backslash\{0,1,2,3\}$, and as $i \to +\infty$, we get $(i-2)/i \to 1$. The function $\rho_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is of class $C_0^\infty(\mathbb{R}^{n+1})$, has a compact support and satisfies: $\int_{\mathbb{R}^{n+1}} \rho_1(t,x)\,\mathrm{d}t\,\mathrm{d}x = 1$; $\rho_\beta(t,x) := (1/\beta^{n+1})\rho_1(t/\beta, x/\beta) \in C_0^\infty(\mathbb{R}^{n+1})$; supp $\rho_1 \subset B_1(\mathbb{R}^{n+1})$, where $B_1(\mathbb{R}^n)$ is the ball in $\mathbb{R}^n$ with centre 0 and radius l.

Following the previously described construction steps, a new function $(t,x) \to F_{2,j}^{\beta,i}(t,x)$, $\beta > 0$, $i \in \mathbb{N}\backslash\{0,1,2,3\}$, must be defined in the subsets P_j, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$ of the set T as follows:

$$F_{2,j}^{\beta,i}(t,x) := \frac{\partial}{\partial t} w_{2,j}^{\beta,i}(t,x) + \min_{u \in U}\left\{\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) f(t,x,u) + L(t,x,u)\right\}, \tag{28}$$

where the function $(t,x) \to w_{2,j}^{\beta,i}(t,x)$ is defined by (27).

It can be clearly seen that the function $w_{2,j}^{\beta,i}(\cdot,\cdot)$, defined and continuous in the subsets P_j, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$, can be extended to their closures $\bar{P}_j \subset T$, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$ by

$$w_{2,j}^{\beta,i}(t,x) := \big(w_{1,j} * \rho_\beta\big)(t,x) - \frac{i-2}{i}\eta_j(b-t),$$

for all $(t,x) \in \bar{P}_j \backslash P_j$, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$.

Clearly, this extended function $w_{2,j}^{\beta,i}(\cdot,\cdot)$ and consequently the function $F_{2,j}^{\beta,i}(\cdot,\cdot)$ will also be continuous in $\bar{P}_j \subset T$, $j \in \{-r,\ldots,-1\} \cup \{1,\ldots,k\}$, because the function $(w_{1,j} * \rho_\beta)(\cdot,\cdot)$ is continuous in the set T, and the functions $f(\cdot,\cdot,\cdot)$ and $L(\cdot,\cdot,\cdot)$ are continuous in the set $T \times U$.

The following description shows how the estimate of the function $F_{2,j}^{\beta,i}(\cdot,\cdot)$ is obtained. The result of this formulation is that the values of this function are arbitrarily close to zero, although they are of different signs. Thus, although the function

$w_{2,j}^{\beta,i}(\cdot,\cdot)$ does not converge to the value function for the Bolza problem (1)–(4) under consideration (with the assumptions (Z)), by shifting the function to the left we obtain a new function $w_{3,j}^{\beta,i}(\cdot,\cdot)$, which does approximate the value function. The correctness of the estimation of the $F_{2,j}^{\beta,i}(\cdot,\cdot)$ values ensures that Theorem 3 is satisfied, and the main result of this paper, i.e. the convergence of the function $w_{3,j}^{\beta,i}(\cdot,\cdot)$ to the value function, is formulated in Theorem 4.

Let us formulate and prove six lemmas, which will simplify and shorten the proof of Theorem 3.

According to the proof of Theorem 3 the fact that the functions $L(\cdot,\cdot,\cdot)$ and $(L*\rho_\beta)(\cdot,\cdot,\cdot)$ have values arbitrarily close to each other is needed, so Lemma 1 should be proved first. This gives an estimate of the difference between the values of these two functions by an arbitrary real number, close to zero in $\bar{P}_j \times U$, $j \in \{-r,\dots,-1\}\cup\{1,\dots,k\}$.

Lemma 1. *Let $L(\cdot,\cdot,\cdot)$ be a function satisfying the assumptions (Z), and $\rho_\beta(\cdot,\cdot)$ be the function of class $C_0^\infty(\mathbb{R}^{n+1})$ defined above. Then for arbitrary $i \in \mathbb{N}\backslash\{0,1,2,3\}$ and η_j, $j \in \{-r,\dots,-1\}\cup\{1,\dots,k\}$ described during construction of the function $w_{1,j}(\cdot,\cdot)$ there exist $\beta_i^j > 0$ such that for all $\beta \le \beta_i^j$ and for all $(t,x,u) \in \bar{P}_j \times U$, $j \in \{-r,\dots,-1\}\cup\{1,\dots,k\}$ the following inequality holds:*

$$\big|L(t,x,u) - (L*\rho_\beta)(t,x,u)\big| < \frac{1}{i}\eta_j.$$

Proof. For $(t,x,u) \in \bar{P}_j \times U$, $j \in \{-r,\dots,-1\}\cup\{1,\dots,k\}$ the following estimation is valid:

$$\begin{aligned}\big|L(t,x,u) &- (L*\rho_\beta)(t,x,u)\big| \\ &= \Big|\int\limits_{B_\beta(\mathbb{R}^{n+1})} \big[L(t,x,u) - L(t-s,x-y,u)\big]\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y\Big| \\ &\le \sup_{\substack{u\in U,\ (t,x)\in\bar{P}_j \\ (s,y)\in B_\beta(\mathbb{R}^{n+1})}} \big|L(t,x,u) - L(t-s,x-y,u)\big|. \end{aligned} \tag{29}$$

The function $L(\cdot,\cdot,\cdot)$ is uniformly continuous in the compact sets $\bar{P}_j \times U$, $j \in \{-r,\dots,-1\}\cup\{1,\dots,k\}$, and hence

$$\sup_{\substack{u\in U,\ (t,x)\in\bar{P}_j \\ (s,y)\in B_\beta(\mathbb{R}^{n+1})}} \big|L(t,x,u) - L(t-s,x-y,u)\big| \to 0 \ \text{ as } \ \beta \to 0.$$

Consequently,

$$\big|L(t,x,u) - (L*\rho_\beta)(t,x,u)\big| \to 0 \ \text{ as } \ \beta \to 0.$$

So, for arbitrary $i \in \mathbb{N}\backslash\{0,1,2,3\}$ and η_j, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ there exist $\beta_i^j > 0$ such that for every $\beta \le \beta_i^j$ and for all $(t,x,u) \in \bar{P}_j \times U$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following is valid:

$$\left|L(t,x,u) - (L * \rho_\beta)(t,x,u)\right| < \frac{1}{i}\eta_j. \tag{30}$$

Indeed, as $L(\cdot,\cdot,\cdot)$ is the function satisfying the Lipschitz condition with the constant $M_L > 0$ with respect to t, x and uniformly with respect to u and satisfying (29), a constant $M_L > 0$ exists such that for all $(t,x,u) \in \bar{P}_j \times U$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following is valid:

$$\left|L(t,x,u) - (L * \rho_\beta)(t,x,u)\right| \le M_L\sqrt{n+1}\beta.$$

Thus, replacing β with $\eta_j/(iM_L\sqrt{n+1})$, we obtain (30), where $\beta_i^j > 0$ should be equal to

$$\beta_i^j = \frac{\eta_j}{iM_L\sqrt{n+1}}.$$

■

Moreover, $\lim_{i\to+\infty} \eta_j/(iM_L\sqrt{n+1}) = 0$ and $\lim_{\beta\to 0^+} \eta_j/(\beta M_L\sqrt{n+1}) = +\infty$, so if the natural number $i \ge 4$ is increasing or the real number $\beta > 0$ is decreasing, then an estimate of $|L(t,x,u) - (L * \rho_\beta)(t,x,u)|$ by an arbitrarily small positive real number $(1/i)\eta_j$ will be obtained.

In the proof of Theorem 3 the fact that the functions $\partial w_{2,j}^{\beta,i}/\partial x(\cdot,\cdot)f(\cdot,\cdot,\cdot)$ and $[((\partial w_{1,j}/\partial x)f(\cdot,\cdot,\cdot)) * \rho_\beta](\cdot,\cdot)$ have values arbitrarily close is required, so Lemma 2 must be proved. This gives an estimate of the difference between the values of these two functions by a real number arbitrarily close to zero in $\bar{P}_j \times U, j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$.

Lemma 2. *Let* $w_{1,j}(\cdot,\cdot)$, $w_{2,j}^{\beta,i}(\cdot,\cdot)$ *and* $\rho_\beta(\cdot,\cdot)$ *be functions defined in the subsets* $\bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ *(see (27)), and let* $f(\cdot,\cdot,\cdot)$ *be a function satisfying the assumptions (Z). Then, for an arbitrary number* $i \in \mathbb{N}\backslash\{0,1,2,3\}$ *and* η_j, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ *described during the construction of the function* $w_{1,j}(\cdot,\cdot)$, *there exist* $\bar{\beta}_i^j > 0$ *such that for all* $\beta \le \bar{\beta}_i^j$ *and for all* $(t,x,u) \in \bar{P}_j \times U, j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ *the following inequality holds:*

$$\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)f(t,x,u) - \left[\left(\frac{\partial w_{1,j}}{\partial x}f(\cdot,\cdot,u)\right) * \rho_\beta\right](t,x)\right| < \frac{1}{i}\eta_j.$$

Proof. Note that $w_{1,j}(\cdot,\cdot)$ is a function satisfying the Lipschitz condition in $\bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$. Accordingly, $|\partial w_{1,j}/\partial x| \le M_{1,j}$ for some constant $M_{1,j} > 0$.

Thus for all $(t,x,u)\in \bar{P}_j\times U, j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ the following is valid:

$$\begin{aligned}&\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)f(t,x,u)-\left[\left(\frac{\partial w_{1,j}}{\partial x}f(\cdot,\cdot,u)\right)*\rho_\beta\right](t,x)\right|\\ &\quad\le\left|\int_{B_\beta(\mathbb{R}^{n+1})}\frac{\partial}{\partial x}w_{1,j}(t-s,x-y)\big(f(t,x,u)-f(t-s,x-y,u)\big)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|\\ &\quad\le M_{1,j}\sup_{\substack{u\in U,\ (t,x)\in\bar{P}_j\\ (s,y)\in B_\beta(\mathbb{R}^{n+1})}}\big|f(t,x,u)-f(t-s,x-y,u)\big|. \qquad (31)\end{aligned}$$

As the function $s(\cdot,\cdot,\cdot)$ is uniformly continuous in the compact sets $\bar{P}_j\times U, j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$, we have

$$\sup_{\substack{u\in U,\ (t,x)\in\bar{P}_j\\ (s,y)\in B_\beta(\mathbb{R}^{n+1})}}\big|f(t,x,u)-f(t-s,x-y,u)\big|\to 0\ \text{ as }\ \beta\to 0,$$

and consequently,

$$\left|\frac{\partial w_{2,j}^{\beta,\,i}}{\partial\,x}(t,x)f(t,x,u)-\left[\left(\frac{\partial w_{1,\,j}}{\partial x}f(\cdot,\cdot,u)\right)*\rho_\beta\right](t,x)\right|\to 0\ \text{ as }\ \beta\to 0.$$

Thus, for arbitrary $i\in\mathbb{N}\backslash\{0,1,2,3\}$ and $\eta_j,\ j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ there exist $\bar{\beta}_i^j>0$ such that for all $\beta\le\bar{\beta}_i^j$ and for all $(t,x,u)\in\bar{P}_j\times U,\ j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ the following holds:

$$\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)f(t,x,u)-\left[\left(\frac{\partial w_{1,j}}{\partial x}f(\cdot,\cdot,u)\right)*\rho_\beta\right](t,x)\right|<\frac{1}{i}\eta_j. \qquad (32)$$

Since $f(\cdot,\cdot,\cdot)$ is the function satisfying the Lipschitz condition with constant $M_f>0$ with respect to t, x and uniformly with respect to u and satisfying (31), a constant $M_f>0$ must exist such that for all $(t,x,u)\in\bar{P}_j\times U, j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ the following holds:

$$\begin{aligned}&\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)f(t,x,u)-\left[\left(\frac{\partial w_{1,j}}{\partial x}f(\cdot,\cdot,u)\right)*\rho_\beta\right](t,x)\right|\\ &\qquad\le M_{1,j}M_f\sqrt{n+1}\beta.\end{aligned}$$

Hence, by replacing β with $\eta_j/(iM_{1,j}M_f\sqrt{n+1})$, we obtain (32), where $\bar{\beta}_i^j>0$ should be

$$\bar{\beta}_i^j=\frac{\eta_j}{iM_{1,j}M_f\sqrt{n+1}}.$$

■

Moreover, $\lim_{i\to+\infty}\eta_j/(iM_{1,j}M_f\sqrt{n+1}) = 0$ and $\lim_{\beta\to0^+}\eta_j/(\beta M_{1,j}M_f \sqrt{n+1}) = +\infty$, so if the natural number $i \geq 4$ is increasing or the real number $\beta > 0$ is decreasing, then an estimate of

$$\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)f(t,x,u) - \left[\left(\frac{\partial w_{1,\,j}}{\partial x}f(\cdot,\cdot,u)\right) * \rho_\beta\right](t,x)\right|$$

by an arbitrarily small positive real number $(1/i)\eta_j$ will be obtained.

In the proof of Theorem 3 the uniform convergence of the sequence $(\partial w_{2,j}^{\beta,i}/\partial x)(t,x)$ to $(\partial w_{1,j}/\partial x)(t,x)$ as $\beta \to 0$, for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ is also required as ensured by the following result.

Lemma 3. *Let $w_{1,j}(\cdot,\cdot)$, $w_{2,j}^{\beta,i}(\cdot,\cdot)$ and $\rho_\beta(\cdot,\cdot)$ be functions defined in the subsets $\bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ (see (27)). Then, for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ we have*

$$\lim_{\beta\to0}\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) = \frac{\partial w_{1,j}}{\partial x}(t,x)$$

and this convergence is uniform.

Proof. According to the definition of the uniform convergence of a function sequence, to prove that this lemma holds, it is sufficient to show that for arbitrary $\varepsilon_j > 0$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ a $\beta_i^j > 0$ exists such that for every $\beta \leq \beta_i^j$ and for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following holds:

$$\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) - \frac{\partial w_{1,j}}{\partial x}(t,x)\right| \leq \varepsilon_j.$$

For all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ we have the following estimate:

$$\begin{aligned}
&\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) - \frac{\partial w_{1,j}}{\partial x}(t,x)\right| \\
&\quad = \left|\left(\frac{\partial w_{1,j}}{\partial x} * \rho_\beta\right)(t,x) - \frac{\partial w_{1,j}}{\partial x}(t,x)\right| \\
&\quad = \left|\int\limits_{B_\beta(\mathbb{R}^{n+1})}\left[\frac{\partial}{\partial x}w_{1,j}(t-s,x-y) - \frac{\partial}{\partial x}w_{1,j}(t,x)\right]\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y\right| \\
&\quad \leq \sup_{\substack{(t,x)\in\bar{P}_j \\ (s,y)\in B_\beta(\mathbb{R}^{n+1})}}\left|\frac{\partial}{\partial x}w_{1,j}(t-s,x-y) - \frac{\partial}{\partial x}w_{1,j}(t,x)\right|. \qquad (33)
\end{aligned}$$

The function $(\partial w_{1,j}/\partial x)(\cdot,\cdot)$ is uniformly continuous in the compact sets $\bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, and hence

$$\sup_{\substack{(t,x)\in\bar{P}_j \\ (s,y)\in B_\beta(\mathbb{R}^{n+1})}}\left|\frac{\partial}{\partial x}w_{1,j}(t-s,x-y) - \frac{\partial}{\partial x}w_{1,j}(t,x)\right| \to 0 \text{ as } \beta \to 0.$$

Consequently

$$\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)-\frac{\partial w_{1,j}}{\partial x}(t,x)\right|\to 0 \ \text{ as } \ \beta\to 0.$$

Therefore, for an arbitrary $\varepsilon_j>0,\ j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ a $\beta_i^j>0$ exists such that for all $\beta\le\beta_i^j$ and for all $(t,x)\in\bar{P}_j,\ j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ the following holds:

$$\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)-\frac{\partial w_{1,j}}{\partial x}(t,x)\right|\le\varepsilon_j.$$

■

To simplify and shorten the notation, the following abbreviations will be used:

$$g_{2,j}^{\beta,i}(t,x,u):=\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)f(t,x,u)+L(t,x,u),$$

$$g_{1,j}(t,x,u):=\frac{\partial w_{1,j}}{\partial x}(t,x)f(t,x,u)+L(t,x,u),$$

for $(t,x,u)\in\bar{P}_j\times U, j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$, where $\beta>0,\ i\in\mathbb{N}\backslash\{0,1,2,3\}$.

It is also useful to employ the following notation:

$$p_{2,j}^{\beta,i}(t,x):=\min_{u\in U}g_{2,j}^{\beta,i}(t,x,u)=g_{2,j}^{\beta,i}\big(t,x,u_{2,j}^{\beta,i}(t,x)\big),$$

$$p_{1,j}(t,x):=\min_{u\in U}g_{1,j}(t,x,u)=g_{1,j}\big(t,x,u_{1,j}(t,x)\big),$$

for $(t,x)\in\bar{P}_j, j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$, where $\beta>0,\ i\in\mathbb{N}\backslash\{0,1,2,3\}$.

Lemma 3 immediately implies the following Conclusion 1 ensuring the uniform convergence of the function sequence $g_{2,j}^{\beta,i}(t,x,u)$ to $g_{1,j}(t,x,u)$ as $\beta\to 0$, for all $(t,x,u)\in\bar{P}_j\times U,\ j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$.

Conclusion 1. *Let $w_{1,j}(\cdot,\cdot)$, $w_{2,j}^{\beta,i}(\cdot,\cdot)$ and $\rho_\beta(\cdot,\cdot)$ be functions defined in the subsets $\bar{P}_j,\ j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ (see (27)), and let $g_{1,j}(\cdot,\cdot,\cdot)$ and $g_{2,j}^{\beta,i}(\cdot,\cdot,\cdot)$ be the functions defined above. Therefore, if for all $(t,x)\in\bar{P}_j,\ j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ we have*

$$\lim_{\beta\to 0}\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)=\frac{\partial w_{1,j}}{\partial x}(t,x)$$

and the convergence is uniform, then, for all $(t,x,u)\in\bar{P}_j\times U,\ j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$,

$$\lim_{\beta\to 0}g_{2,j}^{\beta,i}(t,x,u)=g_{1,j}(t,x,u)$$

and this convergence is also uniform.

Proof. According to the definition of the uniform convergence of function sequences, in order to prove this conclusion, it is sufficient to show that for arbitrary $\varepsilon'_j > 0$, $j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ there exist $\bar{\beta}_i^j > 0$ such that for every $\beta \le \bar{\beta}_i^j$ and for all $(t, x, u) \in \bar{P}_j \times U, j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ the following holds:

$$\left| g_{2,j}^{\beta,i}(t, x, u) - g_{1,j}(t, x, u) \right| \le \varepsilon'_j.$$

Using the definitions of the functions $g_{1,j}(\cdot, \cdot, \cdot)$ and $g_{2,j}^{\beta,i}(\cdot, \cdot, \cdot)$, we obtain

$$\left| g_{2,j}^{\beta,i}(t, x, u) - g_{1,j}(t, x, u) \right| = \left| \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t, x) - \frac{\partial w_{1,j}}{\partial x}(t, x) \right| |f(t, x, u)|$$

for $(t, x, u) \in \bar{P}_j \times U,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$.

According to Lemma 3, for an arbitrary $\varepsilon_j > 0,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ a $\beta_i^j > 0$ exists such that for all $\beta \le \beta_i^j$ and for all $(t, x) \in \bar{P}_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ we have $|(\partial w_{2,j}^{\beta,i}/\partial x)(t, x) - (\partial w_{1,j}/\partial x)(t, x)| \le \varepsilon_j$, and the function $f(\cdot, \cdot, \cdot)$ is bounded by a constant $M > 0$ in $T \times U$. Hence, for all $(t, x, u) \in \bar{P}_j \times U$, $j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$, the following estimate is valid:

$$\left| g_{2,j}^{\beta,i}(t, x, u) - g_{1,j}(t, x, u) \right| \le \varepsilon'_j,$$

if we set $\varepsilon'_j = \varepsilon_j M$.

We have $\varepsilon_j \to 0$ as $\beta \to 0$ (see the proof of Lemma 3) and $\varepsilon'_j = \varepsilon_j M$, and so $\varepsilon'_j \to 0$ as $\beta \to 0$. ■

The uniform convergence of the sequence $p_{2,j}^{\beta,i}(t, x)$ to $p_{1,j}(t, x)$ as $\beta \to 0$ for all $(t, x) \in \bar{P}_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$, is also required and leads to the following result:

Lemma 4. *Let $p_{2,j}^{\beta,i}(\cdot, \cdot)$ and $p_{1,j}(\cdot, \cdot)$ be the functions defined above. Then for all $(t, x) \in \bar{P}_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$, $\lim_{\beta \to 0} p_{2,j}^{\beta,i}(t, x) = p_{1,j}(t, x)$ and the convergence is uniform.*

Proof. Let us first prove the pointwise convergence of the sequence $p_{2,j}^{\beta,i}(t, x)$ to $p_{1,j}(t, x)$ as $\beta \to 0$, for all $(t, x) \in \bar{P}_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$, and then prove its uniform convergence.

1°. To prove this, assume that $\lim_{\beta \to 0} p_{2,j}^{\beta,i}(\bar{t}, \bar{x}) < p_{1,j}(\bar{t}, \bar{x})$ for some pair $(\bar{t}, \bar{x}) \in \bar{P}_j$. Then, for this pair and an arbitrary $\bar{\varepsilon} > 0$ there exists $\delta > 0$ such that for all $\beta \le \bar{\varepsilon}$ the following inequality holds:

$$\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(\bar{t}, \bar{x}) f\big(\bar{t}, \bar{x}, u_{2,j}^{\beta,i}(\bar{t}, \bar{x})\big) + L\big(\bar{t}, \bar{x}, u_{2,j}^{\beta,i}(\bar{t}, \bar{x})\big)$$

$$< \frac{\partial w_{1,j}}{\partial x}(\bar{t}, \bar{x}) f\big(\bar{t}, \bar{x}, u_{1,j}(\bar{t}, \bar{x})\big) + L\big(\bar{t}, \bar{x}, u_{1,j}(\bar{t}, \bar{x})\big) - \delta.$$

It can easily be seen that by replacing in this inequality the control $u_{1,j}(\cdot,\cdot)$, corresponding to the minimum in the formula for $p_{1,j}(\cdot,\cdot)$, with the control $u_{2,j}^{\beta,i}(\cdot,\cdot)$, for this pair $(\bar{t},\bar{x}) \in \bar{P}_j$ the following holds:

$$\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(\bar{t},\bar{x}) f\big(\bar{t},\bar{x},u_{2,j}^{\beta,i}(\bar{t},\bar{x})\big) + L\big(\bar{t},\bar{x},u_{2,j}^{\beta,i}(\bar{t},\bar{x})\big)$$

$$< \frac{\partial w_{1,j}}{\partial x}(\bar{t},\bar{x}) f\big(\bar{t},\bar{x},u_{2,j}^{\beta,i}(\bar{t},\bar{x})\big) + L\big(\bar{t},\bar{x},u_{2,j}^{\beta,i}(\bar{t},\bar{x})\big) - \delta,$$

which implies

$$-\left[\frac{\partial w_{1,j}}{\partial x}(\bar{t},\bar{x}) - \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(\bar{t},\bar{x})\right] f\big(\bar{t},\bar{x},u_{2,j}^{\beta,i}(\bar{t},\bar{x})\big) < -\delta. \tag{a1}$$

According to Conclusion 1 we have

$$\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) - \frac{\partial w_{1,j}}{\partial x}(t,x)\right| |f(t,x,u)| \le \varepsilon_j'$$

for all $(t,x,u) \in \bar{P}_j \times U$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, where $\varepsilon_j' \to 0$ as $\beta \to 0$, so setting $\varepsilon_j' = \delta/2$ for the pair $(\bar{t},\bar{x}) \in \bar{P}_j$ and considering assumption 1°, we get

$$\left[\frac{\partial w_{1,j}}{\partial x}(\bar{t},\bar{x}) - \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(\bar{t},\bar{x})\right] f\big(\bar{t},\bar{x},u_{2,j}^{\beta,i}(\bar{t},\bar{x})\big) \le \delta/2. \tag{a2}$$

Adding the corresponding left-hand sides and similarly the right-hand sides of inequalities (a1) and (a2), we obtain $0 < -\delta/2$. This leads to a contradiction with assumption 1°, i.e. that $\lim\limits_{\beta\to 0} p_{2,j}^{\beta,i}(\bar{t},\bar{x}) < p_{1,j}(\bar{t},\bar{x})$ for some pair $(\bar{t},\bar{x}) \in \bar{P}_j$.

2°. Now let $\lim_{\beta\to 0} p_{2,j}^{\beta,i}(\bar{t},\bar{x}) > p_{1,j}(\bar{t},\bar{x})$ for some pair $(\bar{t},\bar{x}) \in \bar{P}_j$. Then, for this pair and an arbitrary $\bar{\varepsilon} > 0$ there exists $\delta > 0$ such that for all $\beta \le \bar{\varepsilon}$ the following inequality holds:

$$\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(\bar{t},\bar{x}) f\big(\bar{t},\bar{x},u_{2,j}^{\beta,i}(\bar{t},\bar{x})\big) + L\big(\bar{t},\bar{x},u_{2,j}^{\beta,i}(\bar{t},\bar{x})\big)$$

$$> \frac{\partial w_{1,j}}{\partial x}(\bar{t},\bar{x}) f\big(\bar{t},\bar{x},u_{1,j}(\bar{t},\bar{x})\big) + L\big(\bar{t},\bar{x},u_{1,j}(\bar{t},\bar{x})\big) + \delta.$$

Clearly, replacing the control $u_{2,j}^{\beta,i}(\cdot,\cdot)$, corresponding to the minimum in the formula for the function $p_{2,j}^{\beta,i}(\cdot,\cdot)$, with the control $u_{1,j}(\cdot,\cdot)$, for this pair $(\bar{t},\bar{x}) \in \bar{P}_j$ the following holds:

$$-\delta + \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(\bar{t},\bar{x}) f\big(\bar{t},\bar{x},u_{1,j}(\bar{t},\bar{x})\big) + L\big(\bar{t},\bar{x},u_{1,j}(\bar{t},\bar{x})\big)$$

$$> \frac{\partial w_{1,j}}{\partial x}(\bar{t},\bar{x}) f\big(\bar{t},\bar{x},u_{1,j}(\bar{t},\bar{x})\big) + L\big(\bar{t},\bar{x},u_{1,j}(\bar{t},\bar{x})\big),$$

which implies

$$-\left[\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(\bar{t},\bar{x})-\frac{\partial w_{1,j}}{\partial x}(\bar{t},\bar{x})\right] f\big(\bar{t},\bar{x},u_{1,j}(\bar{t},\bar{x})\big)<-\delta. \qquad \text{(b1)}$$

According to Conclusion 1 we have

$$\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)-\frac{\partial w_{1,j}}{\partial x}(t,x)\right| |f(t,x,u)| \le \varepsilon_j'$$

for all $(t,x,u)\in \bar{P}_j\times U$, $j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$, where $\varepsilon_j'\to 0$ as $\beta\to 0$, and hence by substituting $\varepsilon_j'=\delta/2$ for the pair $(\bar{t},\bar{x})\in\bar{P}_j$ and considering assumption 2°, we obtain

$$\left[\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(\bar{t},\bar{x})-\frac{\partial w_{1,j}}{\partial x}(\bar{t},\bar{x})\right] f\big(\bar{t},\bar{x},u_{1,j}(\bar{t},\bar{x})\big)\le\delta/2. \qquad \text{(b2)}$$

Adding the corresponding left- and right-hand sides of inequalities (b1) and (b2), we obtain $0<-\delta/2$, which leads to a contradiction with assumption 2° that $\lim_{\beta\to 0} p_{2,j}^{\beta,i}(\bar{t},\bar{x})>p_{1,j}(\bar{t},\bar{x})$ for some pair $(\bar{t},\bar{x})\in\bar{P}_j$.

As both the assumptions, i.e. 1° that $\lim_{\beta\to 0} p_{2,j}^{\beta,i}(\bar{t},\bar{x})<p_{1,j}(\bar{t},\bar{x})$ for some pair $(\bar{t},\bar{x})\in\bar{P}_j$, and 2° that $\lim_{\beta\to 0} p_{2,j}^{\beta,i}(\bar{t},\bar{x})>p_{1,j}(\bar{t},\bar{x})$ for this pair $(\bar{t},\bar{x})\in\bar{P}_j$, lead to the above contradiction, it is proved that $\lim_{\beta\to 0} p_{2,j}^{\beta,i}(t,x)=p_{1,j}(t,x)$ for all pairs $(t,x)\in\bar{P}_j$, $j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$.

Accordingly, the pointwise convergence of the sequence $p_{2,j}^{\beta,i}(t,x)$ to $p_{1,j}(t,x)$ as $\beta\to 0$ for all $(t,x)\in\bar{P}_j$, $j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ is proved. In order to ensure that this convergence is uniform, it is sufficient to prove that for an arbitrary $\varepsilon_j'>0$, $j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ a $\bar{\beta}_i^j>0$ exists such that for all $\beta\le\bar{\beta}_i^j$ and for all $(\bar{t},\bar{x})\in\bar{P}_j$, $j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ the following inequality holds:

$$\left|p_{2,j}^{\beta,i}(t,x)-p_{1,j}(t,x)\right|\le\varepsilon_j'.$$

From the first part of the proof we have that for an arbitrary $\varepsilon_j'>0$, $j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}$ and an arbitrary, fixed pair $(\bar{t},\bar{x})\in\bar{P}_j$ there can be found $\bar{\beta}_i^j>0$, for which Conclusion 1 is satisfied and the above inequality holds at the point $(\bar{t},\bar{x})\in\bar{P}_j$, where $i\in\mathbb{N}\backslash\{0,1,2,3\}$.

Now consider an arbitrary fixed $\beta\le\bar{\beta}_i^j$ calculated from the pair $(\bar{t},\bar{x})\in\bar{P}_j$, for the following two cases:

(i) Let $(t,x)\in Z_j'$, where

$$Z_j'=\{(t,x)\in\bar{P}_j,\quad j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}\colon\ p_{2,j}^{\beta,i}(t,x)\ge p_{1,j}(t,x)\}.$$

Then, due to the definitions of functions $p_{2,j}^{\beta,i}(\cdot,\cdot)$ and by $p_{1,j}(\cdot,\cdot)$ and replacing the control $u_{2,j}^{\beta,i}(\cdot,\cdot)$, corresponding the minimum in the formula for the function $p_{2,j}^{\beta,i}(\cdot,\cdot)$, with the control $u_{1,j}(\cdot,\cdot)$, we obtain the following condition for $(t,x)\in Z_j'$:

$$\begin{aligned}
0 &\le p_{2,j}^{\beta,i}(t,x) - p_{1,j}(t,x)\\
&= \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big) + L\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)\\
&\quad - \frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u_{1,j}(t,x)\big) - L\big(t,x,u_{1,j}(t,x)\big)\\
&\le \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) f\big(t,x,u_{1,j}(t,x)\big) + L\big(t,x,u_{1,j}(t,x)\big)\\
&\quad - \frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u_{1,j}(t,x)\big) - L\big(t,x,u_{1,j}(t,x)\big)\\
&\le \left[\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) - \frac{\partial w_{1,j}}{\partial x}(t,x)\right] f\big(t,x,u_{1,j}(t,x)\big) \le \varepsilon_j M = \varepsilon_j',
\end{aligned}$$

which is implied directly from Conclusion 1 and assumption (i).

(ii) Now let $(t,x)\in Z_j''$, where

$$Z_j'' = \{(t,x)\in \bar{P}_j, \quad j\in\{-r,\dots,-1\}\cup\{1,\dots,k\}:\ p_{2,j}^{\beta,i}(t,x) < p_{1,j}(t,x)\}.$$

Similarly, due to the definitions of functions $p_{2,j}^{\beta,i}(\cdot,\cdot)$ and $p_{1,j}(\cdot,\cdot)$, and by replacing the control $u_{1,j}(\cdot,\cdot)$, corresponding to the minimum in the formula for the function $p_{1,j}(\cdot,\cdot)$, with the control $u_{2,j}^{\beta,i}(\cdot,\cdot)$, we obtain the following condition for $(t,x)\in Z_j''$:

$$\begin{aligned}
0 &< p_{1,j}(t,x) - p_{2,j}^{\beta,i}(t,x)\\
&= \frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u_{1,j}(t,x)\big) + L\big(t,x,u_{1,j}(t,x)\big)\\
&\quad - \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big) - L\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)\\
&\le \frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big) + L\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)\\
&\quad - \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x) f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big) - L\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)\\
&\le \left[\frac{\partial w_{1,j}}{\partial x}(t,x) - \frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)\right] f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big) \le \varepsilon_j M = \varepsilon_j',
\end{aligned}$$

which is implied directly from Conclusion 1 and assumption (ii).

Thus, in both cases (i) and (ii), i.e. for $(t,x) \in Z'_j \cup Z''_j = \bar{P}_j$, the estimate $|p^{\beta,i}_{2,j}(t,x) - p_{1,j}(t,x)| \le \varepsilon'_j$ is valid, and as β was chosen arbitrarily, this implies the uniform convergence of the sequence $p^{\beta,i}_{2,j}(t,x)$ to $p_{1,j}(t,x)$ as $\beta \to 0$ for all pairs $(t,x) \in \bar{P}_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$. ■

To prove Theorem 3 the uniform convergence of the sequence $u^{\beta,i}_{2,j}(t,x)$ to $u_{1,j}(t,x)$ is required as $\beta \to 0$ for all $(t,x) \in \bar{P}_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, which is guaranted by the following result.

Lemma 5. *Let $p^{\beta,i}_{2,j}(\cdot,\cdot)$ and $p_{1,j}(\cdot,\cdot)$ be the functions defined above and let the following assumptions be satisfied:*

(L1) an $\bar{\alpha}_j > 0$ exists such that for $(t,x) \in \bar{P}_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ we have $|(\partial w_{1,j}/\partial x)(t,x)| \ge \bar{\alpha} j$;

(L2) for every $u_0 \in U$ a subset $U_0 \subset U$ exists and $\alpha_f > 0$ exists such that for all $u_1, u_2 \subset U_0$ and $(t,x) \in \bar{P}_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the inequality $\alpha_f |u_1 - u_2| \le |f(t,x,u_1) - f(t,x,u_2)|$ holds;

(L3) $\bar{\alpha}_j \alpha_f - M_L > 0$, where $M_L > 0$ is a Lipschitz constant for the function $L(\cdot,\cdot,\cdot)$.

Hence, if $p^{\beta,i}_{2,j}(t,x) \to p_{1,j}(t,x)$ as $\beta \to 0,\ (t,x) \in \bar{P}_j$ uniformly, then also $u^{\beta,i}_{2,j}(t,x) \to u_{1,j}(t,x)$ as $\beta \to 0,\ (t,x) \in \bar{P}_j$ uniformly, i.e. if for an arbitrary $\varepsilon'_j > 0$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ a $\bar{\beta}^j_i > 0$ exists such that for every $\beta \le \bar{\beta}^j_i$ and for all $(t,x) \in \bar{P}_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ we have $|p^{\beta,i}_{2,j}(t,x) - p_{1,j}(t,x)| \le \varepsilon'_j$, then for an arbitrary $\varepsilon''_j > 0,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, a $\bar{\bar{\beta}}^j_i > 0$ exists such that for every $\beta \le \bar{\bar{\beta}}^j_i$ and for all $(t,x) \in \bar{P}_j, j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ holds $\big|u^{\beta,i}_{2,j}(t,x) - u_{1,j}(t,x)\big| \le \varepsilon''_j$.

Proof. It can easily be seen that for all $(t,x) \in \bar{P}_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following is true:

$$
\begin{aligned}
&\big|p^{\beta,i}_{2,j}(t,x) - p_{1,j}(t,x)\big| \\
&\qquad = \bigg|\frac{\partial w^{\beta,i}_{2,j}}{\partial x}(t,x) f\big(t,x,u^{\beta,i}_{2,j}(t,x)\big) + L\big(t,x,u^{\beta,i}_{2,j}(t,x)\big) \\
&\qquad\quad - \frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u_{1,j}(t,x)\big) - L\big(t,x,u_{1,j}(t,x)\big)\bigg| \\
&\qquad \ge \bigg|\frac{\partial w^{\beta,i}_{2,j}}{\partial x}(t,x) f\big(t,x,u^{\beta,i}_{2,j}(t,x)\big) - \frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u_{1,j}(t,x)\big)\bigg| \\
&\qquad\quad - \Big|L\big(t,x,u^{\beta,i}_{2,j}(t,x)\big) - L\big(t,x,u_{1,j}(t,x)\big)\Big| \\
&\qquad = \bigg|\frac{\partial w^{\beta,i}_{2,j}}{\partial x}(t,x) f\big(t,x,u^{\beta,i}_{2,j}(t,x)\big) - \frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u^{\beta,i}_{2,j}(t,x)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)-\frac{\partial w_{1,j}}{\partial x}(t,x) f\big(t,x,u_{1,j}(t,x)\big)\Big| \\
&-\Big|L\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)-L\big(t,x,u_{1,j}(t,x)\big)\Big| \\
\geq\ &-\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)-\frac{\partial w_{1,j}}{\partial x}(t,x)\right|\left|f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)\right| \\
&+\left|\frac{\partial w_{1,j}}{\partial x}(t,x)\right|\Big|f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)-f\big(t,x,u_{1,j}(t,x)\big)\Big| \\
&-\Big|L\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)-L\big(t,x,u_{1,j}(t,x)\big)\Big|. \qquad (34)
\end{aligned}
$$

From Lemma 4 we have the following condition:

$$\left|p_{2,j}^{\beta,i}(t,x)-p_{1,j}(t,x)\right|\leq\varepsilon_j'.$$

Also, from Conclusion 1 we have

$$-\left|\frac{\partial w_{2,j}^{\beta,i}}{\partial x}(t,x)-\frac{\partial w_{1,j}}{\partial x}(t,x)\right|\left|f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)\right|\geq-\varepsilon_j'.$$

Furthermore, from assumptions (L1)–(L3) in Lemma 5 the following three conditions are also satisfied:

$$
\begin{aligned}
\left|\frac{\partial w_{1,j}}{\partial x}(t,x)\right| &\geq \bar{\alpha}_j, \\
\Big|f\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)-f\big(t,x,u_{1,j}(t,x)\big)\Big| &\geq \alpha_f\left|u_{2,j}^{\beta,i}(t,x)-u_{1,j}(t,x)\right|, \\
-\Big|L\big(t,x,u_{2,j}^{\beta,i}(t,x)\big)-L\big(t,x,u_{1,j}(t,x)\big)\Big| &\geq -M_L\left|u_{2,j}^{\beta,i}(t,x)-u_{1,j}(t,x)\right|.
\end{aligned}
$$

Thus, due to (34) and the above conditions, the following inequality holds:

$$
\begin{aligned}
\varepsilon_j' &\geq \left|p_{2,j}^{\beta,i}(t,x)-p_{1,j}(t,x)\right| \\
&\geq -\varepsilon_j'+\big(\bar{\alpha}_j\alpha_f-M_L\big)\left|u_{2,j}^{\beta,i}(t,x)-u_{1,j}(t,x)\right|.
\end{aligned}
$$

Hence

$$\left|u_{2,j}^{\beta,i}(t,x)-u_{1,j}(t,x)\right|\leq\frac{2\varepsilon_j'}{\bar{\alpha}_j\alpha_f-M_L}=\varepsilon_j''.$$

According to the proof of Conclusion 1, we have $\varepsilon_j'\to 0$ as $\beta\to 0$, and furthermore, if $\varepsilon_j''=2\varepsilon_j'/(\bar{\alpha}_j\alpha_f-M_L)$, then $\varepsilon_j''\to 0$ as $\beta\to 0$. ■

Before formulating the main theorem in this section, a new function $(t,x)\to\bar{F}_{2,j}^{\beta,i}(t,x)$ must be defined. It must also be proved that, as $\beta\to 0$, this function is uniformly convergent to the function $(t,x) \to F_{1,j}(t,x)$ defined by (22), (25). The uniform convergence of this function is proved in Lemma 6 below.

Let $(t,x) \to u^{\beta,i}_{2,j}(t,x)$ be a function corresponding the minimum in the definition of the function $(t,x) \to F^{\beta,i}_{2,j}(t,x)$, i.e. in (28). Define, in the sets $\bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ a function $(t,x) \to \bar{F}^{\beta,i}_{2,j}(t,x)$ as follows:

$$\bar{F}^{\beta,i}_{2,j}(t,x) := \frac{\partial}{\partial t} w_{1,j}(t,x) + \frac{\partial w_{1,j}}{\partial x}(t,x) f\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) + L\left(t,x,u^{\beta,i}_{2,j}(t,x)\right).$$

Lemma 6. *Let $\bar{F}^{\beta,i}_{2,j}(\cdot,\cdot)$ be the function defined above, where $w_{1,j}(\cdot,\cdot)$ is the function defined by (21) and (24) in the subsets $\bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, and $f(\cdot,\cdot,\cdot)$ and $L(\cdot,\cdot,\cdot)$ are the functions satisfying the assumptions (Z). Then for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following is satisfied:*

$$\lim_{\beta \to 0} \left| \bar{F}^{\beta,i}_{2,j}(t,x) - F_{1,j}(t,x) \right| = 0,$$

and the convergence is uniform.

Proof. Let $(t,x) \to u_{1,j}(t,x)$ be the function realising the minimum in the definition of $(t,x) \to F_{1,j}(t,x)$ defined by (22) and (25). Note that for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following is satisfied:

$$\begin{aligned}
&\left| \bar{F}^{\beta,i}_{2,j}(t,x) - F_{1,j}(t,x) \right| \\
&\quad = \left| \frac{\partial}{\partial t} w_{1,j}(t,x) + \frac{\partial w_{1,j}}{\partial x}(t,x) f\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) + L\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) \right. \\
&\quad\quad \left. - \frac{\partial}{\partial t} w_{1,j}(t,x) - \frac{\partial w_{1,j}}{\partial x}(t,x) f\left(t,x,u_{1,j}(t,x)\right) - L\left(t,x,u_{1,j}(t,x)\right) \right| \\
&\quad \le \left| \frac{\partial w_{1,j}}{\partial x}(t,x) \right| \left| f\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) - f\left(t,x,u_{1,j}(t,x)\right) \right| \\
&\quad\quad + \left| L\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) - L\left(t,x,u_{1,j}(t,x)\right) \right| \\
&\quad \le M_{1,j} M_f \left| u^{\beta,i}_{2,j}(t,x) - u_{1,j}(t,x) \right| + M_L \left| u^{\beta,i}_{2,j}(t,x) - u_{1,j}(t,x) \right| \\
&\quad = \left(M_{1,j} M_f + M_L \right) \left| u^{\beta,i}_{2,j}(t,x) - u_{1,j}(t,x) \right|,
\end{aligned} \tag{35}$$

because the functions $f(\cdot,\cdot,\cdot)$ and $L(\cdot,\cdot,\cdot)$ satisfy the Lipschitz condition with respect to t, x and u, and the function $w_{1,j}(\cdot,\cdot)$ satisfies the Lipschitz condition with respect to t and x.

Lemmas 3–5 imply that if for $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ we have $\lim_{\beta \to 0} (\partial w^{\beta,i}_{2,j}/\partial x)(t,x) = (\partial w_{1,j}/\partial x)(t,x)$, then $\lim_{\beta \to 0} u^{\beta,i}_{2,j}(t,x) = u_{1,j}(t,x)$

and the convergence is uniform. Hence, it follows that for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following is satisfied:

$$\lim_{\beta\to 0} \left|\bar{F}^{\beta,i}_{2,j}(t,x) - F_{1,j}(t,x)\right| = 0. \qquad \blacksquare$$

We now formulate and prove the section's main theorem, which gives the evaluation of the functions $F^{\beta,i}_{2,j}(\cdot,\cdot)$. In the proof of Theorem 3 the results of Lemmas 1–6 are used.

Theorem 3. *Let $w^{\beta,i}_{2,j}(\cdot,\cdot)$ and $F^{\beta,i}_{2,j}(\cdot,\cdot)$ be the functions defined in the sets $\bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ by (27) and (28), respectively, and suppose that the assumptions (L1)–(L3) from Lemma 5 are satisfied. Then for arbitrary $i \in \mathbb{N}\backslash\{0,1,2,3\}$ and μ_j, η_j, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, described during the estimation (26) of the values of the function $F_{1,j}(\cdot,\cdot)$ defined by (22) and (25), and for the numbers $\xi_j = \varepsilon''_j(M_{1,j}M_f + M_L)$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, a $\beta^j_i > 0$ exists such that for all $\beta \le \beta^j_i$ and for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following inequality holds:*

$$-\xi_j - \mu_j + \frac{i-4}{i}\eta_j \;\le\; F^{\beta,i}_{2,j}(t,x) \;\le\; \xi_j.$$

Proof. Note that the formula for the function $F^{\beta,i}_{2,j}(\cdot,\cdot)$ can be transformed in such a way that for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ the following is satisfied:

$$\begin{aligned}
F^{\beta,i}_{2,j}(t,x) &= \frac{\partial}{\partial t} w^{\beta,i}_{2,j}(t,x) + \frac{\partial}{\partial x} w^{\beta,i}_{2,j}(t,x) f\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) + L\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) \\
&= L\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) - (L*\rho_\beta)\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) + \frac{i-2}{i}\eta_j \\
&\quad + \left[\left(\frac{\partial w_{1,j}}{\partial t} + \frac{\partial w_{1,j}}{\partial x} f\left(\cdot,\cdot,u^{\beta,i}_{2,j}\right) + L\left(\cdot,\cdot,u^{\beta,i}_{2,j}\right)\right) * \rho_\beta\right](t,x) \\
&\quad + \frac{\partial}{\partial x} w^{\beta,i}_{2,j}(t,x) f\left(t,x,u^{\beta,i}_{2,j}(t,x)\right) \\
&\quad - \left[\left(\frac{\partial w_{1,j}}{\partial x} f\left(\cdot,\cdot,u^{\beta,i}_{2,j}\right)\right) * \rho_\beta\right](t,x).
\end{aligned} \tag{36}$$

In order to estimate the values of the function $F^{\beta,i}_{2,j}(\cdot,\cdot)$, it is sufficient to estimate the values of the individual terms in (36).

From Lemma 5 and the proof of Lemma 6, the following estimation is valid:

$$\left|\bar{F}^{\beta,i}_{2,j}(t,x) - F_{1,j}(t,x)\right| < \varepsilon''_j(M_{1,j}M_f + M_L),$$

where $\beta > 0$, $i \in \mathbb{N}\backslash\{0,1,2,3\}$, $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$, M_f and M_L are the Lipschitz constants for the functions $f(\cdot,\cdot,\cdot)$ and $L(\cdot,\cdot,\cdot)$, respectively, and the constant $M_{1,j} > 0$ is such that $|(\partial w_{1,j}/\partial x)(\cdot,\cdot)| \le M_{1,j}$.

Therefore, using the estimation (26) of the values of $F_{1,j}(\cdot,\cdot)$ and setting $\xi_j = \varepsilon''_j(M_{1,j}M_f + M_L)$, the following inequality holds:

$$-\xi_j - \mu_j \leq \bar{F}^{\beta,i}_{2,j}(t,x) \leq \xi_j - \eta_j, \quad (t,x) \in \bar{P}_j, \quad j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}. \quad (37)$$

From Lemma 1 we know that for arbitrary $i \in \mathbb{N}\backslash\{0,1,2,3\}$ and numbers η_j, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ bounding the values of the function $F_{1,j}(\cdot,\cdot)$ from above (see (26)) there exist $\beta^j_i > 0$ such that for all $\beta \leq \beta^j_i$ and for all $(t,x,u) \in \bar{P}_j \times U$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ we get

$$\left|L(t,x,u) - (L * \rho_\beta)(t,x,u)\right| < \frac{1}{i}\eta_j.$$

Furthermore, Lemma 2 ensures that for arbitrary $i \in \mathbb{N}\backslash\{0,1,2,3\}$ and the numbers η_j, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ bounding the values of the function $F_{1,j}(\cdot,\cdot)$ from above (see (26)) there exist $\bar{\beta}^j_i > 0$ such that for all $\beta \leq \bar{\beta}^j_i$ and for all $(t,x,u) \in \bar{P}_j \times U$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ we have

$$\left|\frac{\partial w^{\beta,i}_{2,j}}{\partial x}(t,x) f(t,x,u) - \left[\left(\frac{\partial w_{1,j}}{\partial x} f(\cdot,\cdot,u)\right) * \rho_\beta\right](t,x\right| < \frac{1}{i}\eta_j.$$

Let us note that the above conditions are valid for all $\beta \leq \min\{\beta^j_i, \bar{\beta}^j_i\}$, where $i \in \mathbb{N}\backslash\{0,1,2,3\}$ is arbitrary and fixed, and $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$.

Therefore, using the values of all terms in (36), based on the definition of the convolution, inequality (37), and Lemmas 1 and 2, it is possible to estimate the values of the function $F^{\beta,i}_{2,j}(\cdot,\cdot)$ for all $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ as follows:

$$\begin{aligned} -\xi_j - \mu_j + \frac{i-4}{i}\eta_j &\leq -\frac{1}{i}\eta_j + \frac{i-2}{i}\eta_j + ((-\xi_j - \mu_j) * \rho_\beta)(t,x) - \frac{1}{i}\eta_j \\ &\leq F^{\beta,i}_{2,j}(t,x) \\ &\leq \frac{1}{i}\eta_j + \frac{i-2}{i}\eta_j + ((\xi_j - \eta_j) * \rho_\beta)(t,x) + \frac{1}{i}\eta_j = \xi_j. \end{aligned}$$

■

In order to simplify the notation, we formulate the following Conclusion 2 from Theorem 3.

Conclusion 2. *Using $\nu^i_j = \mu_j - \frac{i-4}{i}\eta_j$, the following estimation of the values of the function $F^{\beta,i}_{2,j}(\cdot,\cdot)$ can be obtained:*

$$-\xi_j - \nu^i_j \leq F^{\beta,i}_{2,j}(t,x) \leq \xi_j \quad (38)$$

for $(t,x) \in \bar{P}_j$, $j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ and ξ_j defined above (see (37)).

It should be noted that as $\xi_j = \varepsilon''_j(M_{1,j}M_f + M_L)$ and ε''_j can be arbitrarily small, then also ξ_j can be arbitrarily small. Moreover, as μ_j, η_j, $j \in \{-r,\dots,-1\} \cup$

$\{1,\ldots,k\}$ are arbitrarily small for $r,\ k\to+\infty$, and also $(i-4)/i\to 1$ as $i\to+\infty$, the numbers $\nu_j^i=\mu_j-((i-4)/i)\eta_j$ are also arbitrarily small. The function $F_{2,j}^{\beta,i}(\cdot,\cdot)$ defined in $\bar{P}_j,\ j\in\{-r,\ldots,-1\}\cup\{1,\ldots,k\}$ can have values of different signs, but close to zero.

The above fact implies that the dynamic programming inequality (17) is not satisfied yet. Therefore, a new function $w_{3,j}^{\beta,i}(\cdot,\cdot)$ must be defined in $\bar{P}_j,\ j\in\{-r,\ldots,-1\}\cup\{1,\ldots,k\}$, by shifting the function $w_{2,j}^{\beta,i}(\cdot,\cdot)$ to the left as follows:

$$w_{3,j}^{\beta,i}(t,x):=w_{2,j}^{\beta,i}(t,x)+\xi_j(b-t), \tag{39}$$

where the numbers ξ_j are defined above and the function $w_{2,j}^{\beta,i}(\cdot,\cdot)$ defined by (27) satisfies Theorem 3.

Using the previous construction steps described in this section (and summarised in the algorithm of Section 5), we now define the following function:

$$\begin{aligned} F_{3,j}^{\beta,i}(t,x) &:= \frac{\partial}{\partial t}w_{3,j}^{\beta,i}(t,x)+\min_{u\in U}\left\{\frac{\partial w_{3,j}^{\beta,i}}{\partial x}(t,x)f(t,x,u)+L(t,x,u)\right\} \\ &= F_{2,j}^{\beta,i}(t,x)-\xi_j, \end{aligned} \tag{40}$$

for $(t,x)\in\bar{P}_j,\ j\in\{-r,\ldots,-1\}\cup\{1,\ldots,k\}$.

The main result of this work is formulated in Theorem 4, which ensures the convergence of the function $w_{3,j}^{\beta,i}(\cdot,\cdot)$ to the value function.

Theorem 4. *Let the functions $w_{3,j}^{\beta,i}(\cdot,\cdot)$ and $F_{3,j}^{\beta,i}(\cdot,\cdot)$ be defined in the sets $\bar{P}_j$, $j\in\{-r,\ldots,-1\}\cup\{1,\ldots,k\}$ by (39) and (40), respectively. Then the values of the function $F_{3,j}^{\beta,i}(\cdot,\cdot)$ can be estimated as follows:*

$$-2\xi_j-\nu_j^i\le F_{3,j}^{\beta,i}(t,x)\le 0, \tag{41}$$

where $(t,x)\in\bar{P}_j,\ j\in\{-r,\ldots,-1\}\cup\{1,\ldots,k\}$, and the numbers ξ_j and ν_j^i are defined above.

Proof. Theorem 4 is implied immediately from Conclusion 2, from the definitions of the functions $w_{3,j}^{\beta,i}(\cdot,\cdot)$ and $F_{3,j}^{\beta,i}(\cdot,\cdot)$, and the numbers ξ_j and ν_j^i. ■

Note that both the functions $w_{3,j}^{\beta,i}(\cdot,\cdot)$ and $F_{3,j}^{\beta,i}(\cdot,\cdot)$ defined above satisfy the assumptions of Proposition 3 for arbitrarily small $\beta>0$, because ξ_j and ν_j^i can be arbitrarily small as $j\to+\infty$ and $i\to+\infty$. The inequality (17) from Proposition 3 will be satisfied if we use $\varepsilon=\max\{2\xi_j+\nu_j^i:\ \ j\in\{-r,\ldots,-1\}\cup\{1,\ldots,k\}\}$ for arbitrary and fixed $i\in\mathbb{N}\backslash\{0,1,2,3\}$.

It is worth noting here that the function $w_{3,j}^{\beta,i}(\cdot,\cdot)$ constructed in this paper, belonging to the class $C^1(T)$, is an ε-value function of the Bolza problem (1)–(4), under assumptions (Z). Clearly, with the knowledge of an effective formula defining the ε-value function, the value $w_{3,j}^{\beta,i}(a,c)$ of an approximate minimum for the considered problem can be calculated and hence the difference from the infimum of the Bolza functional can be estimated.

Section 5 presents, in some detail, a numerical algorithm for determining the formula for the ε-value function.

5. Numerical Algorithm

The numerical algorithm proposed here is used for constructing the ε-value function and calculating the value of the approximate minimum of the Bolza functional for the problem under consideration, (1)–(4) with the assumptions (Z).

Algortihm:

1. Read the real numbers a, b, c $(a < b,\ x(a) = c)$ and the natural numbers n, m; define the compact subsets $T \subset [a, b] \times \mathbb{R}^n$ and $U \subset \mathbb{R}^m$.

2. Read the required calculation accuracy $\varepsilon_0 > 0$.

3. Define the functions $(t, x, u) \to f(t, x, u)$, $(t, x, u) \to L(t, x, u)$ described in the Bolza problem (1)–(4). Calculate:

   (i) the Lipschitz constants M_f and M_L for these functions,

   (ii) the constant α_f from Lemma 5,

   (iii) the constant M bounding the function $(t, x, u) \to f(t, x, u)$ in T.

4. Choose a starting function $(t, x) \to w(t, x)$ of class $C^1(T)$ and define the function $(t, x) \to F(t, x)$ of (19) in the set T.

5. Calculate κ_d and κ_g, i.e. the infimum and supremum of the function $(t, x) \to F(t, x)$ in the set T, respectively.

6. Read the step size $h > 0$ of the subinterval of the interval $[\kappa_d, \kappa_g]$.

7. Partition the interval $[\kappa_d, \kappa_g]$ into $r+k$ subintervals with step size h as follows:

   (a) if $\kappa_d < 0 < \kappa_g$, then

   $$\kappa_d = y_{-r} < y_{-r+1} < \cdots < y_{-1} < y_0 < y_1 < \cdots < y_k = \kappa_g,$$

   where $y_0 = 0$,

   $$y_j = y_0 + jh, \quad j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\},$$

   $$r = -\kappa_d/h, \quad k = \kappa_g/h.$$

   (b) if $\kappa_d \geq 0$, then

   $$\kappa_d = y_0 < y_1 < \cdots < y_k = \kappa_g, \quad y_j = y_0 + jh,$$

   $$j \in \{1, \ldots, k\}, \quad k = (\kappa_g - \kappa_d)/h.$$

(c) if $\kappa_g \le 0$, then

$$\kappa_d = y_{-r} < \cdots < y_{-1} < y_0 = \kappa_g, \quad y_j = y_0 + jh,$$
$$j \in \{-r, \ldots, -1\}, \quad r = (\kappa_g - \kappa_d)/h.$$

Remark 2. The following calculations will be completed for Case (a); in Cases (b) and (c), the calculations are analogous.

8. Calculate the subsets $P_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ of the set T.

9. Calculate the numbers $1 < \gamma_j < 2,\ j \in \{1, \ldots, k\}$ and define the function $(t, x) \to w_{1,j}(t, x)$ using (21) in $P_j,\ j \in \{1, \ldots, k\}$.

10. Calculate the numbers $0 < \delta_j < 1,\ j \in \{-r, \ldots, -1\}$ and define the functions $(t, x) \to w_{1,j}(t, x)$ using (24) in $P_j,\ j \in \{-r, \ldots, -1\}$.

11. Calculate the lower bounds $\bar{\alpha}_j$ for $|(\partial w_{1,j}/\partial x)(\cdot, \cdot)|$ and check if the assumptions (L1)–(L3) from Lemma 5 are satisfied.

    If NOT, then choose another starting function and repeat the calculations starting from Step 4.

    If YES, then define the functions $(t, x) \to F_{1,j}(t, x)$ using (22) in $P_j,\ j \in \{1, \ldots, k\}$ and $(t, x) \to F_{1,j}(t, x)$ using (25) in $P_j,\ j \in \{-r, \ldots, -1\}$.

12. Calculate the numbers $\mu_j,\ \eta_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ which will be used in order to estimate (see (26)) the values of the function $(t, x) \to F_{1,j}(t, x)$:

$$\mu_j = \begin{cases} -y_{j-1} + \gamma_j y_j & \text{for} \quad j \in \{1, \ldots, k\}, \\ -y_j + \delta_j y_{j+1} & \text{for} \quad j \in \{-r, \ldots, -1\}, \end{cases}$$

    and

$$\eta_j = \begin{cases} -y_j + \gamma_j y_j & \text{for} \quad j \in \{1, \ldots, k\} \\ -y_{j+1} + \delta_j y_{j+1} & \text{for} \quad j \in \{-r, \ldots, -1\}. \end{cases}$$

13. Calculate the numbers $M_{1,j}$ bounding $|(\partial w_{1,j}/\partial x)(\cdot, \cdot)|$ from above and the number $\beta > 0$ as the minimum of all numbers $\beta_j^i,\ \bar{\beta}_j^i$, for $j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ and an arbitrary and fixed $i \in \mathbb{N} \backslash \{0, 1, 2, 3\}$, using the formulae from Lemmas 1 and 2 (see Section 4).

14. Define the functions $(t, x) \to w_{2,j}^{\beta,i}(t, x)$ using (27) and $(t, x) \to F_{2,j}^{\beta,i}(t, x)$ using (28) in $P_j,\ j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$.

15. Calculate the numbers from Conclusion 2, i.e. $\nu_j^i = \mu_j - ((i-4)/i)\eta_j$, for $j \in \{-r, \ldots, -1\} \cup \{1, \ldots, k\}$ and an arbitrary, fixed $i \in \mathbb{N} \backslash \{0, 1, 2, 3\}$.

16. Calculate the numbers $\varepsilon'_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ from Conclusion 1 based on the formula $\varepsilon'_j = \varepsilon_j M$, where $\varepsilon_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ are the constants from Lemma 3, and $M > 0$ is the constant bounding the function $(t,x,u) \to f(t,x,u)$.

17. Calculate the numbers $\varepsilon''_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ from Lemma 5 based on the formula

$$\varepsilon''_j = \frac{2\varepsilon'_j}{\bar{\alpha}_j \alpha_f - M_L}.$$

18. Calculate the numbers $\xi_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$ from Theorem 3 based on the formula $\xi_j = \varepsilon''_j (M_{1,j} M_f + M_L)$.

19. Define the functions $(t,x) \to w^{\beta,i}_{3,j}(t,x)$ using (39) and $(t,x) \to F^{\beta,i}_{3,j}(t,x)$ using (40) in $P_j,\ j \in \{-r,\dots,-1\} \cup \{1,\dots,k\}$.

20. Calculate the value $w^{\beta,i}_{3,j}(a,c)$ of the approximate minimum of the Bolza functional and the achieved calculation accuracy $\varepsilon = \max\{2\,\xi_j + \nu^i_j :\ j \in \{-\mathrm{r},\dots,-1\} \cup \{1,\dots,k\}\}$, for an arbitrary and fixed $i \in \mathbb{N}\backslash\{0,1,2,3\}$. Check if $\varepsilon > \varepsilon_0$.

21. If YES, then the required calculation accuracy was not achieved. Calculate a new step size h, e.g. by replacing h with $h/2$, and repeat the calculations from Steps 7–20, or choose another starting function $(t,x) \to w(t,x)$ of class $C^1(T)$ and repeat the calculations from Steps 4–20.

    If NO, i.e. if the requirement $\varepsilon > \varepsilon_0$ is not fulfilled, this means that the required calculation accuracy is achieved. Print the results of the calculations and terminate the program.

A computer program for this algorithm has been written in Pascal. Computations were made for the example illustrating the method described in this paper. The results of these calculations, based on an example, are described in Section 6.

6. Example

According to the numerical algorithm proposed in Section 5 some calculations were made to illustrate the power of the method described in this paper for constructing the ε-value function. The following provides a tutorial example of the application of the algorithm.

The approximate minimum of the following Bolza functional is sought:

$$J(x,u) = \int_0^1 \left[x^2(t) + u^2(t)\right] \mathrm{d}t,$$

subject to

$$\dot{x}(t) = u(t) \quad \text{a.e. in } [0,1],$$

$$u(t) \in U, \quad t \in [0,1],$$

$$x(0) = 1.$$

In this example we have

$$a = 0, \;\; b = 1, \;\; c = 1, \;\; n = 1, \;\; m = 1, \;\; U = [1,3], \;\; T = [0,1] \times [1,3],$$

$$f(t,x,u) = u, \quad L = (t,x,u) = x^2 + u^2,$$

with $M_f = 1$, $M = 3$, $\alpha_f = 1$, $M_L = 6$. The calculations are made to the accuracy $\varepsilon_0 = 0.1$.

The chosen starting function $w(\cdot,\cdot)$ of class $C^1(T)$ is as follows:

$$w(t,x) = 9t + x^2 + 6x, \quad (t,x) \in T.$$

The function $F(\cdot,\cdot)$ is then given by the formula

$$F(t,x) = -6x, \quad (t,x) \in T,$$

and its infimum and supremum are respectively equal to

$$\kappa_d = -18, \quad \kappa_g = -6.$$

Let the step size $h > 0$ used for the partition of the interval $[\kappa_d, \kappa_g] = [-18,-6]$ be $h = 3$. The number of the subsets of the interval $[\kappa_d, \kappa_g] = [-18,-6]$ is equal to $r = (\kappa_g - \kappa_d)/h = 4$, so the next calculations will be made for $j \in \{-r, \dots, -1\} = \{-4,-3,-2,-1\}$.

The partition points of the interval $[\kappa_d, \kappa_g] = [-18,-6]$ are calculated as follows:

$$y_j = y_0 + jh, j \in \{-4,-3,-2,-1\}, \quad y_0 = \kappa_g.$$

The subsets P_j, $j \in \{-4,-3,-2,-1\}$ of the set T are calculated as

$$P_j := \{(t,x) \in T: \;\; y_j \le F(t,x) < y_{j+1}\}, \quad j \in \{-4,-3,-2\},$$

$$P_j := \{(t,x) \in T: \;\; y_j \le F(t,x) \le y_{j+1}\}, \quad j = -1.$$

Let the numbers $0 < \delta_j < 1$ be equal to $\delta_j = 0.99$, $j \in \{-4,-3,-2,-1\}$. The functions $w_{1,j}(\cdot,\cdot)$ defined in P_j, $j \in \{-4,-3,-2,-1\}$ are given by the formulae

$$w_{1,j}(t,x) = (9 - \delta_j y_{j+1})t + x^2 + 6x + \delta_j y_{j+1} b.$$

The numbers $\bar{\alpha}_j$ bounding $|(\partial w_{1,j}(t,x))/\partial x| = |2x+6|$ from below for $(t,x) \in P_j$, $j \in \{-4,-3,-2,-1\}$ are equal to $\bar{\alpha}_{-4} = 11$, $\bar{\alpha}_{-3} = 10$, $\bar{\alpha}_{-2} = 9$, $\bar{\alpha}_{-1} = 8$.

Note that assumptions (L1)–(L3) from Lemma 5 are satisfied because $\bar{\alpha}_j \alpha_f - M_L > 0$ for $j \in \{-4,-3,-2,-1\}$. Thus the following construction guarantees that there can be achieved a good result, i.e. the obtaining of an effective formula for the ε-value function.

If assumptions (L1)–(L3) are not satisfied, then the starting function should be replaced by another and all the calculations should be repeated for this new function. Fortunately, the assumptions (L1)–(L3) for Lemma 5 are satisfied, and the calculations can be continued.

The functions $F_{1,j}(\cdot,\cdot)$ are defined in $P_j,\ j \in \{-4,-3,-2,-1\}$ by

$$F_{1,j}(t,x) = -6x - \delta_j y_{j+1}.$$

The numbers $\mu_j,\ \eta_j$ are lower and upper bounds, respectively, for $F_{1,j}(\cdot,\cdot)$, in $P_j,\ j \in \{-4,-3,-2,-1\}$. They can be calculated as follows:

$$\mu_j = -y_j + \delta_j y_{j+1}, \quad \eta_j = -y_{j+1} + \delta_j y_{j+1}.$$

The numbers $M_{1,j}$ bounding $|(\partial w_{1,j}(t,x))/\partial x| = |2x+6|$ from above for $(t,x) \in P_j$, $j \in \{-4,-3,-2,-1\}$ are equal to $M_{1,-4} = 12,\ M_{1,-3} = 11,\ M_{1,-2} = 10,\ M_{1,-1} = 9$.

Since $M_{1,j} > M_L$ and $M_f = 1$, the numbers $\bar{\beta}^i_j$ from Lemma 2 are smaller than β^i_j from Lemma 1. Therefore, it is only necessary to calculate the numbers $\bar{\beta}^i_j = \eta_j/(iM_{1,j}M_f\sqrt{2})$. The calculations were made for $i = 1000$ and $j \in \{-4,-3,-2,-1\}$.

Additional calculations were made for $\beta = \min \bar{\beta}^i_j$. For this case $\beta = 0.0000047140$ was used. In order to construct the function $(t,x) \to w^{\beta,i}_{2,j}(t,x)$ given by the formula:

$$w^{\beta,i}_{2,j}(t,x) := (w_{1,j} * \rho_\beta)(t,x) - \frac{i-2}{i}\eta_j(b-t),$$

the function $(t,x) \to \rho_\beta(t,x)$ defined below (see Adams, 1975) can be used. First define

$$\rho_1(t,x) = \begin{cases} Ke^{-1/1-(t^2+x^2)} & \text{for} \quad \sqrt{t^2+x^2} \le 1, \\ 0 & \text{for} \quad \sqrt{t^2+x^2} > 1, \end{cases}$$

where the constant K is chosen so that $\int_{\mathbb{R}^{n+1}} \rho_1(t,x)\,\mathrm{d}t\,\mathrm{d}x = 1$, and in this example we have $n = 1$. Also note that $\rho_1 :\ \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^\infty_0(\mathbb{R}^{n+1})$ having compact support and such that $\operatorname{supp}\rho_1 \subset B_1(\mathbb{R}^{n+1})$, where $B_1(\mathbb{R}^n)$ is a ball in $\mathbb{R}^n$, with centre 0 and radius 1.

Thus the function $\rho_\beta(t,x) := (1/\beta^{n+1})\rho_1(t/\beta, x/\beta) \in C^\infty_0(\mathbb{R}^{n+1})$ is defined by

$$\rho_\beta(t,x) = \begin{cases} K\dfrac{1}{\beta^2}e^{-\beta^2/(\beta^2-(t^2+x^2))} & \text{for} \quad \sqrt{t^2+x^2} \le \beta, \\ 0 & \text{for} \quad \sqrt{t^2+x^2} > \beta. \end{cases}$$

The next calculations were made using the function $(t,x) \to \rho_\beta(t,x)$ defined above, and for $\beta = 0.0000047140$ and $i = 1000$. The functions $w^{\beta,i}_{2,j}(\cdot,\cdot)$ defined in

$P_j,\ j \in \{-4,-3,-2,-1\}$ are given by the formulae

$$w_{2,j}^{\beta,i}(t,x) = \left(9 - \delta_j y_{j+1} + \frac{i-2}{i}\eta_j\right) t + x^2 + 2(3+\beta)x$$

$$+ \beta(\beta + 15 - \delta_j y_{j+1}) + b\left(\delta_j y_{j+1} - \frac{i-2}{i}\eta_j\right).$$

The functions $F_{2,j}^{\beta,i}(\cdot,\cdot)$ defined in $P_j,\ j \in \{-4,-3,-2,-1\}$ are given by the formulae

$$F_{2,j}^{\beta,i}(t,x) = -2(3+\beta)x - (3+\beta)^2 + 9 - \delta_j y_{j+1} + \frac{i-2}{i}\eta_j.$$

The numbers $\nu_j^i = \mu_j - \frac{i-4}{i}\eta_j$ are calculated for $j \in \{-4,-3,-2,-1\}$ and $i = 1000$. Choosing $\varepsilon_j = 0.001$ and calculating the numbers $\varepsilon_j' = \varepsilon_j M$, $\varepsilon_j'' = 2\varepsilon_j'/\bar{\alpha}_j\alpha_f - M_L$ and $\xi_j = \varepsilon_j''(M_{1,j}M_f + M_L)$ for $j \in \{-4,-3,-2,-1\}$, the functions $w_{3,j}^{\beta,i}(\cdot,\cdot)$ defined in $P_j,\ j \in \{-4,-3,-2,-1\}$ are given by the formulae:

$$w_{3,j}^{\beta,i}(t,x) = \left(9 - \delta_j y_{j+1} + \frac{i-2}{i}\eta_j - \xi_j\right) t + x^2 + 2(3+\beta)x$$

$$+ \beta(\beta + 15 - \delta_j y_{j+1}) + b\left(\delta_j y_{j+1} - \frac{i-2}{i}\eta_j + \xi_j\right).$$

The functions $F_{3,j}^{\beta,i}(\cdot,\cdot)$ defined in $P_j,\ j \in \{-4,-3,-2,-1\}$ are given by

$$F_{3,j}^{\beta,i}(t,x) = -2(3+\beta)x - (3+\beta)^2 + 9 - \delta_j y_{j+1} + \frac{i-2}{i}\eta_j - \xi_j.$$

The desired value $w_{3,j}^{\beta,i}(a,c) = w_{3,j}^{\beta,i}(0,1)$ of the approximate minimum is equal to 1.0452281402. For the step size $h = 3$ $(r = 4)$ the calculation accuracy $\varepsilon = 3.09024$ was achieved.

The required calculation accuracy was not achieved because $\varepsilon > \varepsilon_0 = 0.1$. Therefore, the calculations were repeated for a smaller step size of $h = 0.015$ $(r = 800)$ and the following results were obtained: the value of the approximate minimum was equal to $w_{3,j}^{\beta,i}(0,1) = 1.0422565818$, and the calculation accuracy was given by $\varepsilon = 0.09927$. Clearly, the required accuracy was achieved and the parameter ε reached the stopping criterion $\varepsilon < \varepsilon_0 = 0.1$.

7. Final Conclusions

The theory, illustrated via the above example shows that, using the method described in this paper, a suitably chosen arbitrary starting function $(t,x) \to w(t,x)$ of class $C^1(T)$ can be used to construct an ε-value function for the Bolza optimal-control problem. An arbitrary starting function may not lead to a proper result of construction of the ε-value function, although the example shows that if the assumptions given in Lemma 5 (Section 4) are satisfied, this method guarantees that the effective formula for the ε-value function can be obtained. Furthermore, the method enables the calculation of the value of the approximate minimum for the Bolza functional to be made.

The example of Section 6 also shows that the achieved calculation accuracy will be better if a smaller step size $h > 0$ is used for the partition of the interval $[\kappa_d, \kappa_g]$ into the required subintervals, during construction of the subsets $P_j,\ j \in \{-r, \dots, -1\} \cup \{1, \dots, k\}$ of the set T.

There is some significance in the way in which the numbers $0 < \delta_j < 1,\ j \in \{-r, \dots, -1\},\ 1 < \gamma_j < 2,\ j \in \{1, \dots, k\},\ \varepsilon_j,\ j \in \{-r, \dots, -1\} \cup \{1, \dots, k\}$ (from Lemma 3) and the natural number $i \geq 4$ are chosen. The example illustrates that as the numbers δ_j and γ_j tend to 1, the numbers ε_j become close to 0, and once the natural number i becomes larger (i.e. the number $\beta > 0$ is smaller), then the estimations obtained for the consecutive functions defined in Section 4 improve.

This paper is concerned with the use of the classical dynamic programming method. However, the dual dynamic programming approach could be the subject of a future investigation. Furthermore, whilst this paper is based on a rigorous study of the theory behind the derivation of the approximate minimum of the Bolza functional, future studies could combine this work with numerical strategies described by Polak (1997).

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